VARIANCE ESTIMATION IN STARATIFIED RANDOM SAMPLING IN THE PRESENCE OF TWO AUXILIARY VARIABLES

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(Mathematics – Statistics Option)

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2014

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MS300-0009/12

A Thesis submitted to Pan African University Institute for Basic Sciences, Technology and Innovation in partial fulfillment of the requirements for the degree of Master of Science in Mathematics (Statistics Option)

2014

Declaration

This thesis is my original work and has not been submitted to any other university for examination.

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This thesis has been submitted for examination with our approval as University supervisors.

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Acknowledgements

First of all, I give Glory to God for bringing me this far and giving me good health and knowledge while doing this thesis. My heartfelt gratitude goes to my respectable Supervisors George Otieno Orwa (Phd) and Romanus Odhiambo Otieno (Prof) for their constructive advice, suggestions, ideas and comments in all phases of the thesis. Furthermore, I appreciate my dear classmates and friends for encouraging me to work hard. Finally, I would like to extend my sincere appreciation and thanks to Pan Africa University and Jomo Kenyatta University for kind assistance in many ways.

Dedication

This thesis is dedicated to my family.

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Abstract

The objective of this thesis is to develop improved population variance estimators in the presence of two auxiliary variables under stratified random sampling. New estimators of population variance of the study variable were suggested using prior information on two auxiliary variables**.** The mean square errors of the proposed estimators have been obtained using first order approximation of Taylor series method. Efficiency comparisons of proposed estimators have been discussed and achieved improvement under certain conditions. Results are also supported by numerical analysis. Based on result obtained, the proposed ratio- type variance estimators may be preferred over traditional ratio- type ${s^2}_t$ and sample estimator of population variance ${s^2}_{st,y}$ for the use in practical applications.

CHAPTER ONE

INTRODUCTION

1.1.Back ground of the study

In sample surveys, to estimate unknown population parameter(s) more accurately, it is common to utilize information on auxiliary variable(s) such as, population mean, population standard deviation, and population coefficient of variation and population kurtosis in many situations at the estimation stage to increase precision of estimators. The other methods in sample survey which helps researchers to obtain efficient estimate of unknown population parameter(s) is sampling design. Since estimation depends on sampling design, it is important to choose an appropriate sampling design. Stratified Random Sampling is a method of sampling that involves the division of a population into smaller groups known as strata. In Stratified Random Sampling, the aim is to represent the population by a sample in the most accurate way. In this method the population data are grouped in strata and then the data in each stratum are randomly selected. Therefore, the determination of the number of strata and sample sizes in each stratum is very important to obtain accurate estimates. For the determination of the number of strata, Cingi (1994) states that the optimal number of strata for a large population size can be approximately 10 if there is no prior information about the scheme for the stratification. For the optimal sample sizes of the strata, there are some popular methods, such as the Neyman Allocation and the Best Allocation methods (Cingi and Kadilar 2009). The detailed information about these methods can be found in Cochran,(1977) and in Singh (2003b). While stratifying the data, each of the population data should be located in only one of the strata and the strata consist of all data in the population. In addition, this stratification is done so that the variance in the stratum should be minimum and the variances among the strata should be maximum (Cingi and Kadilar 2009).The aim of stratification in general is to select representative sample and increase precision of estimates. In this thesis, improved population variance estimators were presented under stratified random sampling design.

1.2. Notations

Consider a finite population P $=\{P_1, P_2, P_3, \dots, P_N\}$ of N units. Let the study and two auxiliary variables be denoted by Y, X and Z associated with each P_j (j=1, 2…, N) of the population respectively. Let the population be stratified into K strata with h^{th} stratum containing N_h units, where h=1, 2, 3,…, K such that $\sum_{h=1}^{k} N_h$=N and from the h^{th} stratum, a sample n_h is drawn by simple random sampling without replacement such that $\sum_{h=1}^{k} n_h$=n. Let (y_{hi}, x_{hi}, z_{hi}) denote the observed values of Y, X, and Z on the i^{th} unit of the h^{th} stratum where i=1, 2, …, N_h. The population variance of the study variable (y) and the auxiliary variables are defined as follows.

$$(N-1){S^2}_{st,y} = \sum_{h=1}^{k}\sum_{i=1}^{N_h}(y_{hi}-\bar{Y})^2 = \sum_{h=1}^{k}\sum_{i=1}^{N_h}[(y_{hi}-\bar{Y}_h)+(\bar{Y}_h-\bar{Y})]^2$$

Where $\bar{Y}_h$ is the population mean of the variate of interest in stratum h, and y_{hi} is the value of the i^{th} observation of variate of interest in stratum h. For large sample size, assuming that $N \cong N-1$ and $N_h \cong N_h-1$, then ${S^2}_{st,y} \cong \sum_{h=1}^{k}\omega_h {S^2}_{yh} + \sum_{h=1}^{k}\omega_h(\bar{Y}_h - \bar{Y})^2$

$\bar{y}_h = \frac{1}{n_h} \sum_{h=1}^{k} y_{hi}$ - sample mean of h^{th} stratum.

$\bar{y}_{st} = \sum_{h=1}^{k} \omega_h \bar{y}_h$ - is the estimator of population mean of the study variable.

$\bar{Y}_h = \frac{1}{N_h} \sum_{h=1}^{N_h} y_{hi}$ -population mean of h^{th} stratum.

$S^2{}_{yh} = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)^2$ - population variance of h^{th} stratum.

$s^2{}_{yh} = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (y_{hi} - \bar{y}_h)^2$- estimator of population variance in the h^{th} stratum. Similar expression are defined for the auxiliary variables x and z.

1.3 . *Statement of the problem*

Kadilar and Cingi (2006) suggested a modified population variance estimator under simple random sampling by adapting estimators of Shapir and Yaab (2003) where coefficients of variation of the auxiliary variable X and it's mean $\bar{X}$ and variance ${S_x}^2$ were used. The proposed estimator in their case was found to be more efficient in estimating variance of the study variable. Kadilar and Cingi (2009), proposed ratio-cum-product type estimators using the population variance, coefficient of variation and kurtosis of a single auxiliary variable which have been found more efficient than the traditional sample estimator of population variance in simple random sampling and stratified sampling under realistic condition. Other many scholars mentioned in literature also proposed different estimators utilizing prior information of single auxiliary variable and multi-auxiliary variables to estimate population variance of the study variable in simple random sampling and two-phase sampling design. However, under stratified random sampling design not much has been done to developed estimators of population variance using information of multi-auxiliary variables like under simple and two-phase sampling. In sample survey theory, if information is available on auxiliary variables, information on these are incorporated in estimation of some population characteristics to increases precision of the estimate. Considering the importance of population variance for construction of confidence interval and prior information of auxiliary variables to increase precision of estimate population variance, this thesis sought to estimate population variance of the study variable utilizing prior information on two auxiliary variables in stratified random sampling design.

1.4. Objectives

1.4.1. General Objective of the study

- To develop improved variance estimators in the presence of auxiliary variables.

1.4.2. Specific Objectives

1. To develop an improved variance estimator when two sets of auxiliary information are available with the study variable.

2. To study the consistency properties of the proposed estimators

3. To compare the efficiency of the proposed estimators relative to the classical sample and adapted estimators of population variance.

*1.5.**Organization of the Thesis***

The rest of the thesis is organized as follows. Chapter two present literature reviews which discusses previous works done by other researchers on population variance estimation of a study variable under different sampling designs utilizing available information on auxiliary variable(s). In Chapter Three, some adapted estimators and new proposed estimators of unknown population variance of variable of interest are presented. Derivation of mean square error of proposed estimators and efficiency comparisons of suggested estimators with existing estimators were considered in this study using numerical data. Chapter Four gives results and discussion of the study. Chapter Five of these thesis present the general conclusion of the study and recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.1. Introduction

In this chapter, literature review on population variance estimation is presented from previous studies related to this study. This enabled us to identify shortcoming of previous studies to avoid repetition of work done by others.

2.2. Variance estimation using auxiliary information

In sample surveys, to improve the sampling design and to obtain more efficient estimators of population parameters under study, different type of techniques/ methods for utilizing auxiliary information obtained from previous census or database administration was described. The role of auxiliary information is to increase precision of estimators of unknown population characteristics of interest. Ratio –type estimators improve the precision of estimate of the population variance of a study variable by using prior information on auxiliary variable (s) which is correlated with the study variable Y. For ratio estimators in sampling theory, population information available on the auxiliary variable (s), such as population kurtosis and population coefficient of variation, is commonly used to increase efficiency of population variance estimators**.** Liu (1974) gave a general class of quadratic estimators for variance and obtained a class of unbiased estimators under certain conditions. Das and Tripathi (1978) defined six estimators of population variances using known information on parameters of auxiliary variable. Using prior information on parameters of auxiliary variable/variables, Srivastava and Jhajj(1980, 1995), Isaki (1983), Singh and Kataria (1990), Prasad and Singh (1990,1992), Ahmed et al.(2000) have defined estimators or classes of estimators of $S^2{}_y$. Singh and Singh (2001) defined ratio-type estimator for population variance using multi-auxiliary information and showed that the suggested estimator was more efficient than the usual unbiased estimator and the Isaki (1983) estimator. AlJararha and Ahmed (2002) defined two classes of estimators of $S^2{}_y$ by using prior information on parameter of one of the two auxiliary variables under double sampling scheme. Ahmed et al.(2003) gave some chain ratio-type as well as chain product-type estimators of $S^2{}_y$ under two-phase sampling scheme. Singh and Singh (2003) proposed a regression-type estimator in two-phase sampling for population variance when information on second variable was known and variance of main auxiliary variable was not known. The suggested estimator was more efficient than Chand (1975), Kiregyera (1980, 1984) and usual ratio and regression estimators. Kadilar and Cingi (2005) suggested ratio-type estimators for population variance of variable of interest using population kurtosis and coefficient of variation of auxiliary variable in simple and stratified random sampling. The proposed estimators in their case were more efficient than other estimators considered in their study. Kadilar and Cingi (2006) also proposed a new ratio and regression estimator for the population variance using auxiliary variable in simple random sampling. They obtained the mean square error of the suggested estimator and shows that the proposed estimator of population variance is more efficient than the traditional ratio and regression estimators, suggested by Isaki (2000), under certain conditions. Many other contributions on variance estimation are present in sampling literature. To measure the variations within the values of variable of interest, survey sampling researchers and statistician give attention to estimation of population variance. From the above literature, few previous studies were done in stratified random sampling to estimate population variance of a study variable. This study mainly focuses on population variance estimators using prior knowledge of two auxiliary variables in stratified sampling design.

CHAPTER THREE

METHODOLOGY

3.1. Introduction

In this chapter, new estimators of unknown population variance of study variable using information on two auxiliary variables were proposed and their asymptotic expressions of the mean squared errors and their minimum values are derived up to first degree of approximation and conditions are derived under which the proposed estimators are more efficient than other estimators considered in this study.

3.2. Adapted estimators

Koyuncu and Kadilar (2009), defined the classical ratio estimator to estimate the population mean of the study variable Y in the stratified random sampling when there are two auxiliary variables as follows:

$$\hat{\bar{y}}_{st}=\bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)\left(\frac{\bar{Z}}{\bar{z}_{st}}\right) \text{ --- } (3.1)$$

Where $\bar{X}$ and $\bar{Z}$ are the population mean of the two auxiliary variables and $\bar{x}_{st}$,$\bar{z}_{st}$ and $\bar{y}_{st}$ are sample estimates of the population mean in stratified random sampling scheme. The regression estimator of the population mean $\bar{Y}$ also defined as:

$$\hat{\bar{Y}}_{reg}=\bar{y}_{st}+\beta_1(\bar{X}-\bar{x}_{st})+\beta_2(\bar{Z}-\bar{z}_{st}) \text{ -- } (3.2)$$

Where $\beta_1 = \frac{s_{yx}}{s^2{}_x}$ and $\beta_2 = \frac{s_{yz}}{s^2{}_z}$. Here $s^2{}_x$ and $s^2{}_z$ are the sample variances of x and z respectively, s_{yx} and s_{yz} are the sample covariance's between y and x and between y and z respectively. Adapting the estimator given in (3.1) and (3.2) to estimate the population mean of the study variable and assuming the population variance of the two auxiliary variables are known, we develop the following ratio-product type and regression estimators for the variance.

$$s^2{}_t = s^2{}_{st,y} \left(\frac{S^2{}_x}{s^2{}_{st,x}}\right)\left(\frac{S^2{}_z}{s^2{}_{st,z}}\right) \quad (3.3)$$

$$s^2{}_{reg} = s^2{}_{st,y} + \beta_1 (S^2{}_x - s^2{}_{st,x}) + \beta_2 (S^2{}_z - s^2{}_{st,z}) \quad (3.4)$$

Where $s^2{}_{st,x} = \sum_{h=1}^{k} \omega_h s^2{}_{xh} + \sum_{h=1}^{k} \omega_h (\bar{x}_h - \bar{x}_{st})^2$, $s^2{}_{st,y} = \sum_{h=1}^{k} \omega_h s^2{}_{yh} + \sum_{h=1}^{k} \omega_h (\bar{y}_h - \bar{y}_{st})^2$, and $s^2{}_{st,z} = \sum_{h=1}^{k} \omega_h s^2{}_{zh} + \sum_{h=1}^{k} \omega_h (\bar{z}_h - \bar{z}_{st})^2$, are the sample estimators of the population variance of each variable in stratified sampling scheme when neglecting population correction factor of each stratum. The mean square error of the variance estimator, given in (3.3) and (3.4), is obtained as follows:

$$MSE(s^2{}_t) \cong \frac{S^4{}_x S^4{}_z}{v^2{}_1 v^2{}_2} \left[H_1 + H_2 \right] \quad (3.5)$$

$$MSE(s^2{}_{reg}) \cong \left[H_1 + H_3 \right] \quad (3.6)$$

[see Appendix (A.4) and (B.1)],

Where $v_0 = \sum_{h=1}^{k} \omega_h S^2{}_{yh} + \sum_{h=1}^{k} \omega_h (\bar{Y}_h - \bar{Y})^2$,

$\mathcal{V}_1 = \sum_{h=1}^{k} \omega_h S^2{}_{xh} + \sum_{h=1}^{k} \omega_h (\bar{X}_h - \bar{X})^2$ and

$\mathcal{V}_2 = \sum_{h=1}^{k} \omega_h S^2{}_{zh} + \sum_{h=1}^{k} \omega_h (\bar{Z}_h - \bar{Z})^2$

3.3 The Proposed Estimators

In this section, some variance estimators are proposed using the variance of two auxiliary variables, population kurtosis and coefficient of variation and their combination. Using the ratio-product type estimator given in equation (3.3) instead of estimator given in equation (3.4) the following estimator is suggested.

$$s^2{}_{pr_1} = s^2{}_{st,y} \left(\frac{S^2{}_x}{s^2{}_{st,x}}\right)\left(\frac{S^2{}_z}{s^2{}_{st,z}}\right) + \beta_1(S^2{}_x - s^2{}_{st,x}) + \beta_2 (S^2{}_z - s^2{}_{st,z}) \quad \text{------------------------(3.7)}$$

The MSE of the estimator, given in (3.7), up to first order approximation is found as follows:

$$MSE\left(s^2{}_{pr_1}\right) \cong \frac{1}{v^2{}_1 v^2{}_2}\left[k^2 H_1 + H_4\right] \text{--- (3.8)}$$

Where k= $S^2{}_x S^2{}_z$. The optimal values of β_1 and β_2 in (3.8) which minimize $MSE\left(s^2{}_{pr_1}\right)$ are given by

$$\beta_1{}^* = -\left\{\frac{k v_0 v_2}{v^2{}_1 v^2{}_2} + \frac{k v_1 v_2[(4C_{13}+2C_{14}+2C_{15}+C_{18})(C_6-2C_8-2C_{10}-4C_{11}) - (4C_{12}-4C_{16}+C_{17})(C_1+4C_3+2C_4+2C_9)]}{[(4C_{12}-4C_{16}+C_{17})(4C_2+C_5-4C_7) - (C_6-2C_8-2C_{10}-4C_{11})^2]}\right\}$$

and

$$\beta^*{}_2 = \frac{k\nu_1\nu_2(C_1+4C_3+2C_4+2C_9)-\left(\beta_1{}^*\nu^2{}_1\nu^2{}_2(4C_2+C_5-4C_7)\right)}{\nu^2{}_1\nu^2{}_2(C_6-2C_8-2C_{10}-4C_{11})} -$$

$$\left[\frac{k\nu_0\nu_1(C_6-2C_8-2C_{10}-4C_{11})-k\nu_0\nu_2(4C_{12}-4C_{16}+C_{17})}{\nu^2{}_1\nu^2{}_2(C_6-2C_8-2C_{10}-4C_{11})}\right].$$

The minimum mean square error of $s^2{}_{pr_1}$, to the first order approximation was obtained by substituting the expressions for the optimal values of β_1 and β_2 in (8) and given by

$$MSE\left(s^2{}_{pr_1}\right)_{min} \cong \frac{1}{\nu^2{}_1\nu^2{}_2}\left[k^2 H_1 + H_5\right] \text{--- (3.9)}$$

Motivated by Cingi and Kadilar (2005a, 2006b) and Koyuncu and Kadilar (2009), the following population variance estimators are proposed in the stratified random sampling:

$$s^2{}_{pr_2} = s^2{}_{st,y}\ \frac{(S^2{}_x+C_x)(S^2{}_z+C_z)}{(s^2{}_{st,x}+C_x)(s^2{}_{st,z}+C_z)} \text{-- (3.10)}$$

$$s^2{}_{pr_3} = s^2{}_{st,y}\ \frac{\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)}{\left(s^2{}_{st,x}+\beta_2(x)\right)\left(s^2{}_{st,z}+\beta_2(z)\right)} \text{--- (3.11)}$$

$$s^2{}_{pr_4} = s^2{}_{st,y}\ \frac{\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)}{\left(s^2{}_{st,x} C_x+\beta_2(x)\right)\left(s^2{}_{st,z} C_z+\beta_2(z)\right)} \text{--- (3.12)}$$

$$s^2{}_{pr_5} = s^2{}_{st,y}\ \frac{(S^2{}_x\beta_2(x)+C_x)(S^2{}_z\beta_2(z)+C_z)}{(s^2{}_{st,x}\beta_2(x)+C_x)(s^2{}_{st,z}\beta_2(z)+C_z)} \text{-- (3.13)}$$

The MSE of the estimators, given in (3.10), (3.11), (3.12) and (3.13) is found using the first degree approximation of Taylor series method as follows:

$$MSE\left(s^2{}_{pr_2}\right) \cong \frac{\left((S^2{}_x+C_x)(S^2{}_z+C_z)\right)^2}{\left((\nu_1+C_x)(\nu_2+C_z)\right)^2}\left\{H_1+H_6\right\} \text{--- ------------------------------------- (3.14)}$$

$$MSE(s^2{}_{pr_3}) \cong \frac{\left(\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)\right)^2}{\left((v_1+\beta_2(x))(v_2+\beta_2(z)\ )\right)^2}\left\{H_1+H_7\right\} \text{ ---------------------------------- } (3.15)$$

$$MSE(s^2{}_{pr_4}) \cong \frac{\left(\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)\right)^2}{\left((v_1 C_x+\beta_2(x))(v_2 C_z+\beta_2(z)\ )\right)^2}\left\{H_1+H_8\right\} \text{------------------------------- } (3.16)$$

$$MSE(s^2{}_{pr_5}) \cong \frac{\left((S^2{}_x\beta_2(x)+C_x)(S^2{}_z\beta_2(z)+C_z)\right)^2}{\left((v_1\beta_2(x)+C_x)(v_2\beta_2(z)+C_z\ )\right)^2}\left\{H_1+H_9\right\} \text{-------------------------------- } (3.17)$$

(See Appendix D and E) where C_x and C_z - are population coefficient of variation of the auxiliary variables (X) and (Z) respectively and given by

C_x $=\sum_{h=1}^{k}\omega_h C_{xh}$, and C_z $=\sum_{h=1}^{k}\omega_h C_{zh}$. $\beta_2(x)$ and $\beta_2(z)$ are the population kurtosis of the auxiliary variables (X) and (Z) respectively and defined by

$\beta_2(x)=\sum_{h=1}^{k}\omega_h\beta_2(x_h)$, $\beta_2(z)=\sum_{h=1}^{k}\omega_h\beta_2(z_h)$.

$\mu_{rst\,h} = \frac{1}{N_h}\sum_{h=1}^{N_h}(Y_{hi}-\bar{Y}_h)^r(X_{hi}-\bar{X}_h)^s(Z_{hi}-\bar{Z}_h)^t$, $\lambda_h = \frac{1}{n_h}$, $\omega_h=\frac{N_h}{N}$

$C_{xh} = \frac{S_{xh}}{\bar{X}_h}$ -is population coefficient of variation of the auxiliary variable (X) in stratum h.

$C_{zh} = \frac{S_{zh}}{\bar{Z}_h}$ -is population coefficient of variation of the auxiliary variable (Z) in stratum h.

$\beta_2(y_h)=\frac{\mu_{400\,h}}{\mu^2{}_{200\,h}}$ - is the population kurtosis of the variate of interest in stratum h.

$\beta_2(x_h)=\frac{\mu_{040\,h}}{\mu^2{}_{020\,h}}$ -is the population kurtosis of the first auxiliary variable (X) in stratum h.

$\beta_2(z_h)=\frac{\mu_{004h}}{\mu^2{}_{002h}}$ - is the population kurtosis of the second auxiliary variable (Z) in stratum h. The detailed derivations of all the mean square errors of the estimators considered in this study were presented in Appendix at the end of the thesis.

3.4 Efficiency comparison of the estimators

In this section, comparison of the proposed estimators with other estimators considered in this study and some efficiency comparison condition is carried out under which the proposed estimators are more efficient than the usual sample estimator of population variance and the adapted variance estimators considered in this thesis. These conditions are given as follows:

$MSE(s^2{}_t)$- $MSE\left(s^2{}_{st,y}\right)$ <0 if $H_1 < -\frac{H_2 S^4{}_x S^4{}_z}{S^4{}_x S^4{}_z - v^2{}_1 v^2{}_2}$ ---------------------------------- (3.18)

$MSE\left(s^2{}_{reg}\right)$- $MSE\left(s^2{}_{st,y}\right)$ <0 if $H_3 < 0$ --- (3.19)

$MSE\left(s^2{}_{pr_1}\right)$- $MSE\left(s^2{}_{st,y}\right)$<0 if $H_1 < -\frac{H_4}{S^4{}_x S^4{}_z - v^2{}_1 v^2{}_2}$ --------------------------------- (3.20)

$MSE\left(s^2{}_{pr_1}\right)_{min}$ - $MSE\left(s^2{}_{st,y}\right)$<0 if $H_1 < -\frac{H_5}{S^4{}_x S^4{}_z - v^2{}_1 v^2{}_2}$ ----------------------------- (3.21)

$MSE\left(s^2{}_{pr_2}\right)$- $MSE\left(s^2{}_{st,y}\right)$<0

if $H_1 < -\frac{H_6\left((S^2{}_x + C_x)(S^2{}_z + C_z)\right)^2}{\left((S^2{}_x + C_x)(S^2{}_z + C_z)\right)^2 - \left((v_1 + C_x)(v_2 + C_z)\right)^2}$ --- (3.22)

$MSE\left(s^2{}_{pr_3}\right)$- $MSE\left(s^2{}_{st,y}\right)$<0

if $H_1 < -\frac{H_7\left(\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)\right)^2}{\left((S^2{}_x+\beta_2(x))(S^2{}_z+\beta_2(z))\right)^2-\left((v_1+\beta_2(x))(v_2+\beta_2(z)\ )\right)^2}$ -------------------------------- (3.23)

$MSE\left(s^2{}_{pr_4}\right) - MSE\left(s^2{}_{st,y}\right)$ <0

if $H_1 < -\frac{H_8\left(\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)\right)^2}{\left((S^2{}_x C_x+\beta_2(x))(S^2{}_z C_z+\beta_2(z))\right)^2-\left((v_1 C_x+\beta_2(x))(v_2 C_z+\beta_2(z)\ )\right)^2}$------------------------ (3.24)

$MSE\left(s^2{}_{pr_5}\right)$- $MSE\left(s^2{}_{st,y}\right)$<0

if $H_1 < -\frac{H_9\left((S^2{}_x\beta_2(x)+C_x)(S^2{}_z\beta_2(z)+C_z)\right)^2}{\left((S^2{}_x\beta_2(x)+C_x)(S^2{}_z\beta_2(z)+C_z)\right)^2-\left((v_1\beta_2(x)+C_x)(v_2\beta_2(z)+C_z\ )\right)^2}$------------------------(3.25)

Where H_i for i=2,3,…,9 is the term of each mean square error with out the common multiplier of all terms and H_1. The other method, which is used to compare the performance of the proposed estimators over, $s^2{}_t$ is Percent Relative Efficient (PRE). The Percent Relative Efficiencies (PREs) of the different estimators are computed with respect to $s^2{}_t$ using the formula:

$PRE(s^2{}_t, s^2{}_{pr_i}) = \frac{MSE(s^2{}_t)}{MSE\left(s^2{}_{pr_i}\right)} \times 100$ for i=1, 2, 3, 4, 5 ----------------------------------- (3.26)

An estimator which has smaller mean square error has higher PRE and hence more efficient than other estimators of population variance.

3.5 Empirical study

In this section, the performance of the suggested estimators have been analyzed with respect to the estimators considered in this thesis. To achieve this, the data set of state wise area, production and productivity of major spices in India was used. In this data set, the study variable (Y) is productivity in metric tons , the first auxiliary variable (X) is area in thousand hectares , and the second auxiliary variable (Z) is production in thousand tons. The data are stratified by year and from each stratum, states are selected randomly using the Neyman allocation as

$$n_h = \frac{n N_h S_h}{\sum_{h=1}^{k} N_h S_h} \quad \text{--} \quad (3.27)$$

The standard deviation and size of each stratum is given as follow which is used to compute the sample selected from each stratum.

Table 3. 1. Sdev. and Size of each stratum

Strata	Stratum I	Stratum II	Stratum III
Size	29	29	29
Standard deviation	1.75756	1.7342	1.79672

Thus, the sample size is calculated using the values given in table3.1 and inserting into equation (3.27) as follows.

$$n_1 = \frac{36 \times 29 \times 1.75756}{153.36505} = 11.964 \approx 12$$

$$n_2 = \frac{36 \times 29 \times 1.7342}{153.36505} = 11.8052 \approx 12$$

$$n_3 = \frac{36 \times 29 \times 1.79672}{153.36505} = 12.23 \approx 12$$

Therefore, from each stratum 12 states are selected. The summary of the data is given in the following tables.

Table 3.2. Data Statistics

N_h	n_h	$\overline{X}_h$	$\overline{Y}_h$	$\overline{Z}_h$	C_{xh}	C_{yh}	C_{zh}	$\beta_2(x_h)$	$\beta_2(y_h)$	$\beta_2(z_h)$
29	12	90.2534	2.2252	150.23	1.524	0.745	1.502	6.72	2.411	12.383
29	12	90.6693	2.3486	142.95	1.515	0.781	1.722	6.23	2.476	13.564
29	12	84.9562	2.3434	138.48	1.443	0.766	1.669	5.361	2.584	14.257

Table3. 3. Data statistics of parameters

Parameters	Stratum I	Stratum II	Stratum III
$\theta_h(yx)$	1.643×10^{-5}	1.42×10^{-5}	1.819×10^{-6}
$\theta_h(yz)$	5.6047×10^{-6}	3.9563×10^{-6}	4.2985×10^{-6}
$\theta_h(xz)$	2.774×10^{-9}	2.782×10^{-9}	4.4645×10^{-6}
S_{yx}	50.5266	72.5386	60.66
S_{yz}	24	2.532	0.927
S_{xz}	25758.621	23551.724	20482.7586

Table 3. 4. Values of parameters

Parameters	Values
ν_0	3.378592
ν_1	18960.84
ν_2	64358.93
S^2_x	4400
S^2_z	14945.833
C_x	1.5
C_z	1.631
$\beta_2(x)$	6.5522
$\beta_2(z)$	14.4724
β_1	3.11×10^{-5}
β_2	5.24×10^{-6}
β^*_1	9.61×10^{-6}
β^*_2	4.12×10^{-4}

Table 3. 5. Summary of μ_{rsth}

μ_{rsth}	Stratum I	Stratum II	Stratum III
μ_{300h}	2.890	4.771	4.506
μ_{210h}	-103.789	-127.5024	-121.621
μ_{201h}	-135.134	-197.1155	-194.107
μ_{120h}	-10344.828	-12896.552	-8379.31
μ_{102h}	55517.241	52034.483	41034.483
μ_{030h}	4827586.207	4620689.655	2965517.24
μ_{021h}	5413793.103	4620689.655	3551724.138
μ_{012h}	12206896.552	11068965.517	9413793.103
μ_{003h}	46551724.138	44827586.207	37586206.897

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1. Introduction

In this section, the computed value of the mean square error of each estimator considered in this study is presented. A comparison among the different estimators of S^2_y with respect to their mean squared error is also made empirically and shows that the proposed estimators are more efficient than the usual sample estimator of population variances of the study variable.

4.2. *Discussion*

This section discusses about the results obtained using SPSS and Microsoft excel to analysis the data stated in section3.5. The results are present in the following tables.

Table 4.1. Estimators with their MSE values

Estimators	$S^2_{st,y}$	S^2_t	S^2_{reg}	$S^2_{pr_1}$	$S^2_{pr_{(1)\min}}$	$S^2_{pr_2}$	$S^2_{pr_3}$	$S^2_{pr_4}$	$S^2_{pr_5}$
MSE values	0.420353	0.019	0.4372	0.032284	0.03222	0.01872	0.01772	0.01673	0.01526

Table 4.1 shows that the proposed estimators of S^2_y are more efficient than the traditional estimator of population variance of interest variable in stratified random sampling according to the data set of a population considered in this study. Theoretically, it has been established that, in general, the regression type estimator is more efficient than the ratio-type estimators.

However, in this thesis the regression estimator of S^2_y is not efficient than the sample estimator and the proposed ratio-type estimators of population variance of interest variable.

Table4. 2. PRE of the different estimators with respect to s^2_t

Estimators	$s^2_{st,y}$	s^2_t	s^2_{reg}	$s^2_{pr_1}$	$s^2_{pr_{(1)\min}}$	$s^2_{pr_2}$	$s^2_{pr_3}$	$s^2_{pr_4}$	$s^2_{pr_5}$
PRE	4.52	100	4.3458	58.8532	58.9694	101.4957	107.163	113.5684	124.5085

Table 4.2 reveals that the suggested estimators $s^2_{pr_i}$, for i= 2, 3, 4, 5 have the higher PRE among other estimators considered in this study. So that the suggested estimators in stratified random sampling provide improvement in variance estimation compared to the s^2_t . It is also observed from Table 4.2 that the first proposed estimator, sample and

regression estimators are less efficient than s^2_t. From the above results and discussion, it is observed that incorporating prior information obtained from the two auxiliary variables improves the estimate of population variance in stratified random sampling.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1. Conclusion

In this thesis, new ratio- type variance estimators using known values of Coefficient variation and kurtosis of the auxiliary variables are presented. The mean squared error of the proposed estimators were obtained up to first order approximation and compared with sample estimator and the traditional ratio-type estimator of population variance in the presence of two auxiliary variables. Further, the conditions for which the proposed estimators are more efficient than the traditional estimator were derived. The performance of the proposed estimators was assessed using population data set. Results show that the mean squared error of the proposed estimators, except the first proposed estimator, are less than the mean squared error of the sample estimator and the traditional ratio-type estimator of the population variance. Therefore, the second, third, fourth and fifth proposed estimators are more efficient than the sample and the traditional ratio-type estimator of the population variance.

5.2. Recommendations

1. Based on results obtained, the proposed ratio-type variance estimators may be preferred over traditional ratio- type and sample estimator of population variance for the use in practical applications.

2. In forthcoming studies, we recommended to develop improved variance estimators by adapting the estimators of Rajesh Singh and Mukesh Kumar (2012).

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Appendixes

Appendix A: MSE derivation of ratio- type estimator of variance

The MSE of the ratio type variance estimator in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first degree approximation in the Taylor series method defined by

$MSE(s^2{}_t) \cong \sum_{h=1}^{k} d_h \Sigma_h d'_h$ --- -*(A.1)*

$MSE(s^2{}_t) \cong$

$$\sum_{h=1}^{k} \left\{ \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 \end{bmatrix} \begin{bmatrix} \sigma^2{}_1 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} & \sigma_{18} & \sigma_{19} \\ \sigma_{21} & \sigma^2{}_2 & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} & \sigma_{29} \\ \sigma_{31} & \sigma_{32} & \sigma^2{}_3 & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} & \sigma_{38} & \sigma_{39} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma^2{}_4 & \sigma_{45} & \sigma_{46} & \sigma_{47} & \sigma_{48} & \sigma_{49} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma^2{}_5 & \sigma_{56} & \sigma_{57} & \sigma_{58} & \sigma_{59} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma^2{}_6 & \sigma_{67} & \sigma_{68} & \sigma_{69} \\ \sigma_{71} & \sigma_{72} & \sigma_{73} & \sigma_{74} & \sigma_{75} & \sigma_{76} & \sigma^2{}_7 & \sigma_{78} & \sigma_{79} \\ \sigma_{81} & \sigma_{82} & \sigma_{83} & \sigma_{84} & \sigma_{85} & \sigma_{86} & \sigma_{87} & \sigma^2{}_8 & \sigma_{89} \\ \sigma_{91} & \sigma_{92} & \sigma_{93} & \sigma_{94} & \sigma_{95} & \sigma_{96} & \sigma_{97} & \sigma_{98} & \sigma^2{}_9 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \end{bmatrix} \right\} \text{-------(A.2)}$$

Where $d_h = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 \end{bmatrix}$ *such that*

$$d_1 = \frac{\partial}{\partial a} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}}$$

$$d_2 = \frac{\partial}{\partial b} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}} \ ,$$

$$d_3 = \frac{\partial}{\partial c} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}}$$

$$d_4 = \frac{\partial}{\partial d} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}} \ ,$$

$$d_5 = \frac{\partial}{\partial e} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}}$$

$$d_6 = \frac{\partial}{\partial f} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}} \ ,$$

$$d_7 = \frac{\partial}{\partial g} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}}$$

$$d_8 = \frac{\partial}{\partial h} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}} \ ,$$

$$d_9 = \frac{\partial}{\partial i} h(a,b,c,d,e,f,g,h,i)|_{S^2{}_{yh},\bar{Y}_h,\bar{Y}\ ,S^2{}_{xh},\bar{X}_h,\bar{X}\ ,S^2{}_{zh,}\bar{Z}_h,\bar{Z}} \ and$$

$$\Sigma_h = \begin{bmatrix} \sigma^2_1 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} & \sigma_{18} & \sigma_{19} \\ \sigma_{21} & \sigma^2_2 & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} & \sigma_{29} \\ \sigma_{31} & \sigma_{32} & \sigma^2_3 & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} & \sigma_{38} & \sigma_{39} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma^2_4 & \sigma_{45} & \sigma_{46} & \sigma_{47} & \sigma_{48} & \sigma_{49} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma^2_5 & \sigma_{56} & \sigma_{57} & \sigma_{58} & \sigma_{59} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma^2_6 & \sigma_{67} & \sigma_{68} & \sigma_{69} \\ \sigma_{71} & \sigma_{72} & \sigma_{73} & \sigma_{74} & \sigma_{75} & \sigma_{76} & \sigma^2_7 & \sigma_{78} & \sigma_{79} \\ \sigma_{81} & \sigma_{82} & \sigma_{83} & \sigma_{84} & \sigma_{85} & \sigma_{86} & \sigma_{87} & \sigma^2_8 & \sigma_{89} \\ \sigma_{91} & \sigma_{92} & \sigma_{93} & \sigma_{94} & \sigma_{95} & \sigma_{96} & \sigma_{97} & \sigma_{98} & \sigma^2_9 \end{bmatrix} \text{---------------------}(A.3)$$

Here $h(a,b,c,d,e,f,g,h,i)= h\left(s^2{}_{yh},\ \bar{y}_h, \bar{y}_{st}\ , s^2{}_{xh}, \bar{x}_h, \bar{x}_{st}\ , s^2{}_{zh,} \bar{z}_h, \bar{z}_{st}\right)$ and Σ_h is the variance-covariance matrixes of $h(a,b,c,d,e,f,g,h,i)$. Note that $\bar{X}_{st} = \sum_{h=1}^{k} \omega_h \bar{X}_h = \bar{X}$,

$\bar{Y}_{st} = \sum_{h=1}^{k} \omega_h \bar{Y}_h = \bar{Y}$ and $\bar{Z}_{st} = \sum_{h=1}^{k} \omega_h \bar{Z}_h = \bar{Z}$. According to (A.3), we obtain d_h for the estimator, $s^2{}_t$ as follows,

Let $\nu_0 = \sum_{h=1}^{k} \omega_h S^2{}_{yh} + \sum_{h=1}^{k} \omega_h (\bar{Y}_h\ - \bar{Y}\ )^2$

$\nu_1 == \sum_{h=1}^{k} \omega_h S^2{}_{xh} + \sum_{h=1}^{k} \omega_h (\bar{X}_h\ - \bar{X}\ )^2$

$\nu_2 == \sum_{h=1}^{k} \omega_h S^2{}_{zh} + \sum_{h=1}^{k} \omega_h (\bar{Z}_h\ - \bar{Z}\ )^2$, then we have

$$d_h = \frac{S^2{}_X\ \ S^2{}_z}{\nu_1 \nu_2}$$

$$\begin{bmatrix} \omega_h & 2\omega_h(\bar{Y}_h - \bar{Y}) & -2\omega_h(\bar{Y}_h - \bar{Y}) & -\frac{\nu_0 \omega_h}{\nu_1} & -\frac{2\nu_0 \omega_h (\bar{X}_h - \bar{X})}{\nu_1} & \frac{2\nu_0 \omega_h (\bar{X}_h - \bar{X})}{\nu_1} & -\frac{\nu_0 \omega_h}{\nu_2} \\ & & & & -\frac{2\nu_0 \omega_h (\bar{Z}_h - \bar{Z})}{\nu_2} & \frac{2\nu_0 \omega_h (Z_h - \bar{Z})}{\nu_2} & \end{bmatrix}$$

We obtain the MSE of ${s^2}_t$ using (A.1), as

$$MSE({s^2}_t) \cong \frac{{S^4}_x {S^4}_z}{{v^2}_1 {v^2}_2}\left[H_1 + H_2\right] \text{--} \quad (A.4)$$

Where $H_1 =$

$$\sum_{h=1}^{k} {\omega^2}_h V\left({s^2}_{yh}\right) + 4\sum_{h=1}^{k} {\omega^2}_h \left(\bar{Y}_h - \bar{Y}\right)\left[COV\left(\bar{y}_h, {s^2}_{yh}\right) - COV\left(\bar{y}_{st}, {s^2}_{yh}\right)\right] + 4$$

$$\sum_{h=1}^{k} {\omega^2}_h \left(\bar{Y}_h - \bar{Y}\right)^2 \left[V\left(\bar{y}_h\right) - 2COV\left(\bar{y}_h, \bar{y}_{st}\right) + V\left(\bar{y}_{st}\right)\right]$$

H_2

$$= -4\sum_{h=1}^{k} v_0 {\omega^2}_h (\bar{Y}_h - \bar{Y})\left[\frac{1}{v_1}\left(cov(\bar{y}_h, {s^2}_{xh}) - cov(\bar{y}_{st}, {s^2}_{xh})\right) + \frac{1}{v_2}\left(cov(\bar{y}_h, {s^2}_{zh}) -\right.\right.$$

$$\left.\left. cov(\bar{y}_{st}, {s^2}_{zh})\right)\right] - 2\sum_{h=1}^{k} \frac{v_0 {\omega^2}_h}{v_1} cov\left({s^2}_{xh}, {s^2}_{yh}\right) - 4\sum_{h=1}^{k} \frac{v_0 {\omega^2}_h (\bar{X}_h - \bar{X})}{v_1}\left[cov\left(\bar{x}_h, {s^2}_{yh}\right) -\right.$$

$$cov\left(\bar{x}_{st}, {s^2}_{yh}\right) - \frac{v_0}{v_1}\left(cov(\bar{x}_h, {s^2}_{xh}) - cov(\bar{x}_{st}, {s^2}_{xh})\right) - \frac{v_0}{v_2}\left(cov(\bar{x}_h, {s^2}_{zh}) -\right.$$

$$\left.\left. cov(\bar{x}_{st}, {s^2}_{zh})\right)\right] - 2\sum_{h=1}^{k} \frac{v_0 {\omega^2}_h}{v_2} cov\left({s^2}_{zh}, {s^2}_{yh}\right) - 8\sum_{h=1}^{k} \frac{1}{v_1} v_0 {\omega^2}_h (\bar{Y}_h - \bar{Y})(\bar{X}_h -$$

$$\bar{X})[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_{st})] -$$

$$8\sum_{h=1}^{k} \frac{v_0 {\omega^2}_h (\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})}{v_2}[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_{st})] +$$

$$\sum_{h=1}^{k} \frac{{v^2}_0 {\omega^2}_h}{{v^2}_1} v({s^2}_{xh}) + 2\sum_{h=1}^{k} \frac{{v^2}_0 {\omega^2}_h}{v_1 v_2} cov({s^2}_{xh}, {s^2}_{zh}) + 4\sum_{h=1}^{k} \frac{{v^2}_0 {\omega^2}_h (\bar{X}_h - \bar{X})^2}{{v^2}_1}[v(\bar{x}_h) -$$

$$2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})] +$$

$$\sum_{h=1}^{k} \frac{{v^2}_0 {\omega^2}_h}{{v^2}_2} v({s^2}_{zh}) - 8\sum_{h=1}^{k} \frac{v_0\, {\omega^2}_h (\bar{X}_h - \bar{X})(\bar{Z}_h - \bar{Z})}{v_1 v_2}[cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_h, \bar{z}_h) +$$

$$cov(\bar{x}_{st}, \bar{z}_h) - cov(\bar{x}_{st}, \bar{z}_{st})] + 4\sum_{h=1}^{k} \frac{{v^2}_0 {\omega^2}_h (\bar{Z}_h - \bar{Z})^2}{{v^2}_2}[v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})] -$$

$$4\sum_{h=1}^{k}\frac{\nu_0 \omega^2{}_h(\bar{Z}_h-\bar{Z})}{\nu_2}\Big[cov\big(\bar{z}_h, s^2{}_{yh}\big) - cov\big(\bar{z}_{st}, s^2{}_{yh}\big) -$$

$$\frac{\nu_0}{\nu_1}\big(cov(\bar{z}_h, s^2{}_{xh}) - cov(\bar{z}_{st}, s^2{}_{xh})\big) - \frac{\nu_0}{\nu_2}\big(cov(\bar{z}_h, s^2{}_{zh}) - cov(\bar{z}_{st}, s^2{}_{zh})\big)\Big]$$

Appendix B: MSE derivation of regression type estimator of variance

The MSE of the regression estimator for the population variance in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first-degree approximation in the Taylor series method as follows:

d_h=[ω_h $2\omega_h(\bar{Y}_h-\bar{Y})$ $-2\omega_h(\bar{Y}_h-\bar{Y})$ $-\beta_1\omega_h$ $-2\beta_1\omega_h(\bar{X}_h-\bar{X})$ $2\beta_1\omega_h(\bar{X}_h-\bar{X})$ $-\beta_2\omega_h$ $-2\beta_2\omega_h(\bar{Z}_h-\bar{Z})$ $2\beta_2\omega_h(\bar{Z}_h-\bar{Z})$] *and* Σ_h *,using (A.1) and (A.2) ,*

$MSE(s^2{}_{reg}) \cong [H_1+H_3]$-- (B.1)

Where

$H_3 = -2\beta_1\sum_{h=1}^{k}\omega^2{}_h cov(s^2{}_{xh},s^2{}_{yh})+4\beta_1\sum_{h=1}^{k}\omega^2{}_h(\bar{X}_h-\bar{X})$ $[cov(\bar{x}_{st},s^2{}_{yh})-cov(\bar{x}_h,s^2{}_{yh})+\beta_1(cov(\bar{x}_h,s^2{}_{xh})-cov(\bar{x}_{st},s^2{}_{xh}))+\beta_2(cov(\bar{x}_h,s^2{}_{zh})-cov(\bar{x}_{st},s^2{}_{zh}))]$ $-2\beta_2\sum_{h=1}^{k}\omega^2{}_h cov(s^2{}_{yh},s^2{}_{zh})$ $+4\beta_2\sum_{h=1}^{k}\omega^2{}_h(\bar{Z}_h-\bar{Z})$ $[cov(\bar{z}_{st},s^2{}_{yh})-cov(\bar{z}_h,s^2{}_{yh})+\beta_1(cov(\bar{z}_h,s^2{}_{xh})-cov(\bar{z}_{st},s^2{}_{xh}))+\beta_2(cov(\bar{z}_h,s^2{}_{zh})-cov(\bar{z}_{st},s^2{}_{zh}))]+4\sum_{h=1}^{k}\omega^2{}_h(\bar{Y}_h-\bar{Y})[\beta_1(cov(\bar{y}_{st},s^2{}_{xh})-cov(\bar{y}_h,s^2{}_{xh}))-\beta_2(cov(\bar{y}_h,s^2{}_{zh})+cov(\bar{y}_{st},s^2{}_{zh}))]+2\beta_1\beta_2\sum_{h=1}^{k}\omega^2{}_h cov(s^2{}_{xh},s^2{}_{zh})+8\beta_2\sum_{h=1}^{k}\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{Z}_h-\bar{Z})[cov(\bar{y}_h,\bar{z}_{st})-cov(\bar{y}_h,\bar{z}_h)+cov(\bar{y}_{st},\bar{z}_h)-cov(\bar{y}_{st},\bar{z}_{st})]+8\beta_1\sum_{h=1}^{k}\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{X}_h-\bar{X})[cov(\bar{y}_h,\bar{x}_{st})-cov(\bar{y}_h,\bar{x}_h)+cov(\bar{y}_{st},\bar{x}_h)-cov(\bar{y}_{st},\bar{x}_{st})]+8$ $\beta_1\beta_2\sum_{h=1}^{k}\omega^2{}_h(\bar{X}_h-\bar{X})(\bar{Z}_h-\bar{Z})$ $[cov(\bar{x}_h,\bar{z}_h)-cov(\bar{x}_h,\bar{z}_{st})-cov(\bar{x}_{st},\bar{z}_h)+cov(\bar{x}_{st},\bar{z}_{st})]+\beta^2{}_1\sum_{h=1}^{k}\omega^2{}_h\ v(s^2{}_{xh})+4\beta^2{}_1\sum_{h=1}^{k}\omega^2{}_h(\bar{X}_h-\bar{X})^2$ $[v(\bar{x}_h)-$

$2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})]$ + ${\beta^2}_2 \sum_{h=1}^{k} {\omega^2}_h \; v({s^2}_{zh})$+$4{\beta^2}_2 \sum_{h=1}^{k} {\omega^2}_h (\bar{Z}_h - \bar{Z})^2 \;\; [v(\bar{z}_h) -$

$2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})]$

Appendix C: MSE derivation of first proposed estimator

The MSE of the first suggested estimator ($s^2{}_{pr\,1}$) for the population variance in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first degree approximation in the Taylor series method as follows:

$d_h = \frac{1}{\nu_1\nu_2}[\ \omega_h k \quad 2\omega_h k(\bar{Y}_h - \bar{Y}) \quad - 2\omega_h k(\bar{Y}_h - \bar{Y}) \ -\omega_h(\mathrm{k}\nu_0\nu_1{}^{-1} + \beta_1\nu_1\nu_2) \ - 2\omega_h(\bar{X}_h - \bar{X})(\mathrm{k}\nu_0\nu_1{}^{-1} + \beta_1\nu_1\nu_2) \ \ 2\omega_h(\bar{X}_h - \bar{X})(\mathrm{k}\nu_0\nu_1{}^{-1} + \beta_1\nu_1\nu_2) \quad -\omega_h(\mathrm{k}\nu_0\nu_2{}^{-1} + \beta_2\nu_1\nu_2) \quad - 2\omega_h(\bar{Z}_h - \bar{Z})(\mathrm{k}\nu_0\nu_2{}^{-1} + \beta_2\nu_1\nu_2) \ \ 2\omega_h(\bar{Z}_h - \bar{Z})(\mathrm{k}\nu_0\nu_2{}^{-1} + \beta_2\nu_1\nu_2)\ ]$ *and* Σ_h *using (A.1) and (A.2), we have*

$MSE\left(s^2{}_{pr\,1}\right) \cong \frac{1}{\nu^2{}_1\nu^2{}_2}\left[k^2 H_1 + H_4\right]$---(C.1)

The optimal equation of β_1 and β_2 in (C.1) could be obtained by differentiating (C.1) with respect to β_1 and β_2 and equalizing to zero. i.e

$\frac{\partial}{\partial\beta_1}\left(MSE\left(s^2{}_{pr\,1}\right)\right)=0$

$\beta_1\nu^2{}_1\nu^2{}_2(4C_2 + C_5 - 4C_7) + \beta_2\nu^2{}_1\nu^2{}_2(C_6 - 2C_8 - 2C_{10} - 4C_{11}) + \mathrm{k}\nu_0\nu_1(C_6 - 2C_8 - 2C_{10} - 4C_{11}) + \mathrm{k}\nu_0\nu_2(4C_2 + C_5 - 4C_7) - k\nu_1\nu_2(C_1 + 4C_3 + 2C_4 + 2C_9) = 0$ ------ (C.2)

$\frac{\partial}{\partial\beta_2}\left(MSE\left(s^2{}_{pr\,1}\right)\right)=0$

$\beta_1\nu^2{}_1\nu^2{}_2(C_6 - 2C_8 - 2C_{10} - 4C_{11}) + \beta_2\nu^2{}_1\nu^2{}_2(4C_{12} - 4C_{16} + C_{17}) + \mathrm{k}\nu_0\nu_1(4C_{12} - 4C_{16} + C_{17}) + \mathrm{k}\nu_0\nu_2(C_6 - 2C_8 - 2C_{10} - 4C_{11}) - k\nu_1\nu_2(4C_{13} + 2C_{14} + 2C_{15} + C_{18}) = 0$

-- (C.3)

Multiply equation (C.2) and (C.3) by $(4C_{12} - 4C_{16} + C_{17})$+ and $-(C_6 - 2C_8 - 2C_{10} - 4C_{11})$ respectively and then adding the two equation gives the equation of β_1 as

$$\beta_1 = \beta_1^* = -\left\{\frac{k\nu_0\nu_2}{\nu^2{}_1\nu^2{}_2} + \frac{k\nu_1\nu_2[(4C_{13}+2C_{14}+2C_{15}+C_{18})(C_6-2C_8-2C_{10}-4C_{11})-(4C_{12}-4C_{16}+C_{17})(C_1+4C_3+2C_4+2C_9)]}{[(4C_{12}-4C_{16}+C_{17})(4C_2+C_5-4C_7)-(C_6-2C_8-2C_{10}-4C_{11})^2]}\right\} \text{----- (C.4)}$$

Using (C.4) in (C.2), we have

$$\beta_2 = \beta^*{}_2 = \frac{k\nu_1\nu_2(C_1+4C_3+2C_4+2C_9)-\left(\beta_1{}^*\nu^2{}_1\nu^2{}_2(4C_2+C_5-4C_7)\right)}{\nu^2{}_1\nu^2{}_2(C_6-2C_8-2C_{10}-4C_{11})} - \left[\frac{k\nu_0\nu_1(C_6-2C_8-2C_{10}-4C_{11})-k\nu_0\nu_2(4C_{12}-4C_{16}+C_{17})}{\nu^2{}_1\nu^2{}_2(C_6-2C_8-2C_{10}-4C_{11})}\right] \text{--- (C.5)}$$

Where k $=S^2{}_x S^2{}_z$

$C_1 = \sum_{h=1}^{k} \omega^2{}_h cov\left(s^2{}_{xh}, s^2{}_{yh}\right)$

$C_2 = \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X})^2\ [v(\bar{x}_h) - 2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})]$

$C_3 = \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_h, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h)]$

$C_4 = \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y})\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right)$

$C_5 = \sum_{h=1}^{k} \omega^2{}_h\ v(s^2{}_{xh})$

$C_6 = \sum_{h=1}^{k} \omega^2{}_h cov(s^2{}_{xh}, s^2{}_{zh})$

$C_7 = \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X})\left(cov(\bar{x}_{st}, s^2{}_{xh}) - cov(\bar{x}_h, s^2{}_{xh})\right)$

$C_8 = \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X}) \left(cov(\bar{x}_{st}, s^2{}_{zh}) - cov(\bar{x}_h, s^2{}_{zh})\right)$

$C_9 = \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X}) \left[cov\left(\bar{x}_h, s^2{}_{yh}\right) - cov\left(\bar{x}_{st}, s^2{}_{yh}\right)\right]$

$C_{10} = \sum_{h=1}^{k} \omega^2{}_h (\bar{Z}_h - \bar{Z}) [cov(\bar{z}_{st}, s^2{}_{xh}) - cov(\bar{z}_h, s^2{}_{xh})]$

$C_{11} = \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X})(\bar{Z}_h - \bar{Z}) \quad [cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_h, \bar{z}_h) + cov(\bar{x}_{st}, \bar{z}_{st}) - cov(\bar{x}_{st}, \bar{z}_h)]$

$C_{12} = \sum_{h=1}^{k} \omega^2{}_h (\bar{Z}_h - \bar{Z})^2 \ [v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})]$

$C_{13} = \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) + cov(\bar{y}_{st}, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h)]$

$C_{14} = \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y}) [cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})]$

$C_{15} = \sum_{h=1}^{k} \omega^2{}_h (\bar{Z}_h - \bar{Z}) \left[cov\left(\bar{z}_h, s^2{}_{yh}\right) - cov\left(\bar{z}_{st}, s^2{}_{yh}\right)\right]$

$C_{16} = \sum_{h=1}^{k} \omega^2{}_h (\bar{Z}_h - \bar{Z}) \left[cov\left(\bar{z}_{st}, s^2{}_{yh}\right) - cov\left(\bar{z}_h, s^2{}_{yh}\right)\right]$

$C_{17} = \sum_{h=1}^{k} \omega^2{}_h \ v(s^2{}_{zh})$, $C_{18} = \sum_{h=1}^{k} \omega^2{}_h cov\left(s^2{}_{yh}, s^2{}_{zh}\right)$

$H_4 =$
$\Big\{-2k \sum_{h=1}^{k} \omega^2{}_h (\mathrm{k}\nu_0 \ \nu_1{}^{-1} + \beta_1 \nu_1 \nu_2) \quad cov\left(s^2{}_{xh}, s^2{}_{yh}\right) + 4 \sum_{h=1}^{k} \omega^2{}_h (\mathrm{k}\nu_0 \ \nu_1{}^{-1} + \beta_1 \nu_1 \nu_2)^2 (\bar{X}_h - \bar{X})^2 \ [v(\bar{x}_h) - 2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})] + 4 \sum_{h=1}^{k} \omega^2{}_h (\mathrm{k}\nu_0 \ \nu_2{}^{-1} + \beta_2 \nu_1 \nu_2)^2 (\bar{Z}_h - \bar{Z})^2 \ [v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})] - 8 \sum_{h=1}^{k} k^2 \nu_0 \ \nu_1{}^{-1} \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) + cov(\bar{y}_{st}, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h)] -$
$8 \sum_{h=1}^{k} \mathrm{k}\beta_1 \nu_1 \nu_2 \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) + cov(\bar{y}_{st}, \bar{x}_{st}) -$

$cov(\bar{y}_{st}, \bar{x}_h)] - 8\sum_{h=1}^{k} k^2 \nu_0 \ \ \nu_2{}^{-1} \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) +$

$cov(\bar{y}_{st}, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h)] - 8\sum_{h=1}^{k} k\, \beta_2 \nu_1 \nu_2 \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})[cov(\bar{y}_h, \bar{z}_h) -$

$cov(\bar{y}_h, \bar{z}_{st}) + cov(\bar{y}_{st}, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h)] - 4\sum_{h=1}^{k} k^2 \nu_0 \omega^2{}_h (\bar{Y}_h -$

$\bar{Y}) \left[\nu_1{}^{-1} \left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) + \nu_2{}^{-1}\left(cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})\right)\right] -$

$4\sum_{h=1}^{k} k\nu_1\nu_2\omega^2{}_h (\bar{Y}_h - \bar{Y}) \left[\ \beta_1\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) + \beta_2\left(cov(\bar{y}_h, s^2{}_{zh}) -\right.\right.$

$\left.\left.cov(\bar{y}_{st}, s^2{}_{zh})\right)\right] + \sum_{h=1}^{k}\left(\mathrm{k}\nu_0 \ \nu_1{}^{-1} \ + \beta_1\nu_1\nu_2\right)^2 \omega^2{}_h \ \ v(s^2{}_{xh}) + 2\sum_{\mathrm{h}=1}^{\mathrm{k}} \omega^2{}_{\mathrm{h}}\left(\mathrm{k}\nu_0 \ \ \nu_1{}^{-1} \ +\right.$

$\left.\beta_1\nu_1\nu_2\right)\left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ + \beta_2\nu_1\nu_2\right)cov(s^2{}_{xh}, s^2{}_{zh}) - 4\sum_{h=1}^{k} \omega^2{}_h \left(\mathrm{k}\nu_0 \ \ \nu_1{}^{-1} \ + \beta_1\nu_1\nu_2\right)(\bar{X}_h -$

$\bar{X}) \left[k \left(cov\left(\bar{x}_h, s^2{}_{yh}\right) - cov\left(\bar{x}_{st}, s^2{}_{yh}\right)\right) + \left(\mathrm{k}\nu_0\nu_1{}^{-1} \ + \beta_1\nu_1\nu_2\right)\left(cov(\bar{x}_{st}, s^2{}_{xh}) -\right.\right.$

$\left.\left.cov(\bar{x}_h, s^2{}_{xh})\right) + \left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ + \beta_2\nu_1\nu_2\right)\left(cov(\bar{x}_{st}, s^2{}_{zh}) - cov(\bar{x}_h, s^2{}_{zh})\right)\right] -$

$4\sum_{h=1}^{k} \omega^2{}_h \left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ + \beta_2\nu_1\nu_2\right)(\bar{Z}_h - \bar{Z}) \ \left[k \left(cov\left(\bar{z}_h, s^2{}_{yh}\right) - cov\left(\bar{z}_{st}, s^2{}_{yh}\right)\right) +\right.$

$(\mathrm{k}\nu_0 \ \ \nu_1{}^{-1} \ + \beta_1\nu_1\nu_2)\left(cov(\bar{z}_{st}, s^2{}_{xh}) - cov(\bar{z}_h, s^2{}_{xh})\right) +$

$\left.\left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ + \beta_2\nu_1\nu_2\right)\left(cov(\bar{z}_{st}, s^2{}_{zh}) - cov(\bar{z}_h, s^2{}_{zh})\right)\right] + \sum_{h=1}^{k} \omega^2{}_h \left(\mathrm{k}\nu_0\nu_2{}^{-1} \ +\right.$

$\left.\beta_2\nu_1\nu_2\right)^2 \ v(s^2{}_{zh}) -$

$2\sum_{\mathrm{h}=1}^{\mathrm{k}} \omega^2{}_{\mathrm{h}} k\left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ + \beta_2\nu_1\nu_2\right)cov\left(s^2{}_{yh}, s^2{}_{zh}\right) - 8 \ \sum_{h=1}^{k} \omega^2{}_h \left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ +\right.$

$\left.\beta_2\nu_1\nu_2\right)\left(\mathrm{k}\nu_0 \ \ \nu_2{}^{-1} \ + \beta_2\nu_1\nu_2\right)(\bar{X}_h - \bar{X})(\bar{Z}_h - \bar{Z})[cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_h, \bar{z}_h) +$

$cov(\bar{x}_{st}, \bar{z}_{st}) - cov(\bar{x}_{st}, \bar{z}_h)]\Big\}$

The minimum mean square error of $s^2{}_{pr_1}$, to the first order approximation is obtained by substituting the optimal equation of $\beta_1$and β_2 in (8) and given by

$$MSE\left(s^2{}_{pr\,1}\right)_{min} \cong \frac{1}{v^2{}_1 v^2{}_2}\left[k^2 H_1 + H_5\right] \text{---} \quad (C.6)$$

Where

$H_5 =$

$\Big\{-2k\sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}v_0 v_1{}^{-1} + \beta^*{}_1 v_1 v_2\right) \quad cov\left(s^2{}_{xh}, s^2{}_{yh}\right) + +4\sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}v_0 \;\; v_1{}^{-1} +\right.$

$\left.\beta^*{}_1 v_1 v_2\right)^2(\bar{X}_h - \bar{X})^2 \; [v(\bar{x}_h) - 2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})] + 4\sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}v_0 \;\; v_2{}^{-1} +\right.$

$\left.\beta^*{}_2 v_1 v_2\right)^2(\bar{Z}_h - \bar{Z})^2 \; [v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})] - 8\sum_{h=1}^{k} k^2 v_0 \;\; v_1{}^{-1}\omega^2{}_h(\bar{Y}_h -$

$\bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) + cov(\bar{y}_{st}, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h)] -$

$8\sum_{h=1}^{k}\mathrm{k}\beta^*{}_1 v_1 v_2 \omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) + cov(\bar{y}_{st}, \bar{x}_{st}) -$

$cov(\bar{y}_{st}, \bar{x}_h)] - 8\sum_{h=1}^{k} k^2 v_0 \;\; v_2{}^{-1}\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) +$

$cov(\bar{y}_{st}, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h)] - 8\sum_{h=1}^{k} k\,\beta^*{}_2 v_1 v_2 \omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})[cov(\bar{y}_h, \bar{z}_h) -$

$cov(\bar{y}_h, \bar{z}_{st}) + cov(\bar{y}_{st}, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h)] - 4\sum_{h=1}^{k} k^2 v_0 \omega^2{}_h(\bar{Y}_h -$

$\bar{Y})\left[v_1{}^{-1}\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) + v_2{}^{-1}\left(cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})\right)\right] -$

$4\sum_{h=1}^{k} k v_1 v_2 \omega^2{}_h(\bar{Y}_h - \bar{Y})\left[\beta^*{}_1\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) + \beta^*{}_2\left(cov(\bar{y}_h, s^2{}_{zh}) -\right.\right.$

$\left.\left.cov(\bar{y}_{st}, s^2{}_{zh})\right)\right] + \sum_{h=1}^{k}\left(\mathrm{k}v_0\, v_1{}^{-1} + \beta^*{}_1 v_1 v_2\right)^2 \omega^2{}_h \;\; v(s^2{}_{xh}) +$

$2\sum_{\mathrm{h}=1}^{\mathrm{k}}\omega^2{}_{\mathrm{h}}\left(\mathrm{k}v_0 \;\; v_1{}^{-1} + \beta^*{}_1 v_1 v_2\right)\left(\mathrm{k}v_0 \;\; v_2{}^{-1} + \beta^*{}_2 v_1 v_2\right) cov(s^2{}_{xh}, s^2{}_{zh}) -$

$4\sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}v_0 \;\; v_1{}^{-1} + \beta^*{}_1 v_1 v_2\right)(\bar{X}_h - \bar{X})\left[k\left(cov\left(\bar{x}_h, s^2{}_{yh}\right) - cov\left(\bar{x}_{st}, s^2{}_{yh}\right)\right) +\right.$

$\left(\mathrm{k}v_0\, v_1{}^{-1} + \beta^*{}_1 v_1 v_2\right)\left(cov(\bar{x}_{st}, s^2{}_{xh}) - cov(\bar{x}_h, s^2{}_{xh})\right) +$

$\left.\left(\mathrm{k}v_0 \;\; v_2{}^{-1} + \beta^*{}_2 v_1 v_2\right)\left(cov(\bar{x}_{st}, s^2{}_{zh}) - cov(\bar{x}_h, s^2{}_{zh})\right)\right] - 4\sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}v_0 \;\; v_2{}^{-1} +\right.$

$$\beta^*{}_2\nu_2)(\bar{Z}_h - \bar{Z})\ \Big[k\left(cov(\bar{z}_h, s^2{}_{yh}) - cov(\bar{z}_{st}, s^2{}_{yh})\right) +$$

$$\left(\mathrm{k}\nu_0\ \ \nu_1{}^{-1}\ + \beta^*{}_1\nu_1\nu_2\right)\left(cov(\bar{z}_{st}, s^2{}_{xh}) - cov(\bar{z}_h, s^2{}_{xh})\right) +$$

$$\left(\mathrm{k}\nu_0\ \ \nu_2{}^{-1}\ + \beta^*{}_2\nu_1\nu_2\right)\left(cov(\bar{z}_{st}, s^2{}_{zh}) - cov(\bar{z}_h, s^2{}_{zh})\right)\Big] + \sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}\nu_0\ \ \nu_2{}^{-1}\ +\right.$$

$$\left.\beta^*{}_2\nu_1\nu_2\right)^2 v(s^2{}_{zh}) - 2\sum_{\mathrm{h}=1}^{\mathrm{k}}\omega^2{}_{\mathrm{h}}k\left(\mathrm{k}\nu_0\ \ \nu_2{}^{-1}\ + \beta^*{}_2\nu_1\nu_2\right)cov\left(s^2{}_{yh}, s^2{}_{zh}\right) -$$

$$8\ \sum_{h=1}^{k}\omega^2{}_h\left(\mathrm{k}\nu_0\ \ \nu_1{}^{-1}\ + \beta^*{}_1\nu_1\nu_2\right)\left(\mathrm{k}\nu_0\ \nu_2{}^{-1}\ + \beta^*{}_2\nu_1\nu_2\right)(\bar{X}_h - \bar{X})(\bar{Z}_h -$$

$$\bar{Z})[cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_h, \bar{z}_h) + cov(\bar{x}_{st}, \bar{z}_{st}) - cov(\bar{x}_{st}, \bar{z}_h)]\Big\}$$

Appendix D: MSE of the proposed estimators

The MSE of the proposed estimator $\left(\boldsymbol{s^2}_{\boldsymbol{pr_2}}, \boldsymbol{s^2}_{\boldsymbol{pr_3}}, \boldsymbol{s^2}_{\boldsymbol{pr_4}}, \boldsymbol{s^2}_{\boldsymbol{pr_5}}\right)$ for the population variance in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first degree approximation in the Taylor series method as follows:

$$d_h = \frac{(S^2{}_x + C_x)(S^2{}_z + C_z)}{(v_1 + C_x)(v_2 + C_z)}$$

$$\begin{bmatrix} \omega_h & 2\omega_h(\bar{Y}_h - \bar{Y}) & -2\omega_h(\bar{Y}_h - \bar{Y}) & -\frac{v_0\omega_h}{(v_1+C_x)} & -\frac{2v_0\omega_h\ (\bar{X}_h-\bar{X})}{(v_1+C_x)} & \frac{2v_0\omega_h\ (\bar{X}_h-\bar{X})}{(v_1+C_x)} & -\frac{v_0\omega_h}{(v_2+C_z)} \\ & & & -\frac{2v_0\omega_h(\bar{Z}_h-\bar{Z})}{(v_2+C_z)} & \frac{2v_0\omega_h(Z_h-\bar{Z})}{(v_2+C_z)} & & \end{bmatrix}$$

, Using (A.1) and (A.2), we have:

$$MSE(s^2{}_{pr_2}) \cong \frac{\left((S^2{}_x + C_x)(S^2{}_z + C_z)\right)^2}{\left((v_1 + C_x)(v_2 + C_z)\right)^2} \left\lfloor H_1 + H_6 \right\rfloor \text{--} \quad (D.1)$$

Where $H_6 =$

$$\Big\{-4\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_1+C_x)}\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) - 4\sum_{h=1}^{k}\frac{1}{(v_2+C_z)}\ v_0\omega^2{}_h(\bar{Y}_h -$$

$$\bar{Y})\left(cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})\right) -$$

$$2\sum_{h=1}^{k}\frac{v_0\omega^2{}_h}{(v_1+C_x)}cov\left(s^2{}_{xh}, s^2{}_{yh}\right) - 4\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{X}_h-\bar{X})}{(v_1+C_x)}\Big[cov\left(\bar{x}_h, s^2{}_{yh}\right) - cov\left(\bar{x}_{st}, s^2{}_{yh}\right) -$$

$$\frac{v_0}{(v_1+C_x)}\left(cov(\bar{x}_h, s^2{}_{xh}) - cov(\bar{x}_{st}, s^2{}_{xh})\right) - \frac{v_0}{(v_2+C_z)}\left(cov(\bar{x}_h, s^2{}_{zh}) - cov(\bar{x}_{st}, s^2{}_{zh})\right)\Big] -$$

$$2\sum_{h=1}^{k}\frac{v_0\omega^2{}_h}{(v_2+C_z)}cov\left(s^2{}_{zh}, s^2{}_{yh}\right) -$$

$$8\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{X}_h-\bar{X})}{(v_1+C_x)}[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_{st})] -$$

$$8\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{Z}_h-\bar{Z})}{(v_2+C_z)}[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_{st})] +$$

$$\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h}{(v_1+C_x)^2}v(s^2{}_{xh}) +$$

$$2\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h}{(v_1+C_x)(v_2+C_z\ )}cov(s^2{}_{xh},s^2{}_{zh})+4\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h(\bar{X}_h-\bar{X})^2}{(v_1+C_x)^2}[v(\bar{x}_h)-2cov(\bar{x}_h,\bar{x}_{st})+$$

$$v(\bar{x}_{st})]+\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h}{(v_2+C_z)^2}v(s^2{}_{zh})-8\sum_{h=1}^{k}\frac{v_0\ \omega^2{}_h(\bar{X}_h-\bar{X})(\bar{Z}_h-\bar{Z})}{(v_1+C_x)(v_2+C_z\ )}[cov(\bar{x}_h,\bar{z}_{st})-$$

$$cov(\bar{x}_h,\bar{z}_h)+cov(\bar{x}_{st},\bar{z}_h)-cov(\bar{x}_{st},\bar{z}_{st})]+4\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h(\bar{Z}_h-\bar{Z})^2}{(v_2+C_z)^2}[v(\bar{z}_h)-$$

$$2cov(\bar{z}_h,\bar{z}_{st})+v(\bar{z}_{st})]-4\sum_{h=1}^{k}\frac{v_0\ \ \omega^2{}_h(\bar{Z}_h-\bar{Z})}{(v_2+C_z)}\Big[cov\big(\bar{z}_h,s^2{}_{yh}\big)-cov\big(\bar{z}_{st},s^2{}_{yh}\big)-$$

$$\frac{v_0}{(v_1+C_x)}\big(cov(\bar{z}_h,s^2{}_{xh})-cov(\bar{z}_{st},s^2{}_{xh})\big)-\frac{v_0}{(v_2+C_z\ )}\big(cov(\bar{z}_h,s^2{}_{zh})-cov(\bar{z}_{st},s^2{}_{zh})\big)\Big]\Big\}$$

$$d_h=\frac{\big(S^2{}_x+\beta_2(x)\big)\big(S^2{}_z+\beta_2(z)\big)}{(v_1+\beta_2(x))(v_2+\beta_2(z)\ )}$$

$$\begin{bmatrix}\omega_h & 2\omega_h(\bar{Y}_h-\bar{Y}) & -2\omega_h(\bar{Y}_h-\bar{Y}) & -\frac{v_0\omega_h}{(v_1+\beta_2(x))} & -\frac{2v_0\omega_h\ \ (\bar{X}_h-\bar{X})}{(v_1+\beta_2(x))} & \frac{2v_0\omega_h\ \ (\bar{X}_h-\bar{X})}{(v_1+\beta_2(x))} & -\frac{v_0\omega_h}{(v_2+\beta_2(z)\ )} \\ & & & -\frac{2v_0\omega_h(\bar{Z}_h-\bar{Z})}{(v_2+\beta_2(z)\ )} & \frac{2v_0\omega_h(Z_h-\bar{Z})}{(v_2+\beta_2(z)\ )} & & \end{bmatrix}$$

Using (A.1) and (A.2), we have

$$MSE\big(s^2{}_{pr_3}\ \big)\cong\frac{\Big(\big(S^2{}_x+\beta_2(x)\big)\big(S^2{}_z+\beta_2(z)\big)\Big)^2}{\Big((v_1+\beta_2(x))(v_2+\beta_2(z)\ )\Big)^2}\left\{H_1+H_7\right\} \quad \text{-------------------------------- (D.2)}$$

Where $H_7=$

$$\Big\{-4\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_1+\beta_2(x))}\big(cov(\bar{y}_h,s^2{}_{xh})-cov(\bar{y}_{st},s^2{}_{xh})\big)-$$

$$4\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_2+\beta_2(z))}\big(cov(\bar{y}_h,s^2{}_{zh})-cov(\bar{y}_{st},s^2{}_{zh})\big)-$$

$$2\sum_{h=1}^{k}\frac{v_0\omega^2{}_h}{(v_1+\beta_2(x))}cov\big(s^2{}_{xh},s^2{}_{yh}\big)-$$

$4\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{X}_h-\bar{X})}{(v_1+\beta_2(x))}\Big[cov\big(\bar{x}_h,s^2{}_{yh}\big)-cov\big(\bar{x}_{st},s^2{}_{yh}\big)-\frac{v_0}{(v_1+\beta_2(x))}\big(cov(\bar{x}_h,s^2{}_{xh})-$

$cov(\bar{x}_{st},s^2{}_{xh})\big)-\frac{v_0}{(v_2+\beta_2(z))}\big(cov(\bar{x}_h,s^2{}_{zh})-cov(\bar{x}_{st},s^2{}_{zh})\big)\Big]-$

$2\sum_{h=1}^{k}\frac{v_0\omega^2{}_h}{(v_2+\beta_2(z))}cov\big(s^2{}_{zh},s^2{}_{yh}\big)-8\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{X}_h-\bar{X})}{(v_1+\beta_2(x))}[cov(\bar{y}_h,\bar{x}_h)-$

$cov(\bar{y}_h,\bar{x}_{st})-cov(\bar{y}_{st},\bar{x}_h)+cov(\bar{y}_{st},\bar{x}_{st})]-8\sum_{h=1}^{k}\frac{v_0\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{Z}_h-\bar{Z})}{(v_2+\beta_2(z))}[cov(\bar{y}_h,\bar{z}_h)-$

$cov(\bar{y}_h,\bar{z}_{st})-cov(\bar{y}_{st},\bar{z}_h)+cov(\bar{y}_{st},\bar{z}_{st})]+$

$\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h}{(v_1+\beta_2(x))^2}v(s^2{}_{xh})+$

$2\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h}{(v_1+\beta_2(x))(v_2+\beta_2(z))}cov(s^2{}_{xh},s^2{}_{zh})+4\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h(\bar{X}_h-\bar{X})^2}{(v_1+\beta_2(x))^2}[v(\bar{x}_h)-$

$2cov(\bar{x}_h,\bar{x}_{st})+v(\bar{x}_{st})]+$

$\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h}{(v_2+\beta_2(z))^2}v(s^2{}_{zh})-8\sum_{h=1}^{k}\frac{v_0\ \omega^2{}_h(\bar{X}_h-\bar{X})(\bar{Z}_h-\bar{Z})}{(v_1+\beta_2(x))(v_2+\beta_2(z))}[cov(\bar{x}_h,\bar{z}_{st})-cov(\bar{x}_h,\bar{z}_h)+$

$cov(\bar{x}_{st},\bar{z}_h)-cov(\bar{x}_{st},\bar{z}_{st})]+4\sum_{h=1}^{k}\frac{v^2{}_0\omega^2{}_h(\bar{Z}_h-\bar{Z})^2}{(v_2+\beta_2(z))^2}[v(\bar{z}_h)-2cov(\bar{z}_h,\bar{z}_{st})+v(\bar{z}_{st})]-$

$4\sum_{h=1}^{k}\frac{v_0\ \omega^2{}_h(\bar{Z}_h-\bar{Z})}{(v_2+\beta_2(z))}\Big[cov\big(\bar{z}_h,s^2{}_{yh}\big)-cov\big(\bar{z}_{st},s^2{}_{yh}\big)-\frac{v_0}{(v_1+\beta_2(x))}\big(cov(\bar{z}_h,s^2{}_{xh})-$

$cov(\bar{z}_{st},s^2{}_{xh})\big)-\frac{v_0}{(v_2+\beta_2(z))}\big(cov(\bar{z}_h,s^2{}_{zh})-cov(\bar{z}_{st},s^2{}_{zh})\big)\Big]\Big\}$

$$d_h=\frac{\left(S^2{}_xC_x+\beta_2(x)\right)\left(S^2{}_zC_z+\beta_2(z)\right)}{(v_1C_x+\beta_2(x))(v_2C_z+\beta_2(z))}\begin{bmatrix}\omega_h & 2\omega_h(\bar{Y}_h-\bar{Y}) & -2\omega_h(\bar{Y}_h-\bar{Y}) & -\frac{v_0\omega_hC_x}{(v_1C_x+\beta_2(x))}\\ -\frac{2v_0\omega_hC_x\ \ (\bar{X}_h-\bar{X})}{(v_1C_x+\beta_2(x))} & \frac{2v_0\omega_hC_x\ \ (\bar{X}_h-\bar{X})}{(v_1C_x+\beta_2(x))} & & -\frac{v_0\omega_hC_z}{(v_2C_z+\beta_2(z))}\\ & -\frac{2v_0\omega_hC_z(\bar{Z}_h-\bar{Z})}{(v_2C_z+\beta_2(z))} & \frac{2v_0\omega_hC_z(Z_h-\bar{Z})}{(v_2C_z+\beta_2(z))} & \end{bmatrix}$$

Using (A.1) and (A.2), we have

$$MSE\left(s^2{}_{pr_4}\right) \cong \frac{\left(\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)\right)^2}{\left(\left(v_1 C_x+\beta_2(x)\right)\left(v_2 C_z+\beta_2(z)\ \right)\right)^2}\left\{H_1+H_8\right\} \text{---------------------------- (D.3)}$$

Where $H_8 =$

$-4\sum_{h=1}^{k}\frac{v_0\ C_x\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_1 C_x+\beta_2(x))}\left(cov(\bar{y}_h,s^2{}_{xh})-cov(\bar{y}_{st},s^2{}_{xh})\right)-$

$4\sum_{h=1}^{k}\frac{v_0\ C_z\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_2 C_z+\beta_2(z)\ )}\left(cov(\bar{y}_h,s^2{}_{zh})-cov(\bar{y}_{st},s^2{}_{zh})\right)-$

$2\sum_{h=1}^{k}\frac{C_x v_0\omega^2{}_h}{(v_1 C_x+\beta_2(x))}cov\left(s^2{}_{xh},s^2{}_{yh}\right)-4\sum_{h=1}^{k}\frac{v_0 C_x\omega^2{}_h(\bar{X}_h-\bar{X})}{(v_1 C_x+\beta_2(x))}\Big[cov\left(\bar{x}_h,s^2{}_{yh}\right)-$

$cov\left(\bar{x}_{st},s^2{}_{yh}\right)-\frac{v_0 C_x}{(v_1 C_x+\beta_2(x))}\left(cov(\bar{x}_h,s^2{}_{xh})-cov(\bar{x}_{st},s^2{}_{xh})\right)-$

$\frac{v_0 C_z}{(v_2 C_z+\beta_2(z)\ )}\left(cov(\bar{x}_h,s^2{}_{zh})-cov(\bar{x}_{st},s^2{}_{zh})\right)\Big]-2\sum_{h=1}^{k}\frac{v_0 C_z\omega^2{}_h}{(v_2 C_z+\beta_2(z)\ )}cov\left(s^2{}_{zh},s^2{}_{yh}\right)-$

$8\sum_{h=1}^{k}\frac{v_0 C_x\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{X}_h-\bar{X})}{(v_1 C_x+\beta_2(x))}[cov(\bar{y}_h,\bar{x}_h)-cov(\bar{y}_h,\bar{x}_{st})-cov(\bar{y}_{st},\bar{x}_h)+$

$cov(\bar{y}_{st},\bar{x}_{st})]-$

$8\sum_{h=1}^{k}\frac{v_0 C_z\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{Z}_h-\bar{Z})}{(v_2 C_z+\beta_2(z)\ )}[cov(\bar{y}_h,\bar{z}_h)-cov(\bar{y}_h,\bar{z}_{st})-cov(\bar{y}_{st},\bar{z}_h)+$

$cov(\bar{y}_{st},\bar{z}_{st})]+$

$\sum_{h=1}^{k}\frac{v^2{}_0 C_x{}^2\omega^2{}_h}{(v_1 C_x+\beta_2(x))^2}v(s^2{}_{xh})+2\sum_{h=1}^{k}\frac{C_x C_z v^2{}_0\omega^2{}_h}{(v_1 C_x+\beta_2(x))(v_2 C_z+\beta_2(z)\ )}cov(s^2{}_{xh},s^2{}_{zh})+$

$4\sum_{h=1}^{k}\frac{v^2{}_0 C_x{}^2\omega^2{}_h(\bar{X}_h-\bar{X})^2}{(v_1 C_x+\beta_2(x))^2}[v(\bar{x}_h)-2cov(\bar{x}_h,\bar{x}_{st})+v(\bar{x}_{st})]+$

$\sum_{h=1}^{k}\frac{v^2{}_0 C_z{}^2\omega^2{}_h}{(v_2 C_z+\beta_2(z))^2}v(s^2{}_{zh})-8\sum_{h=1}^{k}\frac{v_0\ C_x C_z\omega^2{}_h(\bar{X}_h-\bar{X})(\bar{Z}_h-\bar{Z})}{(v_1 C_x+\beta_2(x))(v_2 C_z+\beta_2(z)\ )}[cov(\bar{x}_h,\bar{z}_{st})-$

$cov(\bar{x}_h,\bar{z}_h)+cov(\bar{x}_{st},\bar{z}_h)-cov(\bar{x}_{st},\bar{z}_{st})]+4\sum_{h=1}^{k}\frac{v^2{}_0 C_z{}^2\omega^2{}_h(\bar{Z}_h-\bar{Z})^2}{(v_2 C_z+\beta_2(z))^2}[v(\bar{z}_h)-$

$2cov(\bar{z}_h,\bar{z}_{st})+v(\bar{z}_{st})]-4\sum_{h=1}^{k}\frac{v_0 C_z\ \omega^2{}_h(\bar{Z}_h-\bar{Z})}{(v_2 C_z+\beta_2(z))}\Big[cov\left(\bar{z}_h,s^2{}_{yh}\right)-cov\left(\bar{z}_{st},s^2{}_{yh}\right)-$

$\frac{v_0 C_x}{(v_1 C_x+\beta_2(x))}\left(cov(\bar{z}_h, s^2{}_{xh}) - cov(\bar{z}_{st}, s^2{}_{xh})\right) - \frac{v_0 C_z}{(v_2 C_z+\beta_2(z))}\left(cov(\bar{z}_h, s^2{}_{zh}) -\right.$

$\left.cov(\bar{z}_{st}, s^2{}_{zh})\right)\Big]$

$$d_h = \frac{(S^2{}_x\beta_2(x)+C_x)(S^2{}_z\beta_2(z)+C_z)}{(v_1\beta_2(x)+C_x)(v_2\beta_2(z)+C_z\ )}\begin{bmatrix} \omega_h \quad 2\omega_h(\bar{Y}_h-\bar{Y}) \quad -2\omega_h(\bar{Y}_h-\bar{Y}) \quad -\frac{v_0\omega_h\beta_2(x)}{(v_1\beta_2(x)+C_x)} \\ -\frac{2v_0\omega_h\beta_2(x)\ \ (\bar{X}_h-\bar{X})}{(v_1\beta_2(x)+C_x)} \quad \frac{2v_0\omega_h\beta_2(x)\ \ (\bar{X}_h-\bar{X})}{(v_1\beta_2(x)+C_x)} \quad -\frac{v_0\omega_h\beta_2(z)}{(v_2\beta_2(z)+C_z\ )} \\ -\frac{2v_0\omega_h\beta_2(z)(\bar{Z}_h-\bar{Z})}{(v_2\beta_2(z)+C_z\ )} \quad \frac{2v_0\omega_h\beta_2(z)(Z_h-\bar{Z})}{(v_2\beta_2(z)+C_z\ )} \end{bmatrix}$$

Using (A.1) and (A.2), we have

$$MSE(s^2{}_{pr_5}) \cong \frac{\left((S^2{}_x\beta_2(x)+C_x)(S^2{}_z\beta_2(z)+C_z)\right)^2}{\left((v_1\beta_2(x)+C_x)(v_2\beta_2(z)+C_z\ )\right)^2}\left\{H_1 + H_9\right\} \text{------------------------------ (D.4)}$$

Where $H_9 = -4\sum_{h=1}^{k}\frac{v_0\beta_2(x)\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_1\beta_2(x)+C_x)}\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) -$

$4\sum_{h=1}^{k}\frac{v_0\beta_2(z)\omega^2{}_h(\bar{Y}_h-\bar{Y})}{(v_2\beta_2(z)+C_z\ )}\left(cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})\right) -$

$2\sum_{h=1}^{k}\frac{\beta_2(x)v_0\omega^2{}_h}{(v_1\beta_2(x)+C_x)}cov\left(s^2{}_{xh}, s^2{}_{yh}\right) - 4\sum_{h=1}^{k}\frac{v_0\beta_2(x)\omega^2{}_h(\bar{X}_h-\bar{X})}{(v_1\beta_2(x)+C_x)}\Big[cov\left(\bar{x}_h, s^2{}_{yh}\right) -$

$cov\left(\bar{x}_{st}, s^2{}_{yh}\right) - \frac{v_0\beta_2(x)}{(v_1\beta_2(x)+C_x)}\left(cov(\bar{x}_h, s^2{}_{xh}) - cov(\bar{x}_{st}, s^2{}_{xh})\right) -$

$\frac{v_0\beta_2(z)}{(v_2\beta_2(z)+C_z\ )}\left(cov(\bar{x}_h, s^2{}_{zh}) - cov(\bar{x}_{st}, s^2{}_{zh})\right)\Big] - 2\sum_{h=1}^{k}\frac{v_0\beta_2(z)\omega^2{}_h}{(v_2\beta_2(z)+C_z\ )}cov\left(s^2{}_{zh}, s^2{}_{yh}\right) -$

$8\sum_{h=1}^{k}\frac{v_0\beta_2(x)\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{X}_h-\bar{X})}{(v_1\beta_2(x)+C_x)}[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h) +$

$cov(\bar{y}_{st}, \bar{x}_{st})] - 8\sum_{h=1}^{k}\frac{v_0\beta_2(z)\omega^2{}_h(\bar{Y}_h-\bar{Y})(\bar{Z}_h-\bar{Z})}{(v_2\beta_2(z)+C_z\ )}[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) -$

$cov(\bar{y}_{st}, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_{st})] +$

$$\sum_{h=1}^{k}\frac{v^2{}_0\left(\beta_2(x)\right)^2\omega^2{}_h}{(v_1\beta_2(x)+C_x)^2}v(s^2{}_{xh}) + 2\sum_{h=1}^{k}\frac{\beta_2(x)\beta_2(z)v^2{}_0\omega^2{}_h}{(v_1\beta_2(x)+C_x)(v_2\beta_2(z)+C_z\ )}cov(s^2{}_{xh},s^2{}_{zh}) +$$

$$4\sum_{h=1}^{k}\frac{v^2{}_0\left(\beta_2(x)\right)^2\omega^2{}_h(\bar{X}_h-\bar{X})^2}{(v_1\beta_2(x)+C_x)^2}[v(\bar{x}_h) - 2cov(\bar{x}_h,\bar{x}_{st}) + v(\bar{x}_{st})] +$$

$$\sum_{h=1}^{k}\frac{v^2{}_0\left(\beta_2(z)\right)^2\omega^2{}_h}{(v_2\beta_2(z)+C_z)^2}v(s^2{}_{zh}) - 8\sum_{h=1}^{k}\frac{v_0\ \beta_2(x)\beta_2(z)\omega^2{}_h(\bar{X}_h-\bar{X})(\bar{Z}_h-\bar{Z})}{(v_1\beta_2(x)+C_x)(v_2\beta_2(z)+C_z\ )}[cov(\bar{x}_h,\bar{z}_{st}) -$$

$$cov(\bar{x}_h,\bar{z}_h) + cov(\bar{x}_{st},\bar{z}_h) - cov(\bar{x}_{st},\bar{z}_{st})] + 4\sum_{h=1}^{k}\frac{v^2{}_0\left(\beta_2(z)\right)^2\omega^2{}_h(\bar{Z}_h-\bar{Z})^2}{(v_2\beta_2(z)+C_z)^2}[v(\bar{z}_h) -$$

$$2cov(\bar{z}_h,\bar{z}_{st}) + v(\bar{z}_{st})] - 4\sum_{h=1}^{k}\frac{v_0\beta_2(z)\omega^2{}_h(\bar{Z}_h-\bar{Z})}{(v_2\beta_2(z)+C_z)}\Big[cov\big(\bar{z}_h,s^2{}_{yh}\big) - cov\big(\bar{z}_{st},s^2{}_{yh}\big) -$$

$$\frac{v_0\beta_2(x)}{(v_1\beta_2(x)+C_x)}\big(cov(\bar{z}_h,s^2{}_{xh}) - cov(\bar{z}_{st},s^2{}_{xh})\big) - \frac{v_0\beta_2(z)}{(v_2\beta_2(z)+C_z)}\big(cov(\bar{z}_h,s^2{}_{zh}) -$$

$$cov(\bar{z}_{st},s^2{}_{zh})\big)\Big]$$

Appendix E: Moments

$\mu_{rst\,h} = \frac{1}{N_h}\sum_{h=1}^{N_h}(Y_{hi}-\bar{Y}_h)^r(X_{hi}-\bar{X}_h)^s(Z_{hi}-\bar{Z}_h)^t$, $\lambda_h = \frac{1}{n_h}$, $\omega_h = \frac{n_h}{n}=\frac{N_h}{N}$

$\theta_h(yx) = \frac{\mu_{220\,h}}{\mu_{200\,h}\mu_{020\,h}}$, $\theta_h(yz) = \frac{\mu_{202\,h}}{\mu_{200\,h}\mu_{002\,h}}$, $\theta_h(xz) = \frac{\mu_{022\,h}}{\mu_{020\,h}\mu_{020\,h}}$,

$\beta_2(y_h)=\frac{\mu_{400\,h}}{\mu^2{}_{200\,h}}$ - is the population kurtosis of the variate of interest in stratum h.

$\beta_2(x_h)=\frac{\mu_{040\,h}}{\mu^2{}_{020\,h}}$ -is the population kurtosis of the first auxiliary variable (X) in stratum h.

$\beta_2(z_h)=\frac{\mu_{004\,h}}{\mu^2{}_{002\,h}}$ - is the population kurtosis of the second auxiliary variable (Z) in stratum h.

$\sigma_1^2 = v(s^2{}_{yh}) = \lambda_h S^4{}_{yh}(\beta_2(y_h) - 1)$

$\sigma_2^2 = v(\bar{y}_h) = \lambda_h S^2{}_{yh}$

$\sigma_3^2 = v(\bar{y}_{st}) = \sum_{h=1}^{k} \omega^2{}_h \, \lambda_h S^2{}_{yh}$

$\sigma_4^2 = v(s^2{}_{xh}) = \lambda_h S^4{}_{xh}(\beta_2(x_h) - 1)$

$\sigma_5^2 = v(\bar{x}_h) = \lambda_h S^2{}_{xh}$

$\sigma_6^2 = v(\bar{x}_{st}) = \sum_{h=1}^{k} \omega^2{}_h \, \lambda_h S^2{}_{xh}$

$\sigma_7^2 = v(s^2{}_{zh}) = \lambda_h S^4{}_{zh}(\beta_2(z_h) - 1)$

$\sigma_8^2 = v(\bar{z}_h) = \lambda_h S^2{}_{zh}$

$\sigma_9^2 = v(\bar{z}_{st}) = \sum_{h=1}^{k} \omega^2{}_h \, \lambda_h S^2{}_{zh}$

$\sigma_{12} = \sigma_{21} = cov(\bar{y}_h, s^2{}_{yh}) = \lambda_h \mu_{300h}$

$\sigma_{13} = \sigma_{31} = cov(\bar{y}_{st}, s^2{}_{yh}) = \sum_{h=1}^{k} \omega_h \, \lambda_h \mu_{300h}$

$\sigma_{14} = \sigma_{41} = cov(s^2{}_{xh}, s^2{}_{yh}) = \lambda_h S^2{}_{xh} S^2{}_{yh}(\theta_h(yx) - 1)$

$\sigma_{15} = \sigma_{51} = cov(\bar{x}_h, s^2{}_{yh}) = \lambda_h \mu_{210h}$

$\sigma_{16} = \sigma_{61} = cov(\bar{x}_{st}, s^2{}_{yh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{210h}$

$\sigma_{17} = \sigma_{71} = cov(s^2{}_{yh}, s^2{}_{zh}) = \lambda_h S^2{}_{zh} S^2{}_{yh} (\theta_h(yz) - 1)$

$\sigma_{18} = \sigma_{81} = cov(\bar{z}_h, s^2{}_{yh}) = \lambda_h \mu_{201h}$

$\sigma_{19} = \sigma_{91} = cov(\bar{y}_{st}, s^2{}_{yh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{201h}$

$\sigma_{23} = \sigma_{32} = cov(\bar{y}_h, \bar{y}_{st}) = \sum_{h=1}^{k} \omega_h \lambda_h S^2{}_{yh}$

$\sigma_{24} = \sigma_{42} = cov(\bar{y}_h, s^2{}_{xh}) = \lambda_h \mu_{120h}$

$\sigma_{25} = \sigma_{52} = cov(\bar{y}_h, \bar{x}_h) = \lambda_h S_{yxh}$

$\sigma_{26} = \sigma_{62} = \sigma_{35} = \sigma_{53} = cov(\bar{y}_{st}, \bar{x}_h) = cov(\bar{y}_h, \bar{x}_{st}) = \sum_{h=1}^{k} \omega_h \lambda_h S_{yxh}$

$\sigma_{27} = \sigma_{72} = cov(\bar{y}_h, s^2{}_{zh}) = \lambda_h \mu_{102h}$

$\sigma_{28} = \sigma_{82} = cov(\bar{y}_h, \bar{z}_h) = \lambda_h S_{yzh}$

$\sigma_{29} = \sigma_{92} = \sigma_{38} = \sigma_{83} = cov(\bar{y}_h, \bar{z}_{st}) = \sum_{h=1}^{k} \omega_h \lambda_h S_{yzh}$

$\sigma_{34} = \sigma_{43} = cov(\bar{y}_{st}, s^2{}_{xh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{120h}$

$\sigma_{36} = \sigma_{63} = cov(\bar{y}_{st}, \bar{x}_{st}) = \sum_{h=1}^{k} \omega^2{}_h \lambda_h S_{yxh}$

$$\sigma_{37} = \sigma_{73} = cov(\bar{y}_{st}, s^2{}_{zh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{102h}$$

$$\sigma_{39} = \sigma_{93} = cov(\bar{y}_{st}, \bar{z}_{st}) = \sum_{h=1}^{k} \omega^2{}_h \lambda_h S_{yzh}$$

$$\sigma_{45} = \sigma_{54} = cov(\bar{x}_h, s^2{}_{xh}) = \lambda_h \mu_{030h}$$

$$\sigma_{46} = \sigma_{64} = cov(\bar{x}_{st}, s^2{}_{xh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{030h}$$

$$\sigma_{47} = \sigma_{74} = cov(s^2{}_{xh}, s^2{}_{zh}) = \lambda_h S^2{}_{zh} S^2{}_{xh} (\theta_h(zx) - 1)$$

$$\sigma_{48} = \sigma_{84} = cov(\bar{z}_h, s^2{}_{xh}) = \lambda_h \mu_{021h}$$

$$\sigma_{49} = \sigma_{94} = cov(\bar{z}_{st}, s^2{}_{xh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{021h}$$

$$\sigma_{56} = \sigma_{65} = cov(\bar{x}_h, \bar{x}_{st}) = \sum_{h=1}^{k} \omega_h \lambda_h S^2{}_{xh}$$

$$\sigma_{57} = \sigma_{75} = cov(\bar{x}_h, s^2{}_{zh}) = \lambda_h \mu_{012h}$$

$$\sigma_{58} = \sigma_{85} = cov(\bar{x}_h, \bar{z}_h) = \lambda_h S_{xzh}$$

$$\sigma_{59} = \sigma_{95} = \sigma_{68} = \sigma_{86} = cov(\bar{x}_h, \bar{z}_{st}) = cov(\bar{z}_h, \bar{x}_{st}) = \sum_{h=1}^{k} \omega_h \lambda_h S_{xzh}$$

$$\sigma_{67} = \sigma_{76} = cov(\bar{x}_{st}, s^2{}_{zh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{012h}$$

$$\sigma_{69} = \sigma_{96} = cov(\bar{x}_{st}, \bar{z}_{st}) = \sum_{h=1}^{k} \omega^2{}_h \lambda_h S_{xzh}$$

$$\sigma_{78} = \sigma_{87} = cov(\bar{z}_h, s^2{}_{zh}) = \lambda_h \mu_{003h}$$

$\sigma_{79} = \sigma_{97} = cov(\bar{z}_{st}, s^2{}_{zh}) = \sum_{h=1}^{k} \omega_h \lambda_h \mu_{003h}$

$\sigma_{89} = \sigma_{98} = cov(\bar{z}_h, \bar{z}_{st}) = \sum_{h=1}^{k} \omega_h \lambda_h S^2{}_{zh}$