ANALYSIS OF TURBULENT HYDROMAGNETIC FLOW WITH RADIATIVE HEAT OVER A MOVING VERTICAL PLATE IN A ROTATING SYSTEM

DAWIT HABTE GEBRE

MASTER OF SCIENCE IN MATHEMATICS (Computational Option)

PAN AFRICAN UNIVERITY INSTITUITE FOR BASIC SCIENCES
TECHNOLOGY AND INNOVATION

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DAWIT H. GEBRE

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DECLARATION

I declare that this thesis is my own work and has not been submitted in any university or other institution for the award of a degree or any other award.

Student's name: Dawit Habte Gebre

Signature: __________________________ **Date:** ……………………………………………

The student, named above, worked under our supervision and the work reported in this thesis is his own work.

Supervisor's name: Professor Matthew Kinyanjui

Department of Pure and Applied Mathematics

Jomo Kenyatta University of Agriculture and Technology

P.0 Box 62000 – 00200

NAIROBI

Signature: __________________________ **Date:** ……………………………………………

Supervisor's name: Professor Oluwole Daniel **Ma**kinde

Faculty of Military Science, Stellenbosch University,

Private Bag x2, saldanha 7395,

SOUTH AFRICA

Signature: __________________________ **Date:** ……………………………………………

DEDICATION

This thesis is dedicated to my late father, who is now in the Lord's
bosom, and to my Mother for their unfailing and unconditional love.
And also to my fiancé, Merry have never left my side and is very special indeed.
I love you all dearly.

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ABBERIVATION

MHD	Magnethodrodynamic
RANS equations	Reynolds-average Navier-stoke
FDM	Finite difference method
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
BL	Boundary Layer
FS	Free stream
DNS	Direct Numerical Simulation
BVP	Boundary Value problem
IVP	Initial Value Problem

NOMENCLATURE

Symbols	Quantity
J	current density, A/m^2
B	magnetic field intensity, Wb/m^2
t	time, s
E	electric field, V/m
u, v, w	components of velocity in the x, y and z directions, respectively, m/s
P	pressure of the fluid, N/m^2
P_e	the electron pressure, N/m^2
g	acceleration due to gravity, m/s^2
B_0	constant magnetic field intensity, Wb/m^2
k	thermal conductivity, $W/m/K$
T	Absolute Temperature K
C_p	specific heat at constant pressure, $J/kg/K$
x, y, z	dimensional Cartesian co-ordinates
ω_e	electron cyclotron frequency, Hz
T	dimensional temperature, K
q	velocity field
T_∞	temperature of the fluid in the free stream, K
T_w	temperature of the fluid at the plate, K
U, V, W	dimensionless velocity components

x, y, z	dimensionless Cartesian co-ordinates
Gr	Grashof number
M^2	magnetic parameter
E	Eckert number
m	Hall parameter $(\omega_e \tau_e)$
Pr	Prandtl number
R	Rotational Parameter
N_r	Radiation Parameter
p^*	Modified pressure including centrifugal force
U^*	Characteristic velocity
L	Linear Length

Greek Symbols

Symbols	**Quantity**
ρ	fluid density, kg/m^3
μ	coefficient of viscosity, kg/ms
μ_e	magnetic permeability, H/m
$\Delta t, \Delta\eta$	time and distance intervals
υ	kinematic viscosity, m^2/s
τ_e	collision time of electrons, s
β	coefficient of thermal expansion, K^{-1}
σ	electrical conductivity, $\Omega^{-1}m^{-1}$
τ	Shear stress
ω_e	Electron cyclotron frequency, Hz
Ω	Angular velocity $(1/s)$

Ξ	Flow property
τ_e	Collision time of electrons, *s*
δ_{ij}	Kronecker delta
σ_{ij}	Stress tensor
ϕ	Body force per unit mass including gravitation effects
Φ	Viscous dissipation

DIMENSIONLESS QUANTITIES

θ	Dimensionless temperature of the fluid
η	Dimensionless Cartesian co-ordinates
U, V, W	Dimensionless velocity components
E_c	Eckert Number $\left(= \frac{U_0^{\ 2}}{C_p \left(T_w - T_\infty \right)} \right)$
G_r	Grashof number $\left(= \frac{\upsilon g \beta \left(T_w - T_\infty \right)}{U_0^{\ 3}} \right)$
P_r	Prandtl Number $\left(= \frac{\mu C_p}{k} \right)$
M^2	Magnetic Parameter $\left(= \frac{\sigma \mu_e^{\ 2} H_o^{\ 2} \upsilon}{\rho U_0^{\ 2}} \right)$
m	Hall parameter
n	Karmen constant
Nu	Nusselt number
R	Rotational parameter $\left(= \frac{\Omega \upsilon}{U_0^{\ 2}} \right)$
N_r	Radiation Parameter $\left(= \frac{4 \varepsilon^2 \upsilon}{\rho c_p U_0^2} \right)$

ABSTRACT

In this study, the combined effects of magnetic fields, buoyancy force, thermal radiation, viscous and Ohmic heating on turbulent hydromagnetic flow of an incompressible electrically conducting fluid over a moving vertical plate in a rotating system is investigated numerically. The governing equations are reduced to non-linear ordinary differential equations using the time-averaged approach known as Reynolds-averaged Navier–Stokes equations and tackled by employing an efficient Runge-Kutta Fehlberg integration technique coupled with shooting scheme. Graphical results showing the effects of various thermophysical parameters on the velocity, temperature, local skin friction and local Nusselt number are presented and discussed quantitatively. Moreover, After introducing Pseudo time spacing into our model the newly emerging PDE's solved using the finite difference scheme and carried the computations by taking a large time interval.

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CHAPTER ONE

Introduction

1.1 Background Information

Flow of fluids occur in all fields of our natural and technical environment and anyone perceiving their surroundings with open eyes and assessing their significance for themselves and their fellow beings can convince themselves of the far reaching effects of fluid flows. Without fluid flows life, as we know it, would not be possible on Earth, nor could technological processes run in the form known to us and lead to the multitude of products which determine the high standard of living that we nowadays take for granted. Without flows our natural and technical world would be different, and might not even exist at all. Flows are therefore vital following (Franz,2008)

In this section, definition of terms will be given.

A fluid is defined as a substance that deforms continuously when acted on by a shearing stress of any magnitude. A shearing stress(force per unit area) is created whenever a tangential force acts on a surface. Common fluids such as water, oil, and air satisfy the definition of a fluid. that is they will flow when acted on by the searing stress.

Fluid dynamics is the study of force that causes fluid motion. The force acting on a fluid element may be classified as either body force or surface forces. Fluids are classified into two *i.e* Compressible and Incompressible. Fluids whose density does not change significantly with change in pressure or temperature are assumed to be incompressible fluids. When there is significant change in the density of the fluid with change in pressure or temperature then the fluid is considered to be compressible.

The relation between searing stress and rate of shearing strain (velocity gradient) is given by a relationship of the form: (Currie,1974)

$$\tau = \mu \frac{dU}{dZ} \tag{1.1}$$

where the constant of proportionality is designated by μ and known as the *Coefficient of viscosity* of the fluid. Fluid in which the shearing stress in linearly related to the rate of shearing strain are designated as Newtonian fluids. Fluid for which the searing stress is not linearly related to the rate of shearing strain are designated as Non-Newtonian fluids. Of course this study devoted to a Newtonian fluid.

If Ξ is a flow property such as velocity, pressure, mass, density or temperature, then we can have the following types of flows:

$$\text{Steady flow : } \left(\frac{\partial \Xi}{\partial t}\right) = 0 \tag{1.2}$$

$$\text{Unsteady flow: } \left(\frac{\partial \Xi}{\partial t}\right) \neq 0 \tag{1.3}$$

$$\text{Uniform flow : } \left(\frac{\partial \Xi}{\partial t}\right)_{t=t_0} = 0 \tag{1.4}$$

$$\text{Non- Uniform flow : } \left(\frac{\partial \Xi}{\partial t}\right)_{t=t_0} \neq 0 \tag{1.5}$$

1.1.1 Magnetohydrodynamics

Magnetic fields influence many natural and man-made flows. They are routinely used in industry to heat, pump, stir and levitate liquid metals. There is the terrestrial magnetic field which is maintained by fluid motion in the earth's core, the solar magnetic field which generates sunspots and solar flares, and the galactic magnetic field which is thought to influence the formation of stars from interstellar clouds. The study of these flows is called magnetohydrodynamics (MHD), (Davidson, 2001).

MHD is the physical-mathematical framework that concerns the dynamics of magnetic fields in flow of electrically conducting fluids, e.g. in plasmas and liquid metals. The word *magnetohydrodynamics* is comprised of the words *magneto-* meaning magnetic, *hydro-* meaning water (or liquid) and *-dynamics* referring to the movement of an entity by forces. Synonyms of MHD that are less frequently used are the terms *magnetofluiddynamics* and *hydromagnetics*.

One of the most famous scholars associated with MHD was the Swedish physicist Hannes Alfvén (1908-1995), who received the Nobel Prize in Physics (1970) for fundamental work and discoveries in *magnetohydrodynamics* with fruitful applications in different areas of plasma physics.

Formally, MHD is concerned with the mutual interaction of flow of an electrically conducting fluid and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolyte (Davidison, 2001).

If an electrically conducting fluid moves past a magnetic field, there arises an interaction between the flow field and magnetic field. The magnetic field exerts a force on the fluid due to induced currents and the induced currents affect the original magnetic field. There develops a component of electric field in the direction perpendicular to both the electric field and the magnetic field. The production of a potential difference across an electrical conductor when a magnetic field is applied in a direction perpendicular to that of the flow of current. This Phenomenon is known as Hall effect.

1.1.2 Free Convection Flow

In free convection, fluid motion results when body forces act on the fluid in which density gradient exist. The density gradient may be due to temperature gradient existing in the fluid, while the body force is due to gravitational force. In our study we consider free convection flow due to temperature difference. Buoyant or Free convection is a very important mechanism that is operative in a variety of environments from cooling electronic circuit boards in computer to causing large scale circulation in the atmosphere as well as in lakes and oceans that influences that the weather. It is caused by the action of density gradients in conjunction with a gravitational field.

1.1.3 Fluid Pressure

The Science of fluid mechanics deals principally with relationship between fluid motion and the forces causes the flow; Pressure forces exerted by jet enable huge aircraft to takeoff and fly. Since a fluid has no definite shape, its pressure applies in all directions. Fluid pressure can also be amplified through hydraulic mechanisms and changes with the velocity of the fluid. Fluid pressure can be caused by gravity, or forces in a closed container. Forces on the walls of closed channels such as pipe make it possible for fluid like water and blood to be pumped through their distribution network. Fluid forces are classified in two groups namely Body Force and Surface Force.

1.1.4 Body Force

A body force is a force that acts throughout the volume of a body, in contrast to contact forces. Gravity and electromagnetic forces are examples of body forces are examples of body forces. centrifugal forces and coriolis force can also be viewed as a body force. A body force is distinct from a contact force in that the force does not require contact for transmission. Thus, common forces associated with pressure gradients and conductive and convective heat transmission are not body forces as they require contact between systems to exist. Radiation heat transfer is, on the other hand, is a perfect example of a body force.

1.1.5 Surface Forces

This is brought about by the interaction between the fluid and its surrounding. They include all forces that across an internal or external surface element in a material body. Therefore these forces appear only at the surface of the fluid element. Surface forces can be resolved into two components, one along the normal to an elemental area and the other along the plane of the elemental area. A fluid motion will have surface due to viscous stresses.

1.1.6 Heat Transfer

If a fluid dissipates heat or heat is introduced to the flow field, then the study of heat transfer is necessary. Heat transfer is the energy transfer which may take place between bodies as a result of temperature difference. *Conduction* refers to heat transfer that takes place when the temperature gradient exists in a stationary medium, which may be a solid or fluid. In contrast the term *convection* refers to heat transfer that occurs between a surface and a moving fluid when they are at different temperature .If the fluid motion is due to buoyancy effects resulting from variation caused by the temperature difference in the fluid, the heat transfer is said to be *free* and *natural* convection. All surface of finite temperature emits energy in the form of electromagnetic waves, hence in the absences of an intervening medium, there is a net transmission due to electromagnetic waves propagation medium, which can occur in a vacuum as well as in a medium. The emissions may occur from solid surface, liquid and gases. This type of heat transfer is referred to as radiation. *Radiation* is type of heat transfer in which there is a net heat transmission due to electromagnetic wave propagation that take place in a vacuum as well in a medium.

Convection heat transfer may be categorized according to the nature of the flow *i.e.* forced convection, when the flow is caused by some external means such as a fan and free and natural convection, when the flow is as a result of density difference caused by temperature gradients. In the present study we will consider free convection heat transfer.

1.1.8 Boundary Layer

Boundary layers arises to a bounding surface when the influence of a physical quantity is restricted to small regions near confining boundaries. A layer near the surface of a body or solid in which the flow is affected by the viscous forces is called Boundary layer. This phenomenon comes into existence when the non-dimensional diffusion parameter, namely, rotation parameter(reciprocal of Ekman number), magnetic parameter (square of Hartmann number), frequency etc. are large. When a vast amount of viscous and incompressible fluid bounded by a rigid surface is rotating rapidly then there appears a thin boundary layer near the bounding surface based on the balance between Coriolis and viscous force. In analyzing flow problems that involve transfer by convection, boundary layer theory plays a significant role. The Boundary layers may exist when a fluid flows on the surface. These are Thermal, concentration and Velocity boundary layer. A zero velocity is assumed by fluid particles when they come into contact with a surface (no-slip condition). Velocity boundary layer is the region in which the velocity gradient is large, however this isn't necessarily true in all scenarios.

The fluid particle attains a thermal equilibrium state when they come in to contact with as isothermal plate on its surface temperature. Thermal boundary layer is the region on the fluid in which temperature gradients exist.

1.1.9 Turbulent Flow

Turbulent flows occur when small disturbance are present in fluid due to small variation in physical properties of fluid motion, and wall roughness. Laminar flows are characterized by orderliness of the fluid particles, while in turbulent flows path of individual particles of fluids are not straight but are intertwined and cross one another in disorderly manner. In turbulent flow there would be continual variation of velocity and pressure almost at every point in flow region. In the diagram we have presented the velocity profile depicting both laminar and turbulent flow scenarios.

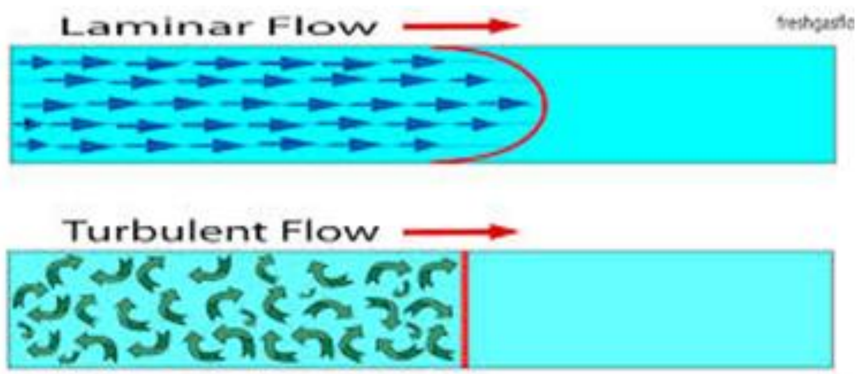

Figure 1. 1: Velocity Profile for Turbulent and laminar flows.

Following Narasimha *et al* (2007), turbulent flow is a flow regime characterized by chaotic variation in the fluid properties, such as low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time. In turbulent flow, drag due to boundary layer increases and the unsteady vortices appear on many scales and interact with each other. Turbulent flows are always unsteady that is the flow various continuously with time even though there is steady downstream motion of the fluid. The unsteadiness of turbulent flow is exhibited and/or manifested by eddies that appears to be random in both magnitude and direction.

The momentum that is carried by the fluid particles and exchanged between layers exerts forces on the fluid in the layer into which the momentum has been transported by the turbulent eddy. Depending on the relative magnitude of the momentum involved, the affected fluid layer may be accelerated. while laminar flows are associated with relatively low Reynolds number turbulent flows corresponds to high values of Reynolds number. Turbulization leads to a rapid mixing of particles in a continuous medium and to an increase in the efficiency of mass, momentum, and heat transfer; in multiphase multi component media, it also contributes to the acceleration of phase transitions and chemical reactions. As the knowledge about the various natural objects in which turbulence plays a significant role, modeling this phenomenon and related hydrodynamic effects acquires a key importance.

In most flows of practical interest especially those of large scale the condition in a boundary layer or wake are more likely to be turbulent than laminar. The overall direction of the flow is well defined but the instantaneous direction of motion of any particle is highly unpredictable, and individual particles distort and rotate as the fluid flows.

Indeed, the early works on fluid dynamics is mostly on laminar flows with very little devotion given to turbulent flows but most flows of engineering importance are turbulent.

MHD turbulent describes in an electrically conducting, magnetized fluid. Strictly speaking, MHD only applies to collision dominated fluids. however, it is often a useful guide to the behavior of magnetized plasmas even in the collision limit. Nature routinely produced MHD turbulence. Ionized gas pervades the region between and within galaxies and inside stars. It is likely that most of the baryonic matter in the universe is in a state of MHD turbulence.

1.2 Statement of the Problem

We shall consider the fluid flow in the presence of a strong magnetic force; moreover in the course of our model formulation the effect of the Hall current will not be neglect. When an electrically conducting fluid flows past a vertical infinite plate in a rotating system in the presences of magnetic field, the motion of the fluid is retarded and the velocity and temperature changed are observed.

The main objective of this present study is to extend the theoretical model of MHD turbulent flow in a rotating system and therefore intends to obtain an approximation solution to the shape of the velocity , temperature profiles and both skin friction & Nuseelt number. We further let the fluid and the plate to be in a state of rigid rotation with uniform angular velocity $\boldsymbol{\Omega}$ about the z-axis taken normal to the plate. The strong magnetic field, B which is assumed to be applied transversely to the direction of the flow.

1.3 Justification

Prominently Fluid mechanics considerations are applied in many fields, especially in engineering. Fluid mechanics has become an essential part of diverse field such as Medicine, Metrology, Astronomy and Oceanography. In general MHD has its application in heat transfer devices. MHD convection flow has many important engineering application in the design of power generator, heat exchangers, pumps and flow meters, in solving space vehicle propulsion, control and re-entry problems; in designing communication and radar system; in creating novel power generating system; in developing confinement schemes for controlled fusion and in design of nuclear cooling reactor and MHD accelerator; Turbulence causes the formation of eddies of many different length scales. Studies related to turbulent flow and heat transfer not only present a mathematical challenge but find several applications in many industrial, engineering and technological processes (Trevethan & Chanso, 2010). For instance, the external flow over all kind of vehicles such as cars, airplanes, ships and submarine are turbulence. The flow conditions in many industrial equipment such as pipes, ducts, precipitators, gas scrubbers, dynamic scraped surface heat exchangers, internal combustion engines and gas turbines are turbulence. In many geophysical flows such as rivers and atmospheric boundary layer, the flow turbulence is dominated by the coherent structure activities and associated turbulent events. In the medical field of cardiology, a stethoscope is used to detect heart sounds and bruits, which are due to turbulent blood flow. Moreover, when flow is turbulent, particles exhibit additional transverse motion which enhances the rate of energy and momentum exchange between them thus increasing the heat transfer and the friction coefficient (Avila *et al* , 2011).

Hydromagnetic flow and heat transfer have received considerable attention in recent years due to its various applications in science, engineering and industries. Melt refining involves magnetic field applications to control excessive heat transfer rate.

Fluid flow involving rotating fluid have their application in some phenomena as we have mentioned earlier. When a large objects such as ships, automobiles and aircrafts move through fluids and the flow of the fluid around them is always turbulent. Turbulent also occurs when a fluid moves through fans, pumps, ducts and pipes.

1.4 Research Hypothesis

1.4.1 Null Hypothesis

The Hall current, other parameter like the Grashoff number, Hall parameter , Heat(radaition) Parameter, Prandtl number and rotational Parameter do not affect the flow variables.

1.5 Research Objective

1.5.1 General Objective

The main objective of this present study is to extend the theoretical model of MHD turbulent flow in a rotating system and to include the combined effects of magnetic fields, buoyancy force, thermal radiation, viscous and Ohmic heating.

1.5.2 Specific Objectives

The specific objective of this study are to determine :

i. Mathematical Formulation and Modeling of the problem .

ii. The velocity , temperature, skin friction, and rate of Heat Transfer profiles for electrically conducting fluid flowing past a vertical infinite plate in a rotating system subjected to strong magnetic field.

iii. The effect of Hall current, Hall Parameter, Radiation parameter, Eckert number, Grsshof number, Prandtl (and turbulence Prandtl) number, Rotation parameter and magnetic parameter on the flow variables.

CHAPTER TWO

Literature Review

2.1 Introduction

In this chapter, we give literature review, in which reassess some recent studies related to our research. In doing so, this section is prominently devoted to a brief literature review of the earlier investigation made on the MHD heat transfer and turbulence flow .

2.2 Literature Review

Studies of fluid flow problems involving electrically conducting fluids has over the recent years received much attentions. In the review of work related to this study, we consider some of the studies in MHD fluid flow involving convection heat transfer, mass transfer, and flows involving rotating fluids in the presence of magnetic field.

According to *Shercliff* , (1965), studies involving hydromagnetic fluid flow date as early as 1830 when Faraday experimented on mercury flowing in a glass tube between the poles of magnet and discovered that a voltage was induced across the magnetic field perpendicular to both the direction of the flow and the magnetic field. *Hertmann* and *Lazarus* (1937) were the first to discuss both experimentally and theoretically the hydromagnetic flow between two parallel plates and studies the flow of mercury in a channel under a transverse magnetic field. Alfve'n(1942) discovered the magneto fluid dynamics waves called Alfve'n waves.

Studies in MHD however become popular in the 1950's due to controlled fusion research and space technology. Sparrow and Cess (1961) discussed the effect of a magnetic field on free convection heat transfer. Chartuverdi (1996) studied the flow of an incompressible viscous fluid past an impulsively started horizontal plate and MHD flow past an infinite plate with a constant and variable suction. He also studied the finite difference of MHD of stokes problem for a vertical infinite plate in a dissipative heat generating fluid with Hall and Ion-slip current.

Model studies of the phenomena of MHD convection with respect to turbulent flow have been made by many authors. Burr *et al*. (2000) presented the analysis of hydromagnetic turbulent flow and heat transfer in a rectangular duct with strong sidewall jets. Ji *et al*. (1997). conducted a numerical investigation into turbulent pipe flow under the influence of an imposed transverse magnetic field. Kenjere and Hanjalic (2000). numerically studied the effects of Lorentz force in turbulence closure models. Kitamura and Hirata (1978). investigated the problem of turbulent heat and momentum transfer for electrically conducting fluid flowing in two-dimensional channel under the influence of an imposed transverse magnetic field. Numerical simulation of large-eddy in conductive flows at low magnetic Reynolds number was reported by (Knaepen and Moin, 2004). Kobayashi (2006). also presented large eddy simulation of hydromagnetic turbulent channel flows with local subgrid-scale model based on coherent structures. Dibaker (2006) discussed the behavior of turbulent flow with variable surface heat and mass transfer past a parallel plate.

Meanwhile, all conducting fluids with a temperature greater than absolute zero emit thermal radiation. Thermal radiation is the emission of electromagnetic waves. It represents a conversion of thermal energy into electromagnetic, energy (Cogley *et al* , 1968). Thermal energy results in kinetic energy in the random movements of molecules in the conducting fluid. Therefore, whenever the temperature of surrounding fluid is high, the radiation effects play a very important role in the flow process, (Makinde and Tshehla, 2014). This situation does occur in many engineering and industrial flow systems. In such cases one has to take into account the effects of radiation and free convection. Aboeldahab and Gendy (2002). studied the radiation effects on MHD free convective flow of a gas past a semi-infinite vertical plate with variable thermophysical properties for higher-temperature difference. Ishak (2011) investigated the thermal radiation effects on hydro-magnetic flow due to an exponentially stretching sheet. Mohammad *et al* (2013), studied the radiation effect on MHD free convection flow along vertical flat plate in presences of Joule heating and heat generation; and they observed that the radiation affect substantially the fluid velocity, temperature and skin frication. Although Emad (2005), Kafousias &

Daskakakis(1985) investigated the effect of both viscous and Joules dissipation on the magnetohydrodynamic convection flow. Palani and Srikanth (2009) investigated the free convection MHD flow with thermal radiation from an impulsively started vertical plate. They observed that velocity increases with a decreasing magnetic parameter Sivaiah *et al* (2010) discussed the radiation effects on MHD free convective flow over a vertical plate with heat and mass transfer.

Furthermore, hydromagnetic flow in a rotating systems as received significant attention of several researchers due to its applications in various technological situations which are governed by the action of Coriolis force. Oceanography, meteorology, atmospheric science and limnology all contain some important and essential features of rotating fluids. An order of magnitude analysis shows that in basic field equations the effects of Coriolis force is more significant as compared to that of inertia and viscous forces. It is worthy to note that Coriolis and magnetohydrodynamic forces are comparable in magnitude and Coriolis force induces secondary flow in the fluid. Seth *et al*. (2012) investigated unsteady hydromagnetic Couette flow of a viscous incompressible and electrically conducting fluid in a rotating system in the presence of uniform transverse magnetic field considering different aspects of the problem. More recently, Ghosh and Ghosh (2008) studied the hydromagnetic rotating flow of a dusty fluid near a pulsating plate with several limiting case studies. Seth and Ghosh (1986) investigated the unsteady Hydromagnetic flow in a rotating channel in the presence of inclined magnetic field. The fluid was viscous, incompressible and electrically conducting. The pressure gradient was applied periodically and magnetic field was applied uniformly. The study observed that flow reversals arose in the direction of pressure gradient. Kinyanjui *et al*. (2012) investigated the effect of Hall current on MHD turbulent flow over a vertical plate in a rotating system. From the literature review, it appears that the combined effects of magnetic fields, buoyancy force, thermal radiation, viscous and Ohmic heating on turbulent hydromagnetic boundary layer flow in a rotating system has not been reported.

The main objective of this present study is to extend the theoretical model of MHD turbulent flow in a rotating system by Kinyanjui *et al*. (2012), and to include the combined effects of magnetic fields, buoyancy force, thermal radiation, viscous and Ohmic heating. The mathematical formulation of the problem based on the Reynolds averaged Navier-Stokes (or RANS) method is established and Numerical scheme is employed to tackle the problem . Both computational and graphical results are presented and discussed quantitatively with respect to various parameters embedded in the system in section three.

CHAPTER THREE

Governing Equation and Mathematical Modelling

3.1 Introduction

It is known that the theory of *Magnetohyrodynamics* (MHD) is important for the description of plasma phenomena especially for the creation of fusion reactors. However, this theory-at least at present - has an ad hoc character, since the fundamental equations of it are accepted from different branches of disciplines of physics. Formally, MHD is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionized gases (plasmas) and strong electrolytes. The mutual interaction of a magnetic field, $\boldsymbol{B}$, and velocity field, $\boldsymbol{q}$, arises partially as a result of the laws of Faraday and Ampere, and partially because of the Lorentz force experienced by a current-carrying body.

In this chapter, we shall describe a mathematical model for the physical problem. However, many physical problems provide mathematical challenges and it is instructive to make some physically meaningful assumptions to reduce the problem into a solvable one. We shall first develop the general physical equations, reduce these equations by making assumptions, adopt the non-dimensional parameters so as to proceed in trimming dawn the complexity along the formulation and finally introduce turbulence. We shall then set up our final governing model equations for the problem.

3.2 Assumptions

In modeling a natural phenomena into mathematical relationship some assumption are made. We make use of the following in our model formulation.

i .Assume the ratio of the square of the fluid velocity V and that of the square of the velocity of light C are too small, i.e $\frac{V^2}{C^2} << 1$

ii . The Fluid flow is going to be restricted to a Turbulent domain.

iii. The fluid shall be considered incompressible hence the density of the fluid is assumed to remain constant.

iv.There is no chemical reaction talking place in the Fluid.

v. There is no externally applied electric current thus the Lorenz Force is given by $\hat{J} \times \hat{B}$ Thus force $\rho e\hat{E} = 0$ due to electric filed induced is negligible i.e. $\hat{E} = 0$

vi.The Displacement current D is negligible with respect to the electric current density $\hat{J}$.

vii. The plate is non-conducting.

$viii$. Liquid metals and ionized gases have permeability μ, so we write $\hat{B} = \mu_e \hat{H}$ in frame of reference.

ix. The Magnetic number is very large.

3.3 Basic Equation of Magnetho-Hydrodynamics in a Rotating Frame

This section considers the Governing equations of the *Magneto-Hydrodynamics* that are obtained from a combination of electromagnetic and hydrodynamics. The combination of the Navier-stokes equation of fluid dynamics and Maxwell's relations of the electromagnetism describes Megnetohydrodynamics flow in a rotating frame. As Maxwell's relation define the property of the electric and magnetic fields, the fundamental laws of electrodynamics are governed by including the Ohm's law(which relates the electric current to induced voltage) is to be modified by including induced current. The effect of rotation, electric and magnetic fields have to be taken to account to modify the basic laws of fluid dynamics comprising of conservation of mass, momentum and energy along with the thermal and caloric equations. Thermodynamics property of an electrically conducting fluid remains same as that of non-conducting fluid if we consider it as non-magnetic and neglect the phenomena like electrostriction. In Magnetohydrodynamics, when fluid velocity is small compared to the velocity of light, the displacement current being very small compared to conduction current, can be neglected. We mention below the basic equations of Magnetohydrodyanmics for the flow of a homogenous, isotopic, electrically conducting and constant density ρ , constant electrical conductivity σ and constant kinematic coefficient of viscosity ν in a rotating frame which rotates with uniform angular velocity Ω relative to an inertia frame.

The equation of continuity

This equation is based on the universal law of conservation of mass, which states that mass is neither created nor destroyed. Mathematical this expressed as:

$$\nabla.\vec{q} = 0 \tag{3.3.1}$$

The momentum equation (equation of motion)

This derived from the second Newton's law of motion which states that the sum of resultant forces is equal to rate of change of momentum of the flow. In tensor form the momentum equation is given by:

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q} + 2\Omega\hat{k} \times \vec{q} = -\frac{1}{\rho}\nabla p^{*} + \phi + \nu\nabla^{2}\vec{q} + \frac{1}{\rho}\vec{J} \times \vec{B} \tag{3.3.2}$$

The Energy equation

The energy equation is derived from the first law of thermodynamics which states that energy is neither created nor destroyed but can be transformed from one form to another. Equation for an incompressible fluid is given by as :

$$\rho C_p \frac{\partial T}{\partial t} + \left(\vec{q}.\nabla\right)T = \kappa\nabla^{2}T + \mu\Phi + \frac{1}{\sigma}J^{2} \tag{3.3.3}$$

The electromagnetic equation that help in model and formulate MHD problem are *Maxwell's* equation and *Ohm's law*. Generally they are well discussed in any book on electromagnetic theory. here they are adopted as presented by (Neff , 1981).

$$\begin{aligned} &\nabla \times \vec{B} = \mu_e \vec{J} \\ &\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &\nabla \times \vec{B} = 0 \\ &\nabla \cdot \vec{B} = 0 \end{aligned} \tag{3.3.4}$$

The constitutive field relation

$$\vec{B} = \mu_e \vec{H} \tag{3.3.5}$$

Ohm's law for a moving conductor

$$\vec{J} = \sigma[\vec{E} + \vec{u} \times \vec{B}] \tag{3.3.6}$$

and Ohm's law for a moving conductor taking Hall current into account

$$\vec{J} + \frac{\omega_e \tau_e}{B_0}(\vec{J} \times \vec{B}) = \sigma[\vec{E} + \vec{u} \times \vec{B}] \tag{3.3.7}$$

$\phi = F_g$ in (3.3.3) is the body force per unit mass including gravitational effects. In free conviction flow, the fluid motion is as result of buoyancy forces due to presence of fluid density gradient. We also need to take note that body forces act in the direction of the flow. Density gradient is caused by temperature gradient of the fluid; and the body force is agreed to be:

$$F_g = \rho g \tag{3.3.8}$$

The pressure gradient $\frac{dp}{dx_i}$ (this gradient will be zero at free stream due the fact seen at the boundary condition) in the x direction results from the change in elevation up the plate thus

$$\frac{dp}{dx} = -\rho_\infty g \tag{3.3.9}$$

And, the electromagnetic force may be written as:

$$\hat{F}_e = \rho_e \hat{E} + \hat{J} \times \hat{B} \tag{3.3.10}$$

In most flow problems the electrostatic force $\rho_e \hat{E}$ is negligibly small as compared to the electromagnetic force $\hat{J} \times \hat{B}$, since there is no externally applied electric current; hence

$$\hat{F}_e = \hat{J} \times \hat{B} \tag{3.3.11}$$

According to many text books the density difference $(\rho_\infty - \rho)$ may be expressed in terms of the volume coefficient of expansion β defined

$$\beta = \frac{1}{\mathbb{v}}\left[\frac{\partial \mathbb{v}}{\partial T}\right]_p = \frac{1}{\mathbb{v}}.\frac{\mathbb{v} - \mathbb{v}_\infty}{T - T_\infty} = \frac{\rho_\infty - \rho}{\rho(T - T_\infty)}$$

or

$$\beta\rho(T - T_\infty) = \rho_\infty - \rho \qquad (3.3.12)$$

Applying the vector cross product rule to simplify $\frac{\hat{J}\times\hat{B}}{\rho}$

$$\frac{\hat{J}\times\hat{B}}{\rho} = \frac{1}{\rho}\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ J_x & J_y & J_z \\ 0 & 0 & B_0 \end{pmatrix} = \frac{1}{\rho}\left[B_0 J_y \hat{i} - B_0 J_x\right] \qquad (3.3.13)$$

Hence the above simplification results a Laplace force to have component along x and y; $i.e.$ $(\hat{J}\times\hat{B})_x$ and $(\hat{J}\times\hat{B})_y$ respectively.

The Coriolis force $2\Omega\hat{k}\times\vec{q}$ can be simplifies as :

$$2\Omega\hat{k}\times\hat{q} = 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega_z \\ u & v & 0 \end{vmatrix} = 2\left[-v\Omega_z\hat{i} + u\Omega_z\hat{j}\right] \qquad (3.3.14)$$

Then Coriolis force has two components $-2v\Omega_z\hat{i}$ and $2u\Omega_z\hat{j}$ along x and y respectively.

And from equation (3.3.3) All the effect due to viscous stress in the energy equation are described by the dissipation function Φ which, after considerable algebra, can be shown to be equal to Versteeg (1995)

$$\Phi = \mu\left(\left(2\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2\right) + \left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 + \left(\frac{du}{dz} + \frac{dw}{dx}\right)^2 + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2\right) + \lambda(div.u)^2 \qquad (3.3.15)$$

The dissipation function is non negative since it only consist of squared terms and represents a source of internal energy due to deformation work on the fluid particle. This work is extracted from the mechanical activity which causes the motion and is converted into internal energy or heat.

Given the scenario discussed this far, the governing equation in the absences the chaotic and/or fluctuation outlined as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\\
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2v\Omega_z=\nu\nabla^2 u+\beta g(T-T_\infty)+\frac{J_y B_0}{\rho}\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2v\Omega_z=\nu\nabla^2 v+\frac{J_x B_0}{\rho}\\
&\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\alpha\nabla^2 T+\mu\Phi+\frac{1}{\sigma}J^2+\frac{\partial q_r}{\partial x_i}
\end{aligned} \tag{3.3.16}$$

In the above equation (3.3.16) The Laplace force components can be simplified so that it can be more convenient to work with as we continuing in the model formulation. The desired simplification cab be carried out by making use of the Ohm's law.

Cothran *et al* (2005) discussed in low collisionality plasma the structure of Ohm's law as modified by kinetic effects is of a special importance in understanding reconnection; moreover the generalized Ohm's law which accompanying the so-called Hall current effect and written as :

$$E+u\times B=\eta J+\frac{1}{ne}\left(J\times B-\nabla.P_e\right)+\frac{m_e}{ne^2}\frac{\partial J}{\partial t} \tag{3.3.17}$$

The ηJ term may be due to classical collisional resistivity or "turbulent resistivity" due to fluctuation. The Hall term $\frac{1}{ne}J\times B$ associated with differential flow of ions and electrons, becomes appreciable at the ion inertia scale $P_{ii}=\frac{C}{\omega_{pi}}$. The electron pressure tensor term is formally of the order of $\beta_e P_{ii}$ (where β_e is the ratio of electron pressure to magnetic pressure). The final term in (3.3.17), the electron inertia term is appreciable at the electron inertial scale C/ω_{p_i}. For the ideal MHD, $E + u \times B = 0$.

Moreau, (1990) has suggested the generalized Ohm's law including the effect of Hall current is written as

$$J + \frac{\omega_e \tau_e}{B_0}(J \times B) = \sigma(E + \mu_e q \times B + \frac{1}{e\eta_e}\nabla P_e) \quad (3.3.17)$$

for partially ionized fluid the electron pressure gradient may be neglected; In our case we consider $E = 0$ (3.4.17) reduce to

$$J = \sigma\mu_e(q \times B) - \frac{\omega_e \tau_e}{B_0}(J \times B) \qquad (3.3.18)$$

Note that μ_e is a constant, in some MHD textbooks, they represent $\mu_e B_0 = H_0$ while others use $\mu_e H_0 = B_0$. All this are constant representing the magnetic field strenght and does not make any different to the model problem.

Applying the cross product on equation (3.3.18) as:

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \sigma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & 0 \\ 0 & 0 & B_0 \end{vmatrix} - \frac{\omega_e \tau_e}{B_0} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ J_x & J_y & J_z \\ 0 & 0 & B_0 \end{vmatrix} \qquad (3.3.18)$$

and yields for J_x & J_y from the system and note that $B_0 = \mu_e H_0$

$$\begin{aligned} J_x &= \sigma B_0 v - \omega_e \tau_e(\omega_e \tau_e J_x - \sigma B_0 u) = \sigma B_0 v - \omega_e{}^2 \tau_e{}^2 J_x + \omega_e \tau_e \sigma B_0 u \\ J_x &+ \omega_e{}^2 \tau_e{}^2 J_x = \sigma B_0 v + \omega_e \tau_e \sigma B_0 u \\ J_x &= \frac{\sigma B_0 v + \omega_e \tau_e \sigma B_0 u}{(1 + \omega_e{}^2 \tau_e{}^2)} \\ J_x &= \frac{\sigma B_0 (v + mu)}{(1 + m^2)} \end{aligned} \qquad (3.3.19)$$

and similarly for J_y

$$J_y = m\left(\frac{\sigma B_0(v+mu}{(1+m^2)}\right) - \sigma B_0 u$$

$$= \frac{m\sigma B_0(v+mu)-(1+m^2)\sigma B_0 u}{1+m^2} \tag{3.3.20}$$

$$= \frac{-\sigma B_0(u-mv)}{1+m^2}$$

hence

$$\begin{pmatrix} J_x & J_y \end{pmatrix} = \begin{pmatrix} \dfrac{\sigma B_0(v+mu)}{(1+m^2)} & \dfrac{-\sigma B_0(u-mv)}{1+m^2} \end{pmatrix} \tag{3.3.20}$$

The Joule heating term $\frac{1}{\sigma}J^2$ also needs to be simplify using (3.3.20) as :

$$\frac{1}{\sigma}J^2 = \left(J_x{}^2 + J_y{}^2\right) = \left(\frac{\sigma B_0{}^2(v+mu)^2}{(1+m^2)^2} \quad + \frac{\sigma B_0{}^2(u-mv)^2}{(1+m^2)^2}\right)$$

$$= \frac{\sigma B_0{}^2\left((v+mu)^2+(u-mv)^2\right)}{(1+m^2)^2} \tag{3.3.21}$$

Now, we need make use of (3.3.20 - 21) so as to modify (3.3.16) as :

$$\frac{\partial u}{\partial t} + u\frac{du}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2v\Omega_z = v\nabla^2 u + \beta g(T - T_\infty) - \frac{\sigma B_0^2(u-mv)}{\rho(1+m^2)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2u\Omega_z = v\nabla^2 v - \frac{\sigma B_0^2(v+mu)}{\rho(1+m^2)}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}{}_z = \alpha\nabla^2 T + \frac{\mu}{\rho C_p}\phi - \frac{1}{\rho C_p}\frac{dq_r}{dz_i} - \frac{\sigma B_0^2(v+mu)^2+(u-mv)^2}{\rho C_p(1+m^2)^2} \tag{3.3.21}$$

3.4 The Geometry and Physics of the Problem in the Present Study

The current problem deals with a turbulent MHD flow and heat transfer past an infinite vertical plate in rotating system. A turbulent flow of an incompressible electrically conducting fluid past infinite plate which is subjected to a magnetic field applied in the normal direction is considered. Moreover the flow is in a rotating system. By faraday's law of electromagnetic induction, we have that a conductor moving in a magnetic field has an electric current induced in it. An ionized fluid flows through a magnetic field, then current is induced in the fluid and this per unit area is the current density, J . The Hall effect usually occurs when the Lorentz force acts on a charge current in a conductor in the presence of perpendicular magnetic field. The introduced heat in the flow field results a heat dissipation. Heat transfer is a form of energy transfer. According Kafousias *et.al* (1985) neglecting the terms representing the Joule heating and viscous dissipation from the energy equation may lead to erroneous(or non-satisfactory) solution of the problem hence in the present problem we have considered it and the energy equation will accompany both effects.

The effect of the turbulence brought us the additional term $\tau^R = -\rho(\overline{u_i ' u_j '})$ as we formulate the fluid flow problem. The τ^R is the so-called Reynolds-stress tensor Joel (2014). And its deemed τ^R to be approximate with an appropriate scheme so that we can use it in the Navier-Stokes equation and the model can be implementable.

Consider the hydromagnetic turbulent flow of an incompressible electrically conducting fluid past a moving vertical plate in a rotating system with thermal radiation in the presence of a uniform transverse magnetic field of strength B_0. Choose the coordinate system in such a way that x-axis is along the plate in upward direction, z - axis normal to plane of the plate in the fluid and y - axis perpendicular to x z- plane. The uniform transverse magnetic field B_0 is applied in a direction which is parallel to z - axis. It is assumed that no applied or polarization voltages exist consequently electric field is zero. The magnetic Reynolds number is very small and the induced magnetic field produced by fluid motion is negligible in comparison to applied one. The plate moves along x-axis with velocity U_0 and both the plate and the fluid rotate in unison with uniform angular velocity Ω about z – axis as shown in figure 3.1.

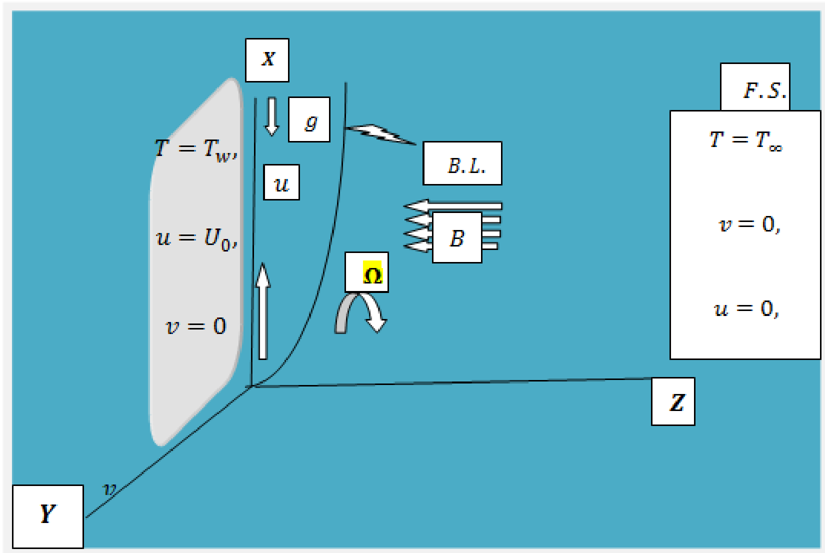

Figure 3. 1: Geometry of the problem

The vertical plate is kept at a higher temperature than the fluid i.e. $T_w > T_\infty$. The fluid flow being studied is free convectional and takes place along the x-axis is under the action of transverse variable magnetic field. The boundary layer thickness is along the z-axis hence the velocity components will changes along it.

The flow is taken place in two dimensions in general; as a result it's deemed the flow variables to be depend on two coordinate space and the geometry of the flow ought depicted on 3D coordinate space. The velocity u and v are depends on x, z, t .The velocity v does not change along y. For the magnetic field, for it to have an effect on the flow it must have been applied in the direction perpendicular to the flow region.

The velocity vector is $\vec{q} = (u, v, 0)$, where $u = u(x, z, t)$ and $v = v(x, z, t)$ in fact based on the theoretical concept, before the flow became turbulent the velocity w along z was there. the magnetic field $B = (0, 0, B_0)$. The condition over the surface of the plate which is at, $z = 0$, the plate is moving and as same time rotating hence we set $u = U_0$ $v = 0$ as the boundary condition, and the temperature of the plate maintained constant at fixed temperature thus, $T_w = T$ Also far away from the plate i.e. $z \to \infty$, we set the boundary condition as $u = v = 0$ and $T = T_\infty$. Due to gravitation which occurred as a result of the geometry of the present problem, the effect of buoyancy force must taken to account in the formulation. Its noteworthy that in this study, one of our major task is to determine how the flow variables would behave, knowing them in the boundary regions.

3.5 Turbulence Modelling

All flows encountered in engineering applications, from simple ones to complex three dimensional ones, become unstable above a certain Reynolds number. In turbulent flow the hydrodynamic and thermodynamic characteristics undergo chaotic fluctuation and hence, vary highly irregularly in space and time.(from the smallest turbulent eddies characterized by Kolmogorov micro-scales, to the flow features comparable with the size of the geometry). A turbulent(outwardly disordered) regime of fluid motion arises as a laminar flow loses its stability when the dimensionless Reynolds number $R = UL/\upsilon$, exceeds some critical value Re_{cr} or turbulence arises either from the growth with small perturbation in a laminar flow or from the convective instability of motion. Re is the most general characteristics of a turbulized fluid.

According to Joel (2014), there are several possible approaches for the numerical simulation of turbulent flows. The first and most intuitive one, is by directly numerically solving the governing equations over the whole range of the turbulent scales (temporal and spatial). This deterministic approach is referred as Direct Numerical Simulation (DNS). In DNS, a fine enough mesh and small enough time-step size must be used so that all of the turbulent scales are resolved. Although some simple problems have been solved using DNS, it is not possible to tackle industrial problems due to that prohibitive computer cost imposed by the mesh and time-step requirements. Hence, this approach is mainly used for benchmarking, research and academic applications.

Another approach used to model turbulent model flows is Large Eddy Simulation (LES). Here, large scale turbulent structures are directly simulated whereas the small turbulent scales are filtered out and modeled by turbulence models called subgrid scale models. According to turbulent theory, small scale eddies are more uniform and have more or less common characteristics; therefore, modeling small scale turbulence appears more appropriate, rather than resolving it. the computational cost of LES is less than that of DNS.

Today's workhorse for industrial and research turbulence modeling application is the Reynolds Averaged Navier-Stokes (RANS) approach. In this approach, the RANS equations are derived by decomposing the flow variables of the governing equation into time-mean (obtained over an appropriate time interval) and fluctuating part, and then time averaging the entire equations. Time averaging the governing equations gives rise to new terms, these new quantities must be related to the mean flow variables through turbulence models. This process introduces further assumption. The turbulences models are primarily developed based on experiment data obtained from relatively simple flows under controlled conditions. This in turn limits the range of applicability of the turbulence models. That is, no single RANS turbulence model is capable of providing accurate solution over a wide range of flow condition and geometries. Hereafter, we limit our discussion to Reynolds averaging.

3.5.1 Reynolds Averaging

In turbulent flow, the transport phenomena variables $(i.e., u, v, w, T, p, etc)$ always vary with time. The instantaneous velocity value for a general flow variable say velocity u for a turbulent flow of moving fluid, provided for any location (x, y, z) can be expressed as summation of its *Mean* and its *Fluctuation* due to the small perturbation:

$u = \bar{u}(x, y, z, t) + u'(x, y, z, t)$ where

$$\bar{u}(x, y, z, t) = \frac{1}{\Delta t}\left(\int_{t}^{t+\Delta t} u(x, y, z, t)dt\right) \tag{3.5.1}$$

is the *Time-averaged* velocity at point (x, y, z, t). In the turbulence flow the mean fluctuation defined as: $i.e$

$$\bar{u}' = \frac{1}{\Delta t}\left(\int_{t}^{t+\Delta t} u'(x, y, z, t)dt\right) = 0 \tag{3.5.2}$$

The time interval for the *Time-averaged,* Δt, must be very long compared with the duration of fluctuation. The mean value of the fluctuation must be zero,

Similarly, the velocity components in the y- and z- direction can be expressed as:

$$v = \bar{v}(x,y,z,t) + v'(x,y,z,t)$$
$$w = \bar{w}(x,y,z,t) + w'(x,y,z,t)$$
$$T = \bar{T}(x,y,z,t) + T'(x,y,z,t)$$

and apply the time-averaging on time dependent momentum and energy balance equations under laminar flow scenario such that the time averaging of fluctuating part (with prime symbol) is zero. The Reynolds's averaging rules shall be used to transform equations governing laminar flow to turbulent flow. Time averaging of transport phenomena equations should provide the net effect of the turbulent perturbation. Following Scott (2004) the term involving

$$\frac{\partial \bar{u}'}{\partial t} = \frac{\partial \bar{v}'}{\partial t} = \frac{\partial \bar{T}'}{\partial t} = 0 \qquad (3.5.3)$$

3.5.1 Time Averaged Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial(\bar{u} + u')}{\partial x} + \frac{\partial(\bar{v} + v')}{\partial y} + \frac{d(\bar{w} + w')}{\partial z} = 0$$

Integrating over period $0 \to \Delta t$

$$\frac{\partial(\bar{u})}{\partial x} + \frac{d(\bar{v})}{\partial y} + \frac{d(\bar{w})}{\partial z} = 0 \qquad (3.5.4)$$

3.5.2 Time-Averaged Momentum Equation

x-direction momentum equation is :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{dy} + w\frac{\partial u}{\partial z} - 2v\Omega_z = v\nabla^2 u + \beta g(T - T_\infty) - \frac{\sigma B_0^2 (u - mv)}{\rho(1+m^2)} \qquad (3.5.5)$$

Multiplying continuity equation by u as:

$$u \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$ and adding to (3.5.5) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - 2v\Omega_z = \nu\nabla^2 u + \beta g(T - T_\infty) - \frac{\sigma B_0^2 (u - mv)}{\rho(1+m^2)}$$

Averaging over period $0 \to \Delta t$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \overline{u^2}}{\partial x} + \frac{\partial (\overline{uv})}{\partial y} + \frac{\partial (\overline{uw})}{\partial z} - 2\bar{v}\Omega_z = \upsilon\nabla^2 u + \beta g(\bar{T} - T_\infty) - \frac{\sigma B_0^2 (\bar{u} - m\bar{v})}{\rho(1+m^2)}$$

(3.5.6)

Then, Consequently from (3.5.6)

$$\frac{\partial \overline{u^2}}{\partial x} + \frac{\partial (\overline{uv})}{\partial y} + \frac{\partial (\overline{uw})}{\partial z} - 2\bar{v}\Omega_z = \upsilon\nabla^2 u + \beta g(\bar{T} - T_\infty) - \frac{\sigma B_0^2 (\bar{u} - m\bar{v})}{\rho(1+m^2)} \qquad (3.5.7)$$

Substituting for $u = \bar{u} + u'(t), v = \bar{v} + v'(t)$, $w = \bar{w} + w'(t)$ and $T = \bar{T} + T'(t)$ in (3.5.7).

$$\frac{\partial \overline{(\bar{u} + u')^2}}{\partial x} + \frac{\partial \overline{((\bar{u} + u')(\bar{v} + v'))}}{\partial y} + \frac{\partial \overline{((\bar{u} + u')(\bar{w} + w'))}}{\partial z} - 2\left(\overline{\bar{v} + v'}\right)\Omega_z = \nu\nabla^2 (\overline{\bar{u} + u'}) +$$
$$\beta g((\overline{\bar{T} + T'}) - T_\infty) - \frac{\sigma B_0^2 ((\overline{\bar{u} + u'}) - m(\overline{\bar{v} + v'}))}{\rho(1+m^2)}$$

then

$$\left[\frac{\partial \overline{u^2}}{\partial x} + \frac{\partial \left(u'\right)^2}{\partial x} \right] + \left[\frac{\partial (\overline{uv})}{\partial y} + \frac{\partial (\overline{u'v'})}{\partial y} \right] + \left[\frac{\partial (\overline{uw})}{\partial z} + \frac{d(\overline{u'w'})}{dz} \right] - 2\bar{v}\Omega_z = \nu\nabla^2 \bar{u} + \beta g(\bar{T} - T_\infty) - \frac{\sigma B_0^2 (\bar{u} - m\bar{v})}{\rho(1+m^2)}$$

from chain law:

$$\frac{\partial \overline{u^2}}{\partial x} = 2\bar{u}\frac{\partial \bar{u}}{\partial x}, \quad \frac{\partial (\overline{uv})}{\partial y} = \bar{u}\frac{\partial (\bar{v})}{\partial y} + \bar{v}\frac{\partial (\bar{u})}{\partial y} \text{ and } \frac{\partial (\overline{uw})}{\partial y} = \bar{u}\frac{\partial (\bar{w})}{\partial y} + \bar{w}\frac{\partial (\bar{u})}{\partial y} \quad \text{then}$$

$$\left[2\bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\partial \left(u'\right)^2}{\partial x} \right] + \left[\bar{u}\frac{\partial (\bar{v})}{\partial y} + \bar{v}\frac{\partial (\bar{u})}{\partial y} + \frac{\partial (\overline{u'v'})}{\partial y} \right] + \left[\bar{u}\frac{\partial (\bar{w})}{\partial y} + \bar{w}\frac{\partial (\bar{u})}{\partial y} + \frac{d(\overline{u'w'})}{dz} \right] - 2\bar{v}\Omega_z =$$
$$\nu\nabla^2 \bar{u} + \beta g(\bar{T} - T_\infty) - \frac{\sigma B_0^2 (\bar{u} - m\bar{v})}{\rho(1+m^2)}$$

(3.5.8)

Multiplying the time-averaged (3.5.4) equation by $\bar{u}$ as:

$$\bar{u}\left[\frac{d(\bar{u})}{dx}+\frac{d(\bar{v})}{dy}+\frac{d(\bar{w})}{dz}=0\right] \tag{3.8.9}$$

Subtracting equation (3.5.8) with (3.8.9) equation, it yields :

$$\left[\bar{u}\frac{\partial\bar{u}}{\partial x}+\frac{\partial(u')^2}{\partial x}\right]+\left[\bar{v}\frac{\partial(\bar{u})}{\partial y}+\frac{\partial(\overline{u'v'})}{\partial y}\right]+\left[\bar{w}\frac{\partial(\bar{u})}{\partial y}+\frac{d(\overline{u'w'})}{dz}\right]-2\bar{v}\Omega_z=\nu\nabla^2\bar{u}+\beta g(\bar{T}-T_\infty)-\frac{\sigma B_0^2(\bar{u}-m\bar{v})}{\rho(1+m^2)}$$

Then:

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial(\bar{u})}{\partial y}+\bar{w}\frac{\partial(\bar{u})}{\partial y}-2\bar{v}\Omega_z=\nu\nabla^2\bar{u}+\beta g(\bar{T}-T_\infty)-\frac{\sigma B_0^2(\bar{u}-m\bar{v})}{\rho(1+m^2)}-\frac{\partial(u')^2}{\partial x}-\frac{\partial(\overline{u'v'})}{\partial y}-\frac{d(\overline{u'w'})}{dz} \tag{3.5.10}$$

y-direction momentum equation is :

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2u\Omega_z=\nu\nabla^2 v-\frac{\sigma B_0^2(v+mu)}{\rho(1+m^2)} \tag{3.5.11}$$

with similar procedure as we did for the x direction momentum :

$$\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{w}\frac{\partial\bar{v}}{\partial z}+2\bar{u}\Omega_z=\upsilon\nabla^2\bar{v}-\frac{\sigma B_0{}^2(\bar{v}+m\bar{u})}{\rho(1+m^2)}-\frac{\partial(v')^2}{\partial x}-\frac{\partial(\overline{u'v'})}{\partial y}-\frac{\partial(\overline{v'w'})}{\partial z} \tag{3.5.12}$$

3.5.3 Time-Averaged Energy Equation

we need to recall the energy equation stated under equation (3.3.21)

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{dT}{dy}+w\frac{\partial T}{\partial z}{}_z=\alpha\nabla^2 T+\frac{\mu}{\rho C_p}\phi-\frac{1}{\rho C_p}\frac{dq_r}{dz_i}-\frac{\sigma B_0^2(v+mu)^2+(u-mv)^2}{\rho C_p(1+m^2)^2}$$

Then

$$\bar{u}\frac{\partial T}{\partial x}+\bar{v}\frac{\partial\bar{T}}{\partial y}+\bar{w}\frac{\partial\bar{T}}{\partial z}=\alpha\nabla^2 T-\frac{\partial(\overline{u'T'})}{\partial x}-\frac{\partial(\overline{v'T'})}{\partial y}-\frac{\partial(\overline{w'T'})}{\partial z}+\frac{\mu}{\rho C_p}\phi-\frac{1}{\rho C_p}\frac{\partial q_r}{\partial z_i}+$$
$$\frac{\sigma B_0^2(v+mu)^2+(u-mv)^2}{\rho C_p(1+m^2)^2} \tag{3.5.13}$$

$\alpha = \dfrac{\kappa}{\rho C_p}$

It is possible that from (3.5.7) and (3.5.12) to obtain the

Molecular shear stress :

$\mu \dfrac{\partial \bar{u}}{\partial z}$ & $\mu \dfrac{\partial \bar{v}}{\partial z}$

The eddy shear stress : $-\rho \bar{u}'\bar{w}' \equiv \rho \varepsilon_m \dfrac{d\bar{u}}{dz}$ ε_m is momentum eddy diffusivity or turbulent eddy viscosity. ε_m is a characteristics flow field and not a physical property of the fluid.

Note that velocity fluctuations $u'w'$ are assumed to be induced by $\dfrac{d\bar{u}}{dz}$ and

From (3.5.12) it possible to obtain

Molecular heat flux

The eddy heat flux : $-\rho \bar{T}'\bar{w}' \equiv \rho \varepsilon_H \dfrac{d\bar{T}}{dz}$ ε_H is eddy thermal diffusivity and $\varepsilon_H = \dfrac{\varepsilon_m}{P_{rt}}$

where $P_{rt} \equiv$ turblent prandtl number $= \dfrac{\varepsilon_m}{\varepsilon_H}$

The Reynolds averaged approach for a turbulence modeling requires to capture and treat for the Reynolds stress; to do so this we adopted the *Boussinsq approximation* (Boussinesq,1903)

$$\tau_t = -\rho \bar{u}'\bar{v}' = \varepsilon_m \frac{\partial \bar{u}}{\partial y} \qquad (3.5.14)$$

ε_m is not a property of the fluid like μ but depends on the mean velocity u. we use the semi empirical methods to resolve the Reynolds shear stress terms in the model equation; and that lead us to the study and use of the *Prandtly mixing length hypothesis* which for a long time has been an important tool in the analysis of turbulent boundary layers.

The Reynold shear $-\rho\bar{u}'\bar{v}'$ represents the flux of x-mmomentum in the direction of y. Prandtly assumed that this momentum was transported by eddies which moved in the y-direction over a distance l without interaction(*i.e.* momentum is assumed to be conserved over distance l) and then mixed with existing fluid at the new location (McComb,1990)

Prandtl, from His experiment deduce that :

$$\rho\bar{u}'\bar{v}' = -\rho l^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{3.5.15}$$

At this stage further assumption has made *i.e.* $l = ny$ where n is the *Von Karman* constant, $n = 0.4$ (McComb, 1990).

And, thus

$$\rho\bar{u}'\bar{v}' = -\rho n^2 y^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{3.5.16}$$

Both Boussinesq approximation and Prandtl mixing length hypothesis Bejan, (1995) are also applied to resolve the time-averaging of product of fluctuating part i.e.

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{v}}{\partial t} = \frac{\partial \bar{T}}{\partial t} = 0,\ \ \overline{u'w'} = -n^2 z^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2,\ \ \overline{v'w'} = -n^2 z^2 \left(\frac{\partial \bar{v}}{\partial z}\right)^2, \\
&\overline{T'w'} = -\varepsilon_H \frac{\partial \bar{T}}{\partial z},\ \ \varepsilon_H = \frac{\varepsilon_m}{\Pr_t}, \\
&\varepsilon_m = n^2 z^2 \frac{\partial \bar{u}}{\partial z},\ \ \frac{\partial^2}{\partial x^2} << \frac{\partial^2}{\partial z^2}, \frac{\partial \bar{u}'^2}{\partial x} << \frac{\partial \overline{u'w'}}{\partial z},\ \ \frac{\partial \bar{v}'^2}{\partial x} << \frac{\partial \overline{v'w'}}{\partial z}, \\
&\frac{\partial \overline{u'T'}}{\partial x} << \frac{\partial \overline{T'w'}}{\partial z},
\end{aligned} \tag{3.5.17}$$

The terms ε_M and ε_H are known as the momentum eddy diffusivity and the thermal eddy diffusivity.

Thus, the approximation of the terms due to the turbulence effect for our model shall be :

$$-\rho\overline{u'w'} \approx \rho n^2 z^2 \left(\frac{d\bar{u}}{dz}\right)^2 \approx \rho\varepsilon_m \frac{d\bar{u}}{dz}$$

$$-\rho\overline{w'v'} \approx \rho n^2 z^2 \left(\frac{d\bar{v}}{dz}\right)^2 \approx \rho\varepsilon_m \frac{d\bar{v}}{dz}$$

$$-\overline{w'T'} \approx \varepsilon_m \frac{d\bar{T}}{dz} \varepsilon_H = \frac{\varepsilon_m}{P_{r_t}} = \frac{n^2 z^2}{P_{r_t}}\left(\frac{d\bar{u}}{dz}\right) \tag{3.5.18}$$

$$\varepsilon_m = n^2 z^2 \frac{d\bar{u}}{dz} = n^2 z^2 \frac{d\bar{v}}{dz}$$

$$-\overline{w'T'} \approx \frac{n^2 z^2}{P_{r_t}} \frac{d\bar{T}}{dz}$$

Taking into consideration the assumptions made above together with the application of Reynolds averaged Navier-Stokes (or RANS) methods, the governing equations for hydromagnetic turbulent boundary layer flow of a viscous, incompressible, electrically conducting fluid with Hall effects and thermal radiations, under Boussinesq approximation, in a rotating frame of reference are given by [Kitamura,1978 Makinde, 2014 Kinyanjui *et al* 2012 and Bejan, 1995] as :

$$-2\Omega\bar{v} = \upsilon\frac{d^2\bar{u}}{dz^2} + \frac{d}{dz}\left(n^2 z^2 \left(\frac{d\bar{u}}{dz}\right)^2\right) + g\beta(\bar{T} - T_\infty) - \frac{\sigma B_0^2}{\rho}\left(\frac{\bar{u} - m\bar{v}}{1+m^2}\right), \tag{3.5.19}$$

$$2\Omega\bar{u} = \upsilon\frac{d^2\bar{v}}{dz^2} + \frac{d}{dz}\left(n^2 z^2 \left(\frac{d\bar{v}}{dz}\right)^2\right) - \frac{\sigma B_0^2}{\rho}\left(\frac{\bar{v} + m\bar{u}}{1+m^2}\right), \tag{3.5.20}$$

$$\alpha\frac{d^2\bar{T}}{dz^2} + \frac{d}{dz}\left(\frac{n^2 z^2}{\mathrm{Pr}_t}\frac{d\bar{u}}{dz}\frac{d\bar{T}}{dz}\right) - \frac{1}{\rho c_p}\frac{dq_r}{dz} + \frac{\upsilon}{c_p}\left[\left(\frac{d\bar{u}}{dz}\right)^2 + \left(\frac{d\bar{v}}{dz}\right)^2\right] + \frac{\sigma B_0^2}{\rho c_p}\left[\frac{(\bar{u} - m\bar{v})^2 + (\bar{v} + m\bar{u})^2}{(1+m^2)^2}\right] = 0 \tag{3.5.21}$$

Together with the Boundary Condition

$$\bar{u} = U_0,\ \bar{v} = 0,\ \bar{T} = T_w, \quad \text{at} \ \ z = 0, \tag{3.5.22}$$

$$\bar{u} = 0,\ \bar{v} = 0,\ \bar{T} = T_\infty, \quad \text{as} \quad z \to \infty, \tag{3.5.23}$$

where $\bar{u}$ and $\bar{v}$ are the mean velocity components, $\bar{T}$ is the mean temperature, T_w is the plate surface temperature, T_∞ is the free stream temperature, g is acceleration due to gravity, n is the von Karman constant (=0.4), α is the thermal diffusivity coefficient, ρ is the fluid density,σ is the fluid electrical conductivity, m is Hall parameter, υ is the kinematic viscosity, t is the time, c_p is the specific heat at constant pressure, β is the thermal expansion coefficient and Pr_t is the turbulence Prandtl number. It is assumed that the fluid medium is optically thin with relatively low density and following Cogley *et al*. (1968), the radiative heat flux q_r is given as

$$\frac{\partial q_r}{\partial z} = -4\varepsilon^2(\bar{T} - T_\infty), \tag{3.5.24}$$

where

$$\varepsilon^2 = \int_0^\infty \mathrm{K}_\lambda \frac{\partial e_{\lambda_\upsilon}}{\partial x} d\lambda \tag{3.5.25}$$

where $\varepsilon \ll 1$ is the radiation absorption coefficient, K_λ is the absorbtion coefficient, λ is the wave length and $e_{\lambda_p}$is the plank's function. .Again , it is important to note that in order to obtain the turbulent flow model equations (3.5.19)- (3.5.21), we have decomposed the flow variables u, v and T into time-mean and fluctuating part.

The Turbulent Prandtl Number - $p_{r_t}$is highly important for predication of heat transfer coefficient for fluid. The classical approach for obtaining the transport mechanism for the heat transfer problem follows the laminar approach, namely, the momentum and thermal transport mechanisms are related by a factor, the Prandtl number, hence combining the molecular and eddy viscosities one obtain the Boussinesq relation for shear stress $\tau / \rho = (\upsilon + \varepsilon_m / \mathrm{Pr}_t)\frac{du}{dz}$ and the analogues relation for heat flux $q / \rho C_p = (\alpha + \varepsilon_m / \mathrm{Pr}_t)\frac{dT}{dz}$ thus if one knows the eddy diffusivity and turbulent prandtl number, p_{r_t},the heat transfer problem can be solved, Simpson *et al* (1970).

Hence we need to get an appropriate model for Pr_t. various researcher suggested a model for it as Tyldesley and Silver (1968) whereby they consider high turbulence intensity:

$$\mathrm{Pr}_t = \frac{2+9\,\mathrm{Pr}_t}{3+9\,\mathrm{Pr}_t} \tag{3.5.26}$$

and as it is presented by Marchello and Toor (1963) for high turbulence intensity, the turbulent Prandtl number is given as.

$$\mathrm{Pr}_t = \sqrt{\mathrm{P}_r} \tag{3.5.27}$$

this will be the one which we are going to adopt.

3.6 Non-Dimensionalization

Non-dimensional analysis has got a vital importance in the study & model formulation for problems arises from areas of Hydrodynamics and MHD in general. As the word suggest it makes equations and the solution of the problem independent of units. We have already formulate our problem mathematical and came up with the following Dimensional model equations.

3.6.1Governing Equations in Dimensional Form

$$-2\Omega v U_0 = \upsilon \frac{U_0{}^2}{\upsilon^2}\frac{d^2u}{dz^2} + \frac{d}{dz}\left(n^2z^2\left(\frac{du}{dz}\right)^2\right) + g\beta(T-T_\infty) - \frac{\sigma B_o{}^2}{\rho}\left(\frac{u-mv}{1+m^2}\right) \tag{3.6.1}$$

$$2\Omega u = \upsilon \frac{d^2v}{dz^2} + \frac{d}{dz}\left(n^2z^2\left(\frac{dv}{dz}\right)^2\right) - \frac{\sigma B_o{}^2}{\rho}\left(\frac{v-mu}{1+m^2}\right) \tag{3.6.2}$$

$$\alpha\frac{d^2T}{dz^2} + \frac{d}{dz}\left(\frac{n^2z^2}{\mathrm{Pr}_t}\frac{du}{dz}\frac{dT}{dz}\right) + 4\varepsilon^2(T-T_\infty) + \frac{\mu}{\rho C_p}\left(\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right) + \frac{\sigma B_o{}^2}{\rho C_p}\left(\frac{(u-mv)^2+(v+mu)^2}{(1+m^2)^2}\right) \tag{3.6.3}$$

The boundary conditions given by:

$$u = U_0, v = 0, T = T_0, \text{ at } z = 0 \quad (3.6.4)$$

$$u = 0, v = 0, T = T_{\infty,} \text{ as } z \to \infty \quad (3.6.31)$$

In equations (3.5.28) -(3.5.30) we dropped the bars above the flow variables for to make easy the process of non dimensinalizing the model.

3.6.2 Non Dimensional Parameters and Their Physical Significance

Dimensional groups are useful in our present investigation since;

i. The analysis of these dimensional groups helps in experimental investigation of reducing the number of variables in the problem. The result of the analysis is to replace an unknown relation between variables by a relationship between in the number of variables greatly reduces the labor of experimental investigation.

ii. Dimensional presentation of experimental data is independent of the units employed and should, be internationally intelligible and convenient to use.

3.6.1Common Non–Dimensional Numbers and Parameters in MHD

Equations (3.6.28) -(3.6.30) can be simplified further by employing common non–dimensional numbers and parameters. In this section we describe a few of these numbers that will be used in the current problem. We shall denote velocity, pressure, length, time and magnetic field by the letters U, P, L, t and B respectively.

3.6.1.1 Prandtl Number

This number describes the ratio of momentum diffusivity to heat diffusivity. A high Prandtl number indicates that heat diffuses very slowly relative to momentum while a low Prandtl number indicates that heat diffuses very fast relative to momentum. A Prandtl number of about 1 implies that heat and momentum are diffused within the material at almost the same rate. The Prandtl number is given by:

$$Pr = \frac{\mu C_p}{k} \quad (3.6.4)$$

3.6.1.2 Grashof Number

The Grashof number is a dimensionless number in fluid dynamic and heat transfer which approximate the ratio of buoyancy forces to viscous forces. It frequently arises in the study of situations involving natural convection. Therefore Grashof number can be thought of as Reynolds number with the velocity of natural convection replacing the velocity in Reynolds number's formula. It is given by:

$$Gr = \frac{\upsilon g \beta \left(T_w - T_\infty\right)}{U_0^{\ 3}} \tag{3.6.5}$$

3.6.1.3 Eckert Number

The Eckert number provides a measure of the kinetic energy of the flow relative to the enthalpy difference. It is a dimensionless quantity useful in determining the relative importance in a heat transfer situation of the kinetic energy of a flow. It is given by:

$$Ec = \frac{U_0^{\ 2}}{C_p \left(T_w - T_\infty\right)} \tag{3.6.6}$$

3.6.3.4 Magnetic Parameter

This is the ratio of the magnetic force to the viscous force. The magnetic parameter is given by:

$$M^2 = \frac{\sigma \mu_e^{\ 2} H_o^{\ 2} \upsilon}{\rho U_0^{\ 2}} \tag{3.6.7}$$

3.6.3.5 Rotational Parameter

The rotational parameter is the ratio of angular kinetic energy to translational kinetic energy and is given by:

$$R = \frac{\Omega \upsilon}{U_0^{\ 2}} \tag{3.6.8}$$

3.6.3.6 Radaition Parameter

$$Nr = \frac{4\varepsilon^2 \upsilon}{\rho c_p U_0^2}$$

(3.6.9)

To allow for independence of units and scales, dimensionless groups are employed. We define the following non–dimensional variables for the MHD problem:

$$\eta = \frac{zU_0}{\upsilon}, U = \frac{\bar{u}}{U_0}, V = \frac{\bar{v}}{U_0}, \theta = \frac{\bar{T} - T_\infty}{T_w - T_\infty}, M = \frac{\sigma B_0^2 \upsilon}{\rho U_0^2}, m = w_e \tau_e,$$

$$R = \frac{\Omega \upsilon}{U_0^2}, \Pr = \frac{\upsilon}{\alpha}, Ec = \frac{U_0^2}{c_p (T_w - T_\infty)}, Gr = \frac{g\beta\upsilon(T_w - T_\infty)}{U_0^3}, Nr = \frac{4\varepsilon^2 \upsilon}{\rho c_p U_0^2},$$

(3.6.10)

We, now carry out the non dimensinaliztion process using set of equation of (3.6.10)

It's clear that

$v = VU_0$, $z = \frac{\eta \upsilon}{U_0}$ and $\frac{d^2 u}{dz^2} = \frac{d}{d\eta}\left(\frac{du}{dz}\right)\frac{d\eta}{dz}$ using chain rule.

Equation (3.6.1) becomes:

$$-2\Omega v U_0 = \upsilon \frac{U_0^{\ 2}}{\upsilon^2}\frac{d^2 U}{dz^2} + \frac{U_0^{\ 5}}{\upsilon^3}\frac{\upsilon^2}{U_0^{\ 2}}\frac{d}{dz}\left(n^2\eta^2\left(\frac{dU}{d\eta}\right)^2\right) + g\beta(T - T_\infty) - \frac{\sigma B_o^{\ 2}}{\rho}U_0\left(\frac{U - mV}{1 + m^2}\right)$$

(3.6.11)

dividing (3.6.11) through by $\upsilon / U_0^{\ 3}$

$$\frac{-2\Omega\upsilon}{U_0^{\ 2}}V = \frac{d^2 U}{dz^2} + \frac{d}{dz}\left(n^2\eta^2\left(\frac{dU}{d\eta}\right)^2\right) + \frac{g\beta(T - T_\infty)\upsilon}{U_0^{\ 3}} - \frac{\sigma\upsilon B_o^{\ 2}}{\rho U_0}\left(\frac{U - mV}{1 + m^2}\right) \qquad (3.6.12)$$

upon inserting the appropriate parameters given above in (3.6.12) the last equation can be written as :

$$-2RV = \frac{d^2 U}{d\eta^2} + 2n^2\eta\left(\frac{dU}{d\eta}\right)^2 + 2n^2\eta^2\frac{dU}{d\eta}\frac{d^2 U}{d\eta^2} + Gr\theta - M\left(\frac{U - mV}{1 + m^2}\right) \qquad (3.6.13)$$

$$\frac{d^2U}{d\eta^2}=M\left(\frac{U-mV}{1+m^2}\right)-2RV-Gr\theta-2n^2\eta\left(\frac{dU}{d\eta}\right)^2-2n^2\eta^2\frac{dU}{d\eta}\frac{d^2U}{d\eta^2} \tag{3.6.14}$$

Similarily Equation (3.6.2)

$$2RU=\frac{d^2V}{d\eta^2}+2n^2\eta\left(\frac{dV}{d\eta}\right)^2+2n^2\eta^2\frac{dV}{d\eta}\frac{d^2V}{d\eta^2}-M\left(\frac{U-mU}{1+m^2}\right) \tag{3.6.15}$$

$$\frac{d^2V}{d\eta^2}=M\left(\frac{V-mU}{1+m^2}\right)+2RV-2n^2\eta\left(\frac{dV}{d\eta}\right)^2-2n^2\eta^2\frac{dV}{d\eta}\frac{d^2V}{d\eta^2} \tag{3.6.16}$$

From Equation (3.6.3)

$$\frac{d^2T}{dz^2}=\frac{d}{dz}\left[\frac{dT}{d\theta}\frac{d\theta}{d\eta}\frac{d\eta}{dz}\right]\frac{d\eta}{dz} \tag{3.6.17}$$

but $T=\theta(T_w-T_\infty)+T_\infty$ hence $\dfrac{dT}{d\theta}=(T_w-T_\infty)$

$$\frac{d^2T}{dz^2}=\frac{d}{d\eta}\left[(T_w-T_\infty)\frac{d\theta}{d\eta}\frac{U_0}{\upsilon}\right]\frac{U_0}{\upsilon}$$

$$=\frac{U_0{}^2}{\upsilon^2}(T_w-T_\infty)\frac{d^2\theta}{d\eta} \tag{3.6.18}$$

Then

$$\alpha\frac{U_0{}^2}{\upsilon^2}(T_w-T_\infty)\frac{d^2\theta}{d\eta}+\frac{U_0{}^2}{\upsilon}(T_w-T_\infty)\frac{d}{d\eta}\left(\frac{n^2\eta^2}{\Pr_t}\frac{dU}{d\eta}\frac{d\theta}{d\eta}\right)+\frac{1}{\rho C_p}4\varepsilon^2(T-T_\infty)+$$
$$\frac{\mu}{\rho C_p}\frac{U_0{}^4}{\upsilon^2}\left(\left(\frac{dU}{d\eta}\right)^2+\left(\frac{dV}{d\eta}\right)^2\right)+\frac{\sigma B_0{}^2}{\rho C_p}U_0{}^4\left(\frac{(V+mU)^2+(U-mV)^2}{(1+m^2)^2}\right)=0 \tag{3.6.19}$$

dividing (3.6.19) through by $\dfrac{\upsilon}{(T-T_\infty)U_0{}^2}$ and consider $P_r=\dfrac{\upsilon}{\alpha}$, $\upsilon=\dfrac{\mu}{\rho}$

$$\frac{\alpha}{\upsilon}\frac{d^2\theta}{d\eta}+\frac{2n^2\eta}{\Pr_t}\frac{dU}{d\eta}\frac{d\theta}{d\eta}+\frac{n^2\eta^2}{\Pr_t}\frac{d^2U}{d\eta^2}\frac{d\theta}{d\eta}+\frac{n^2\eta^2}{\Pr_t}\frac{dU}{d\eta}\frac{d^2\theta}{d\eta^2}+\frac{4\varepsilon^2(T-T_\infty)}{U_0{}^2\rho C_p(T_w-T_\infty)}+$$
$$\frac{U_0{}^2}{(T_w-T_\infty)C_p}\left(\left(\frac{dU}{d\eta}\right)^2+\left(\frac{dV}{d\eta}\right)^2\right)+\frac{\sigma\upsilon B_0{}^2}{\rho C_p(T_w-T_\infty)}\left(\frac{(V+mU)^2+(U-mV)^2}{(1+m^2)^2}\right)=0$$

(3.6.19)

3.7 Final Set of Governing Equations

Using the non–dimensional quantities defined in the previous section, we are able to simplify the equation of motion (3.6.1 - 3.6.2) and the energy equation (3.6.3) and obtain the following final equations for hydromagnetic turbulent boundary layer flow of a viscous, incompressible, electrically conducting fluid with Hall effects and thermal radiations as:

$$-2RV = \frac{d^2U}{d\eta^2} + \frac{d}{d\eta}\left(n^2\eta^2\left(\frac{dU}{d\eta}\right)^2\right) + Gr\theta - M\left(\frac{U-mV}{1+m^2}\right), \tag{3.7.1}$$

$$2RU = \frac{d^2V}{d\eta^2} + \frac{d}{d\eta}\left(n^2\eta^2\left(\frac{dV}{d\eta}\right)^2\right) - M\left(\frac{V+mU}{1+m^2}\right), \tag{3.7.2}$$

$$\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2} + \frac{d}{d\eta}\left(\frac{n^2\eta^2}{\Pr_t}\frac{dU}{d\eta}\frac{d\theta}{d\eta}\right) + Nr\theta + Ec\left[\left(\frac{dU}{d\eta}\right)^2 + \left(\frac{dV}{d\eta}\right)^2\right] + MEc\left[\frac{(U-mV)^2+(V+mU)^2}{(1+m^2)^2}\right] = 0 \tag{3.7.3}$$

With the Boundary conditions

$$U = 1,\ V = 0,\ \theta = 1, \quad \text{at } \eta = 0, \tag{3.7.4}$$

$$U = 0,\ V = 0,\ \theta = 0, \quad \text{as } \eta \to \infty, \tag{3.7.5}$$

where M is the magnetic field parameter, Nr is the radiation parameter, R is the rotational parameter, Ec is the Eckert number, Gr is the Grashof number and Pr is the Prandtl number. Following Marchello and Toor (1963), for high turbulence intensity, the turbulent Prandtl number is given in term of the fluid Prandtl number as $\Pr_t = \sqrt{\Pr}$

3.8 Other quantities of interest

Other quantities of interest are the skin friction (C_f) coefficient and the Nusselt number (Nu) at the plate surface which are given as

3.8.1 Skin Friction

Friction between a fluid and the surface of a solid moving through it or between a moving fluid and its enclosing surface. The boundary layer normally generates a drag on the plate as a result of the viscous stresses which are developed at the wall. This drag is normally referred to as skin friction. Skin friction occurs from the interaction amid the fluid and the skin of the body, and is directly associated to the wetted surface, the area of the facade of the body that is in contact with the fluid. Hence the skin friction mathematically give as :

$$\tau_{wx} = -\mu \left.\frac{du}{dz}\right|_{z=0}, \quad \tau_{wy} = -\mu \left.\frac{dv}{dz}\right|_{z=0} \quad \text{and}$$

$$q_w = -k \left.\frac{dT}{dz}\right|_{z=0}. \tag{3.8.1}$$

3.8.2 Nusselt number : in heat transfer at a boundary within a fluid, the Nusselt number is the ratio of convective to conductive heat transfer across the boundary. In this context, convection includes both advection and diffusion. A thermal boundary layer develops if the fluid free stream temperature and the surface temperatures differ. A temperature profile exists due to the energy exchange resulting from this temperature difference.

$$Nu = \frac{q_w}{k(T_w - T_\infty)} = -\left.\frac{d\theta}{d\eta}\right|_{\eta=0} \tag{3.8.2}$$

CHAPTER FOUR

Numerical Technique

4.1 Introduction

Many real life problems generally do not have "analytical" solutions. Mathematics being one of the scientific research disciplines that lead to real life situations requires numerical techniques to accomplish non-analytical solutions. The mathematical formulation of the problem is the reduction of the physical problem to a set of either algebraic or differential equations subject to certain assumptions. The process of modeling of physical systems in the real world should generally follow the path illustrated schematically in the chart below:

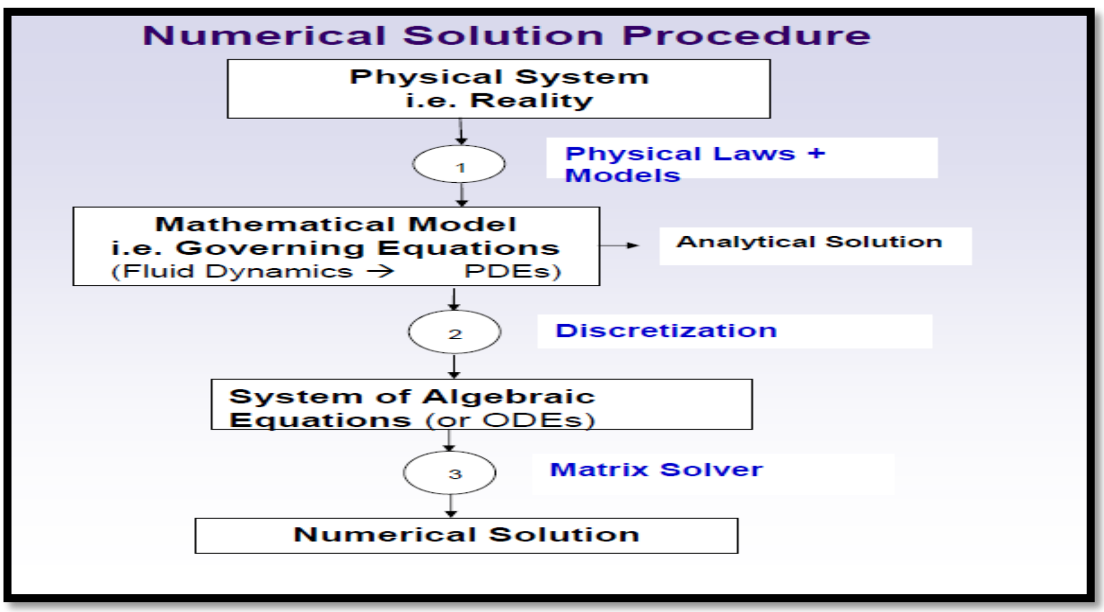

Figure 4. 1: Schematic Illustration for the Numerical Procedure

4.2 Numerical Implementations of The Model Equation

The nonlinear dimensionless model equations : (3.7.1), (3.7.2) and (3.7.3) together with the boundary conditions constitute a BVP (3.7.4 - 3.7.5) can be easily solved directly using ***appropriate finite difference*** numerical technique or by using ***shooting method coupled with Runge-Kutta Fehlberg integration scheme***. This method involves, transforming equations (3.7.1) & (3.7.3) into a set of IVP which contain unknown initial values that need to be determined by guessing, after which Runge-Kutta Fehlberg integration scheme is employed to integrate the set of initial valued problems until the given boundary conditions are satisfied. The entire computation procedure is implemented on computer using a program written in MAPLE language. From the process of numerical computation, the mean velocity and mean temperature are obtained. The numerical values are used to compute the skin friction coefficient and the Nusselt number as given in equation (3.8.1) and (3.8.2).

The second possible numerical approach is the finite difference scheme; In order to employ FDM, a Pseudo time space have been introduced to the model equations (3.7.1) - (3.7.3). Its noteworthy that the introduction of PTS results a non linear PDE which can be solved by dicrtizing the solution domain in to finite grid points. Its deemed that for large time interval the newly emerging coupled PDE's has behaves to converge to the original non linear ODE.

Basically, According to Kelley and David (1998) Pseudo time-stepping, probably better known as pseudo-transient continuation, is the technique of solving for the steady-state solution of time-evolving PED by setting an initial guess and using a time-stepper to evolve the solution forward. It tends to succeed where standard globalization strategies fail by taking advantage of the natural structure of the problem.

To the governing equation (3.7.1) - (3.7.3) we have assigned ***pseudo*** time derivative similar to that described as and together with the boundary conditions outlined as follows :

$$\frac{\partial U}{\partial t} = 2RV + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial}{\partial \eta}\left(n^2\eta^2\left(\frac{\partial U}{\partial \eta}\right)^2\right) + Gr\theta - M\left(\frac{U - mV}{1+m^2}\right) \tag{4.2.1}$$

$$\frac{\partial V}{\partial t} = -2RU + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial}{\partial \eta}\left(n^2\eta^2\left(\frac{\partial V}{\partial \eta}\right)^2\right) - M\left(\frac{V + mU}{1+m^2}\right) \tag{4.2.2}$$

$$\frac{\partial\theta}{\partial t}=\frac{1}{\Pr}\frac{d^2\theta}{d\eta^2}+\frac{\partial}{\partial\eta}\left(\frac{n^2\eta^2}{\Pr_t}\frac{\partial U}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right)+Nr\theta+Ec\left[\left(\frac{\partial U}{\partial\eta}\right)^2+\left(\frac{\partial V}{\partial\eta}\right)^2\right]+MEc\left[\frac{(U-mV)^2+(V+mU)^2}{(1+m^2)^2}\right]$$

(4.2.3)

With Boundary Condition :

$$U=1,\ V=0,\ \theta=1,\ \textbf{at}\ \eta=0 \qquad (4.2.4)$$

$$U=0,\ \ V=0,\ \ \theta=0,\ \textbf{as}\ \eta\to\infty\ \textbf{:}\ \text{for all}\ t\geq 0 \qquad (4.2.5)$$

OR:

$$\left.\begin{array}{l}U(t,0)=1\\V(t,0)=0\\\theta(t,0)=1\end{array}\right\}\eta=0,\qquad \left.\begin{array}{l}U(t,\infty)=1\\V(t,\infty)=0\\\theta(t,\infty)=1\end{array}\right\}\eta\to\infty,t\geq 0 \qquad (4.2.6)$$

Initial Condition :

$$\left.\begin{array}{l}U(0,\eta)=0\\V(0,\eta)=0\\\theta(0,\eta)=0\end{array}\right\},t=0 \qquad (4.2.6)$$

The set of governing equations: (4.1), (4.2) and (4.3) involving a pseudo time derivative in it, cannot be solved analytically since they are coupled and highly nonlinear. With the help of the boundary conditions set out in relations (4.4)-(4.5) , The easiest and most appropriate difference scheme to implement for the system of equations is the explicit method. It is apparent that both velocity and temperature are functions of time (t) and space (η). We, therefore, need to discretize the time and space coordinates to form a solution mesh.

The space co-ordinate is subdivided into $N-1$ intervals of equal length $\Delta\eta$ so that there are N nodal points. On the other hand, the temporal co-ordinate is subdivided into $K-1$ intervals of equal length Δt so that there are K nodal points. Each of the nodal points is labeled by a pair of indices, j and k. Below is a schematic diagram of the representative mesh.

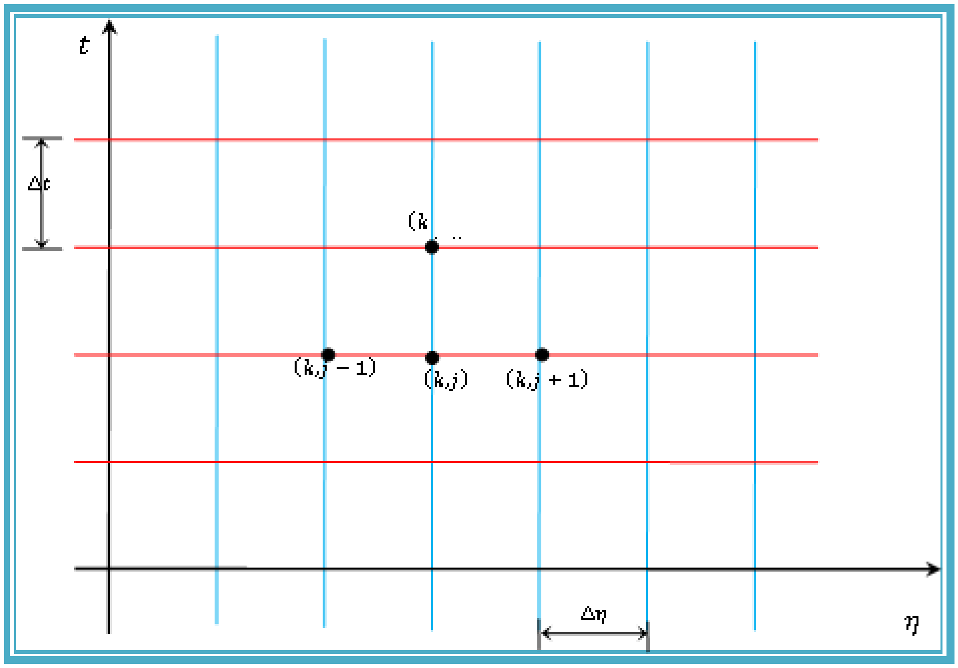

Figure 4. 2: Grid of important points in explicit finite difference method

This method is highly stable and the most accurate method. It is, therefore, the most recommended finite difference method for estimating solutions to partial differential equations. Although fewer time steps can be used with this method to achieve similar accuracy with the explicit method, it is also the most difficult to implement. In the next subsequent section the discrete model equations has presented.

4.3 Difference Equations for the Present Model

All the flow variables of interest in the present problem has involved both temporal and spatial partial derivatives in the same equation. There are known methods in solving such equations. These are the fully explicit method, the fully implicit and the mixed implicit–explicit methods. For the governing equations as set out in (4.1)-(4.3), the easiest and most appropriate difference scheme to implement for the system of equations the explicitly method. The system of equations describes the evolution of velocity (both primary and secondary) and temperature. Below we evaluate the finite difference schemes for these fluid properties.

4.3.1 Velocity

For The primary Velocity :

The velocity along the $x-axis$, $U(t,\eta)$ is the Primary velocity. Using the finite difference formulas defined in equations and employing the explicit difference method we write:

$$\frac{U_i^{k+1}-U_i^k}{\Delta t}=2RV_i^k+\frac{\left(U_{i+1}^k-2U_i^k+U_{i-1}^k\right)}{(\Delta\eta)^2}+$$
$$2n^2\left(\eta_i\left[\frac{(U_{i+1}^k-U_i^k)}{2\Delta\eta}\right]^2+(\eta_i)^2.\left[\frac{(U_{i+1}^k-U_i^k)}{2\Delta\eta}\right].\left[\frac{\left(U_{i+1}^k-2U_i^k+U_{i-1}^k\right)}{(\Delta\eta)^2}\right]\right)+$$
$$Gr\theta_i^k+\frac{M}{1+m^2}\left(U_i^k-mV_i^k\right)$$

(4.3.1)

Multiplying through by Δt and using the ratios $r_1=\frac{\Delta t}{(\Delta\eta)^2}, and, r_2=\frac{\Delta t}{(\Delta\eta)^3}$ equation (4.3.1) simplifies to:

$$U_i^{k+1}=U_i^k+2R\Delta tV_i^k++\frac{M.\Delta t}{1+m^2}\left(U_i^k-mV_i^k\right)+Gr\Delta t\theta_i^k+r_1\left[\left(U_{i+1}^k-2U_i^k+U_{i-1}^k\right)\right]+$$
$$2n^2\left(\frac{r_1}{4}\eta_i\left[(U_{i+1}^k-U_i^k)\right]^2+\frac{r_2}{2}(\eta_i)^2\left(U_{i+1}^k-U_i^k\right).\left(U_{i+1}^k-2U_i^k+U_{i-1}^k\right)\right)$$

(4.3.2)

And

For the secondary velocity, $V(t,\eta)$ we proceed as follows.

$$\frac{V_i^{k+1}-V_i^k}{\Delta t}=-2RU_i^k+\frac{\left(V_{i+1}^k-2V_i^k+V_{i-1}^k\right)}{\Delta\eta^2}+$$
$$2n^2\left(\eta_i\left[\frac{(V_{i+1}^k-V_i^k)}{2\Delta\eta}\right]^2+(\eta_i)^2.\left[\frac{(V_{i+1}^k-V_i^k)}{2\Delta\eta}\right].\left[\frac{\left(V_{i+1}^k-2V_i^k+V_{i-1}^k\right)}{\Delta\eta^2}\right]\right)-\frac{M}{1+m^2}\left(V_i^k+mU_i^k\right)$$

(4.3.3)

Multiplying equation (4.3.3) through by Δt and using the ratios $r_1=\frac{\Delta t}{(\Delta\eta)^2}, and, r_2=\frac{\Delta t}{(\Delta\eta)^3}$ we obtain:

$$V_i^{k+1} = U_i^k - 2R\Delta t U_i^k - \frac{M.\Delta t}{1+m^2}\left(V_i^k + mU_i^k\right) + r_1\left(V_{i+1}^k - 2V_i^k + V_{i-1}^k\right) +$$

$$2n^2\left(\frac{r_1}{4}\eta_i(V_{i+1}^k - V_i^k)^2 + \frac{r_2}{2}(\eta_i)^2(V_{i+1}^k - V_i^k).\left(V_{i+1}^k - 2V_i^k + V_{i-1}^k\right)\right) \tag{4.3.4}$$

4.3.2 Temperature

For the temperature, $\theta(t,\eta)$, we have the difference equation given as:

$$\frac{\theta_i^{k+1} + \theta_i^k}{\Delta t} = \frac{1}{P_r}\frac{\left(\theta_{i+1}^k - 2\theta_i^k + \theta_{i-1}^k\right)}{\left(\Delta\eta\right)^2} + \frac{n^2}{\mathrm{Pr}_t}\frac{2\eta_i}{4(\Delta\eta)^2}\left(\theta_{i+1}^k - \theta_i^k\right).\left(U_{i+1}^k - U_i^k\right) +$$

$$\frac{(\eta_i)^2}{2\left(\Delta\eta\right)^3}\left(U_{i+1}^k - 2U_i^k + U_{i-1}^k\right).\left(\theta_{i+1}^k - \theta_i^k\right) + \frac{(\eta_i)^2}{4\left(\Delta\eta\right)^3}\left(U_{i+1}^k - U_i^k\right).\left(\theta_{i+1}^k - 2\theta_i^k + \theta_{i-1}^k\right) + N_r.\theta_i^k$$

$$+\frac{E_c}{4\left(\Delta\eta\right)^2}.\left[\left(U_{i+1}^k - U_i^k\right) + \left(V_{i+1}^k - V_i^k\right)\right] + \frac{M}{1+m^2}E_c\left(U_i^k - mV_i^k\right) + \left(V_i^k + mU_i^k\right) \tag{4.3.5}$$

Multiplying equation (4.3.5) through by Δt and using the ratios $r_1 = \frac{\Delta t}{(\Delta\eta)^2}, and, r_2 = \frac{\Delta t}{(\Delta\eta)^3}$ we obtain:

$$\theta_i^{k+1} = \theta_i^k - \frac{r_1}{2P_r}\left(\theta_{i+1}^k - 2\theta_i^k + \theta_{i-1}^k\right) + \frac{r_1 \cdot n^2}{2\,\mathrm{Pr}_t}.\begin{bmatrix} \eta_i\left(\theta_{i+1}^k - \theta_i^k\right).\left(U_{i+1}^k - U_i^k\right) + \\ \frac{r_2.(\eta_i)^2}{4}\left(U_{i+1}^k - 2U_i^k + U_{i-1}^k\right).\left(\theta_{i+1}^k - \theta_i^k\right) + \\ \frac{r_2.(\eta_i)^2}{4}\left(U_{i+1}^k - U_i^k\right).\left(\theta_{i+1}^k - 2\theta_i^k + \theta_{i-1}^k\right) \end{bmatrix}$$

$$+N_r \cdot \Delta t \cdot \theta_i^k + \frac{r_1 \cdot E_c}{4}.\left[\left(U_{i+1}^k - U_i^k\right) + \left(V_{i+1}^k - V_i^k\right)\right] + \frac{M \cdot \Delta t}{1+m^2}E_c\left(U_i^k - mV_i^k\right) + \left(V_i^k + mU_i^k\right) \tag{4.3.6}$$

Equations (4.3.2), (4.3.4) and (4.3.11) are the difference form equations for the primary, secondary velocity and temperature variables. In this form they can be evaluated by a computer to obtain solutions time levels.

Its noteworthy that the abovementioned equations in difference forms, will take very long to carry out the numerical computation manually; particularly when there a small step size are used within the solution domain. It's possible to handle the implementation in the mathematical package which performs the computations.

In the upcoming chapter we dealt with the result, observations and discussion with respect to the effect of the embedded parameters for our model.

CHAPTER FIVE

Results and Discussion

5.1 Overview

In order to understand the physical situation of the problem and effects of various parameters controlling the flow regime, we have computed the numerical values of mean velocities, mean temperature, the skin friction and the Nusselt number with respect to each parameter variation as shown in figures 2-31. It is important to note that $Gr > 0$ corresponds to cooling of the plate by convection current while $Gr < 0$ implies heating of the plate by convection current. Moreover, the cooling problem is

Table 5. 1:Values for Computation showing the skin friction and Nusselt number.

M	***m***	***R***	***Gr***	***Ec***	***Nr***	***Pr***	C_{fx}	C_{fy}	***Nu***
1.0	0.1	1.0	0.1	0.1	1.0	0.71	1.21183514	0.81877576	0.83925868
2.0	0.1	1.0	0.1	0.1	1.0	0.71	1.51494067	0.70318128	0.82053306
3.0	0.1	1.0	0.1	0.1	1.0	0.71	1.78458544	0.63077322	0.80364695
1.0	0.5	1.0	0.1	0.1	1.0	0.71	1.21244422	0.94064112	0.84088432
1.0	1.0	1.0	0.1	0.1	1.0	0.71	1.14983533	1.02958701	0.84583895
1.0	0.1	2.0	0.1	0.1	1.0	0.71	1.55777192	1.28916619	0.82453955
1.0	0.1	3.0	0.1	0.1	1.0	0.71	1.84502594	1.62487996	0.80871950
1.0	0.1	1.0	0.5	0.1	1.0	0.71	1.04984232	0.86729541	0.83701858
1.0	0.1	1.0	1.0	0.1	1.0	0.71	0.84543532	0.92903706	0.83237814
1.0	0.1	1.0	-0.5	0.1	1.0	0.71	1.45275334	0.74715631	0.84034948
1.0	0.1	1.0	-1.0	0.1	1.0	0.71	1.65199584	0.68831179	0.83930426
1.0	0.1	1.0	0.1	0.5	1.0	0.71	1.20895408	0.82012949	0.52261924
1.0	0.1	1.0	0.1	1.0	1.0	0.71	1.20534494	0.82182793	0.12720102
1.0	0.1	1.0	0.1	0.1	2.0	0.71	1.21843079	0.81542480	1.41337547
1.0	0.1	1.0	0.1	0.1	3.0	0.71	1.22393362	0.81296122	2.06881414
1.0	0.1	1.0	0.1	0.1	1.0	4.00	1.23139532	0.81022002	3.49966374
1.0	0.1	1.0	0.1	0.1	1.0	7.10	1.23772125	0.80862109	6.20529525

often encountered in engineering applications such as cooling of electronic components and nuclear reactors. Numerical values in table 1 depict the effect of parameter variation on skin friction and Nusselt number. From the table, it is observed that the primary skin friction at the plate surface increases with an increase in parameter values of M, R, Nr, Pr, $Gr < 0$ and decreases with an increase in parameter m, Ec and $Gr > 0$. The secondary skin friction increases with parameter m, R, Ec, Gr>0 and decreases with parameter M, Nr, Pr, Gr <0. An increase in parameter m, Nr and Pr increases the Nusselt number while an increase in parameter values of M, R, Ec and gr decreases the nusselt number.

5.2 Effects of Parameter Variation on Mean Velocities

Figures 5.1-5.7 illustrate the effects of parameter variation on the primary mean velocity component. Generally the primary mean velocity is highest at the moving plate surface and tends zero free stream value far away from the place surface satisfying the prescribed boundary conditions. An increase in the fluid rotation, magnetic field intensity, hall current, thermal radiation and Prandtl number decreases the momentum boundary layer thickness as shown in figures 5.1-5.5, consequently the primary mean velocity decreases. In figure 5.6, it is observed that the momentum boundary layer thickness increases with cooling of the plate by convectional current (Gr>0) due to buoyancy force and decreases with heating of the plate by convectional current (Gr < 0). As the viscous heating increases (Ec), the primary mean velocity increases leading to a rise in momentum boundary layer thickness as shown in figure 5.7 Figures 5.8-5.14 depict the secondary mean velocity for some selected values of the parameters. Interestingly, flow reversal is generally observed with its peak value within the boundary layer regime and zero secondary mean velocity at both the plate surface and free stream satisfying the prescribed boundary conditions. As the values of parameter R, Nr, M and Pr increase due to combined effects of fluid rotation, magnetic field and thermal radiation, the fluid flows toward the plate surface with a decrease in the reverse flow intensity as shown in figures 5.8-5.12 This is expected, since increase in magnetic field enhances the effect of Lorentz force which acts as a resistance to the flow, leading to a decrease in the momentum boundary layer thickness. Figure 14 shows that the secondary mean flow reversal and its peak values increase with cooling (Gr > 0) and decreases with heating (Gr < 0) of the plate surface by convectional current due to buoyancy force. Increase viscous heating (Ec) increases the secondary mean flow reversal and the momentum boundary layer thickness as shown figure 5.14.

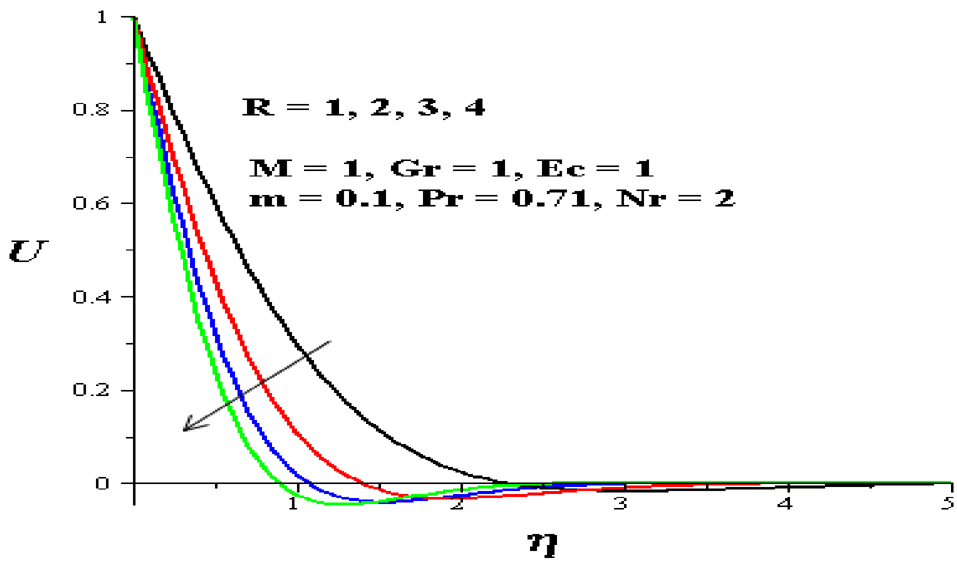

Figure 5. 1: Primary velocity profiles with increasing R.

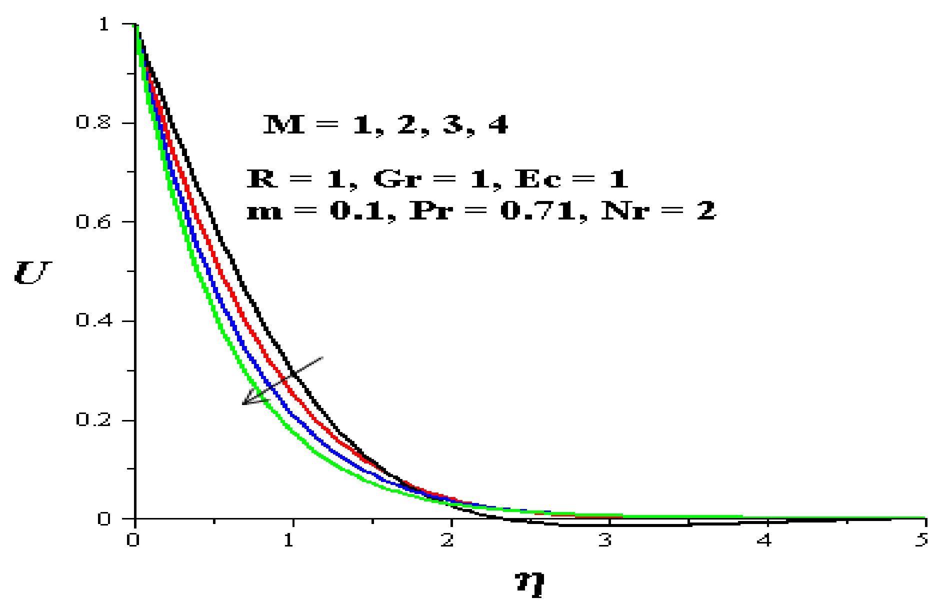

Figure 5. 2: Primary velocity profiles with increasing M.

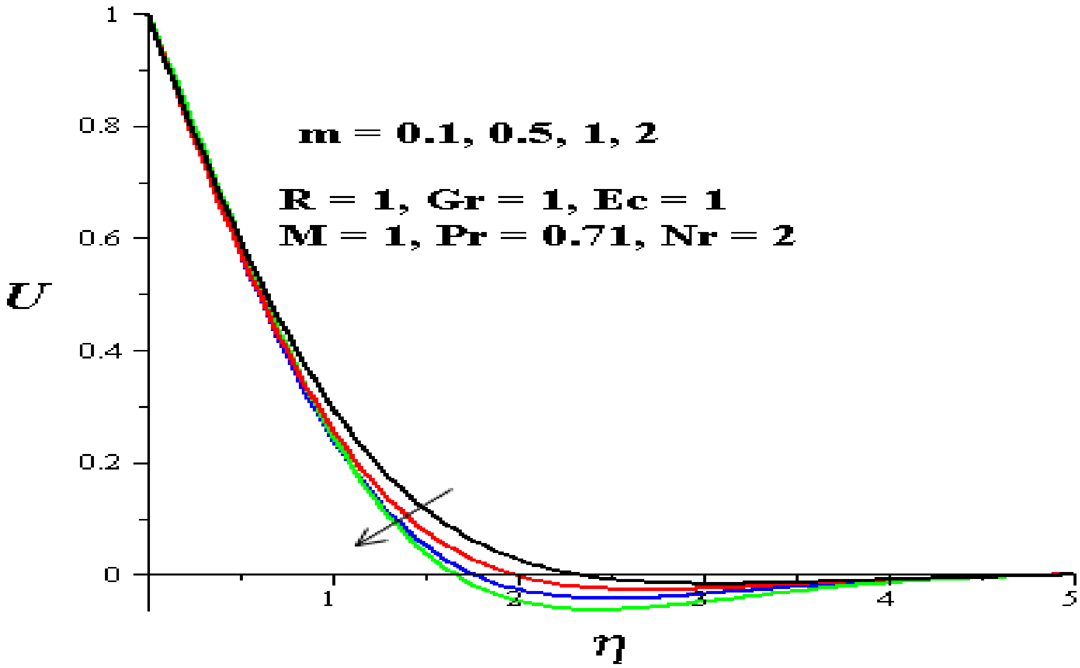

Figure 5. 3: Primary velocity profiles with increasing m.

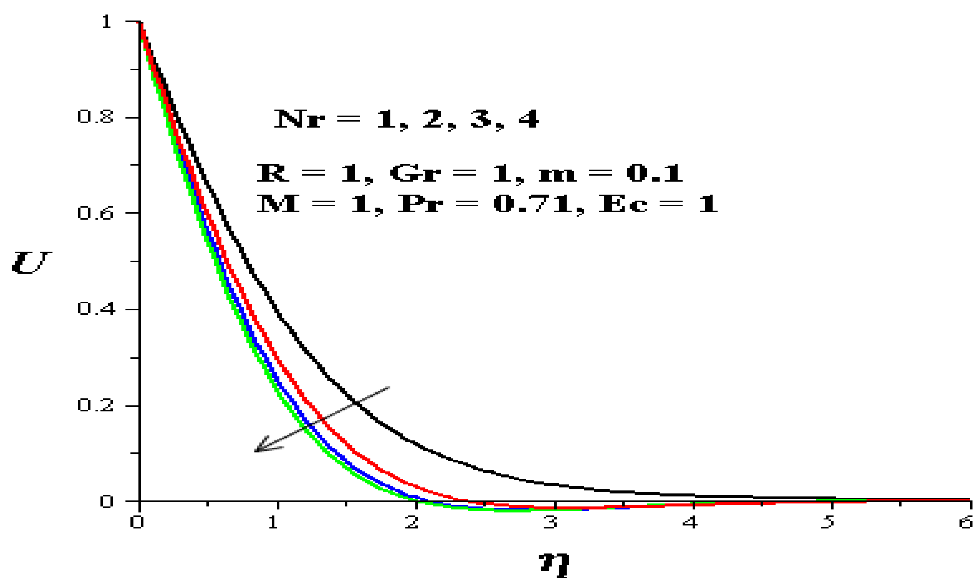

Figure 5. 4: Primary velocity profiles with increasing Nr.

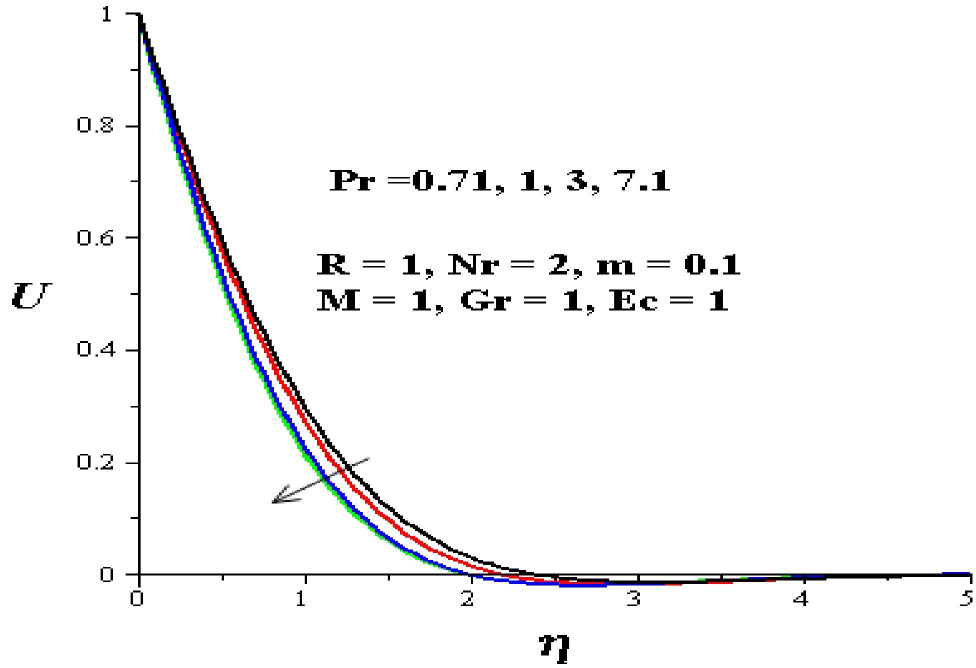

Figure 5. 5: Primary velocity profiles with increasing Pr.

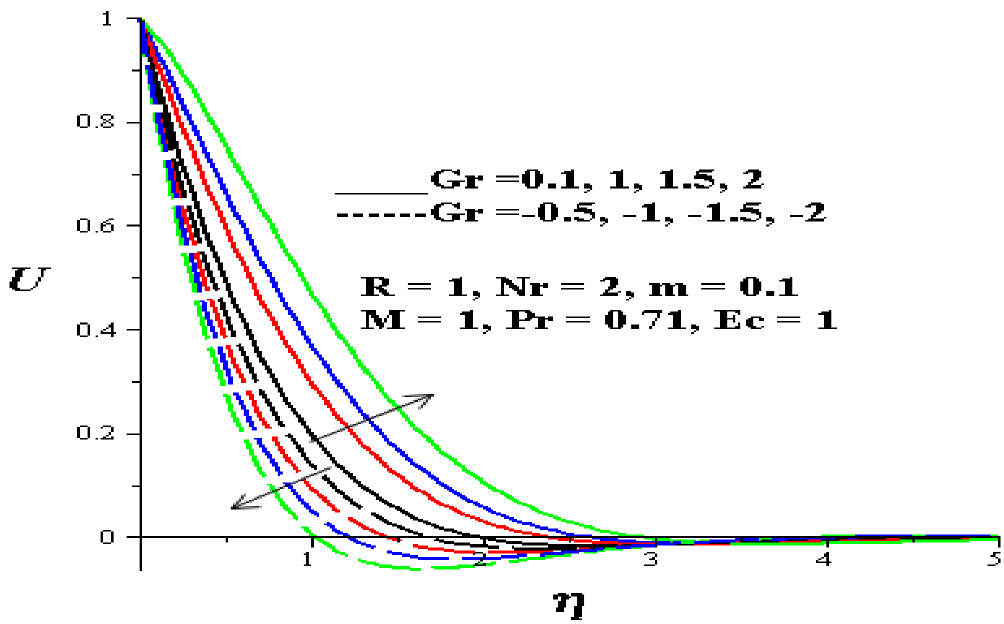

Figure 5. 6: Primary velocity profiles with increasing Gr.

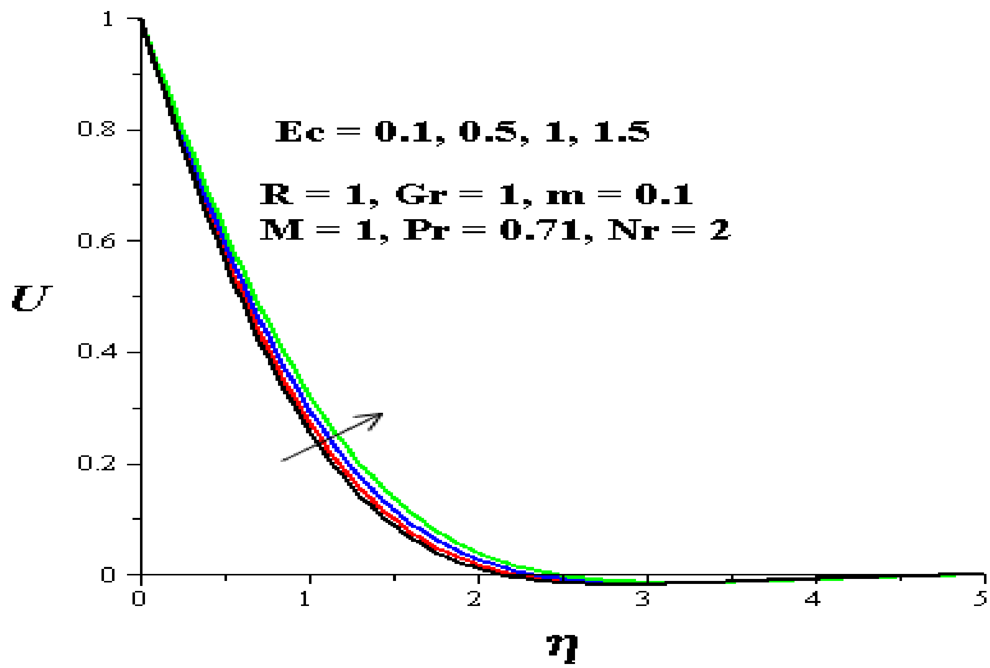

Figure 5. 7: Primary velocity profiles with increasing Ec.

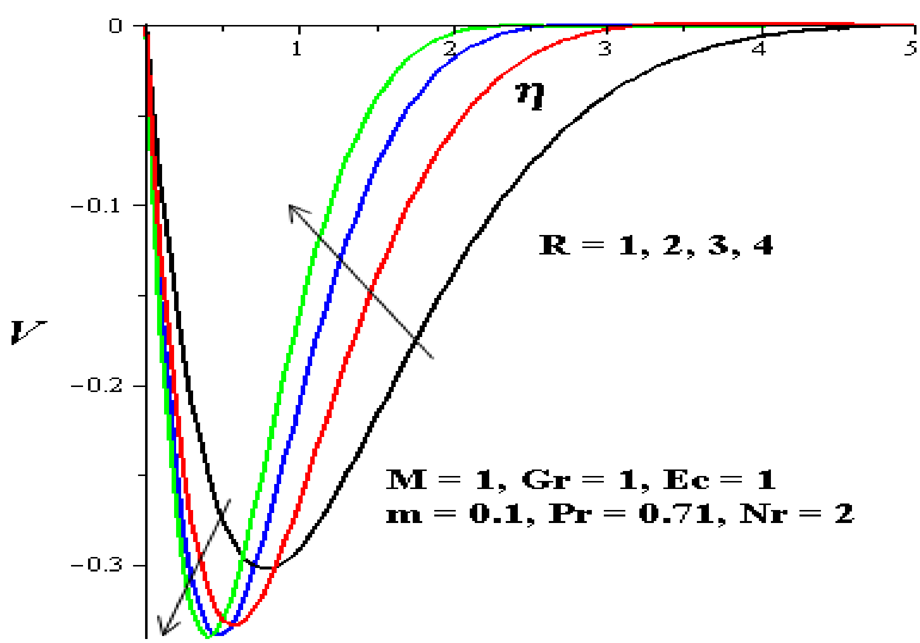

Figure 5. 8: Secondary velocity profiles with increasing R.

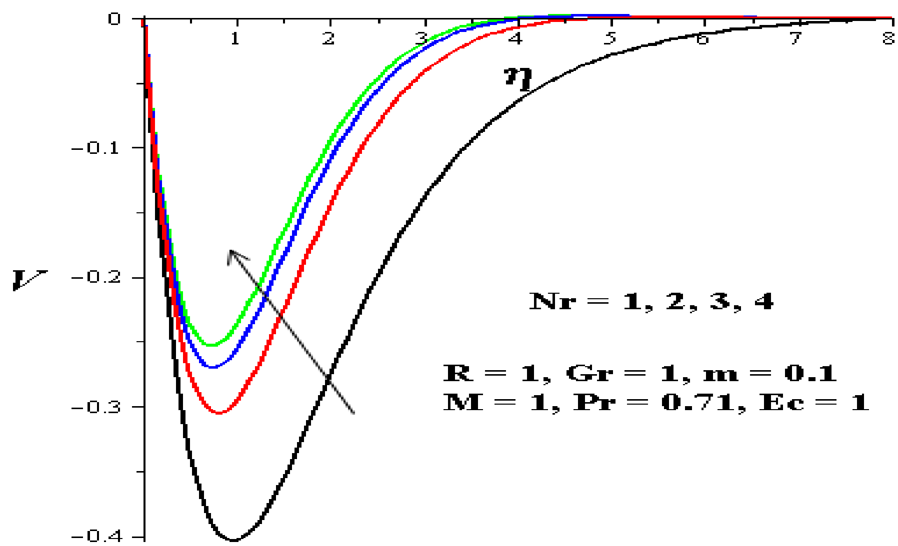

Figure 5. 9: Secondary velocity profiles with increasing Nr.

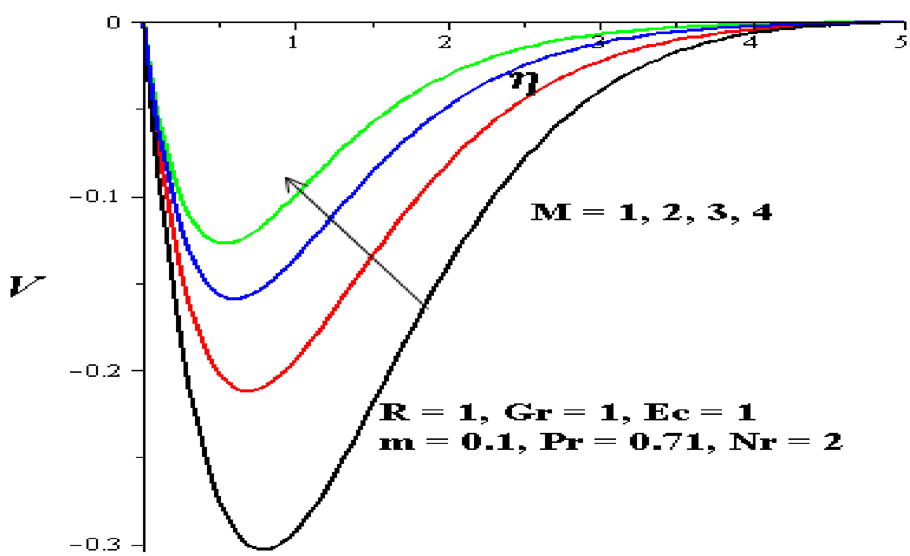

Figure 5. 10: Secondary velocity profiles with increasing M.

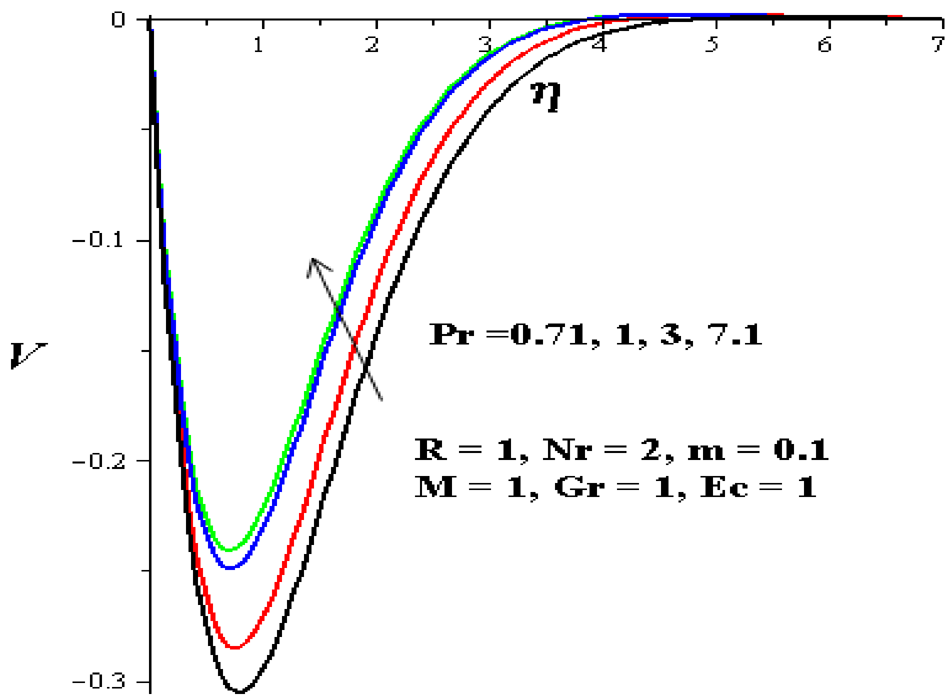

Figure 5. 11: Secondary velocity profiles with increasing Pr.

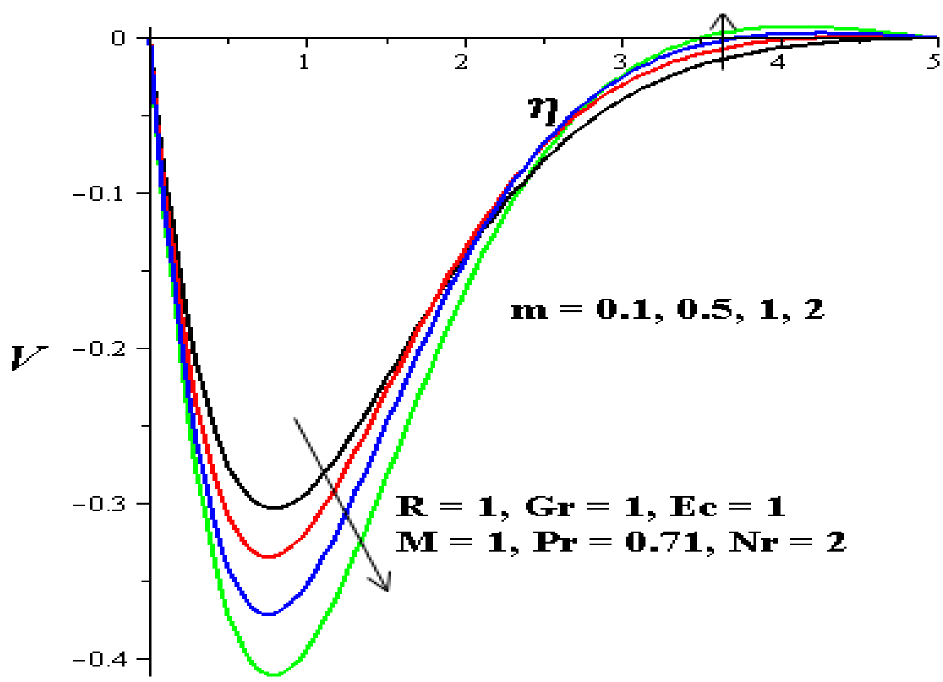

Figure 5. 12: Secondary velocity profiles with increasing m.

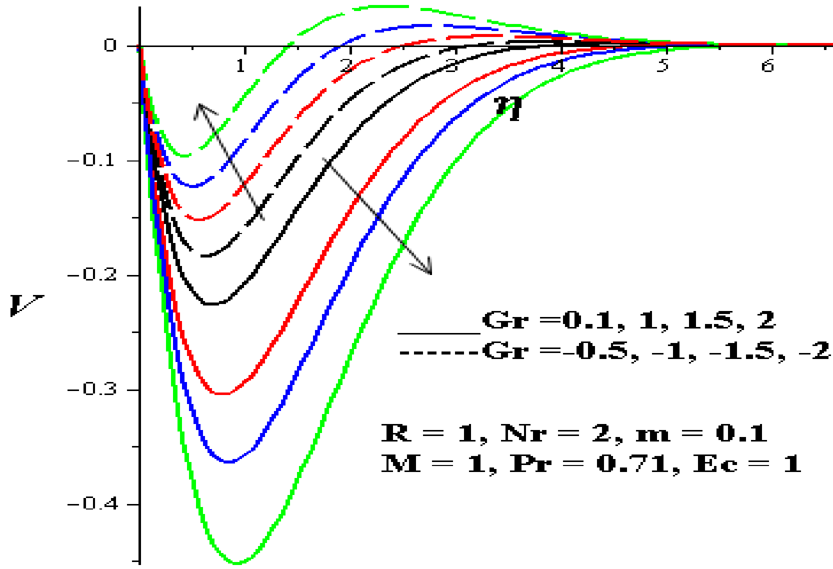

Figure 5. 13: Secondary velocity profiles with increasing Gr.

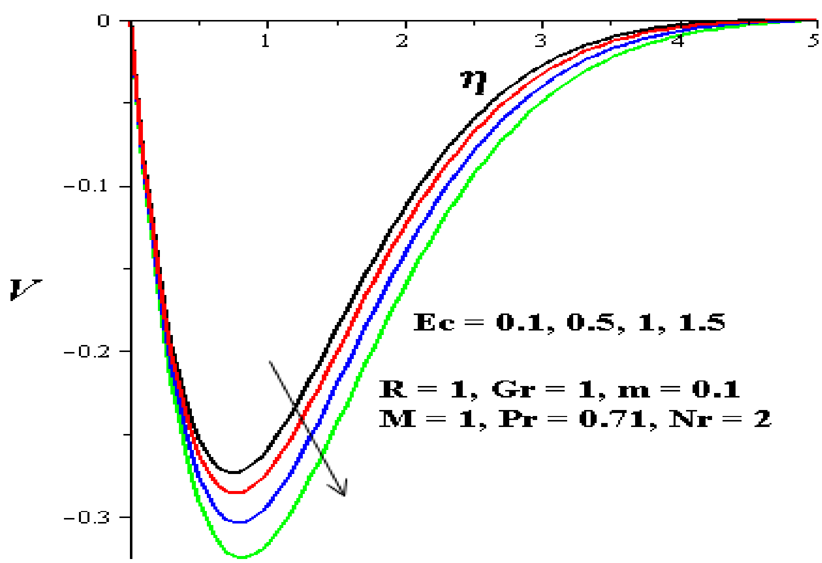

Figure 5. 14: Secondary velocity profiles with increasing Ec.

5.3 Effects of Parameter Variation on Mean Temperature Profiles

Figures 5.15-5.21 illustrate the effects of various thermophysical parameters on the mean temperature profiles. It is noteworthy that the mean temperature decreases from the plate surface to the prescribed free stream zero value far away the plate. Meanwhile, as the parameter values of *R*, *M* and *Ec* increase, the thermal boundary layer thickness increases near the plate surface as shown in figures 5.15-5.17. This may be attributed to the combined effects of fluid rotation, viscous and Ohmic heating, and leading to a rise in the mean temperature. In figures 5.18-5.20, a decrease in the thermal boundary layer thickness in observed with increasing parameter values of *Nr*, *m* and *Pr*, consequently, the mean temperature decreases. Figure 5.21 shows that the mean temperature within the boundary layer regime is enhanced by plate cooling while the heating of the plate by buoyancy force decreases the boundary layer thickness. This can be explained by the fact that the heat is transferred from the plate to the fluid by buoyancy force during cooling leading to a rise in the mean temperature.

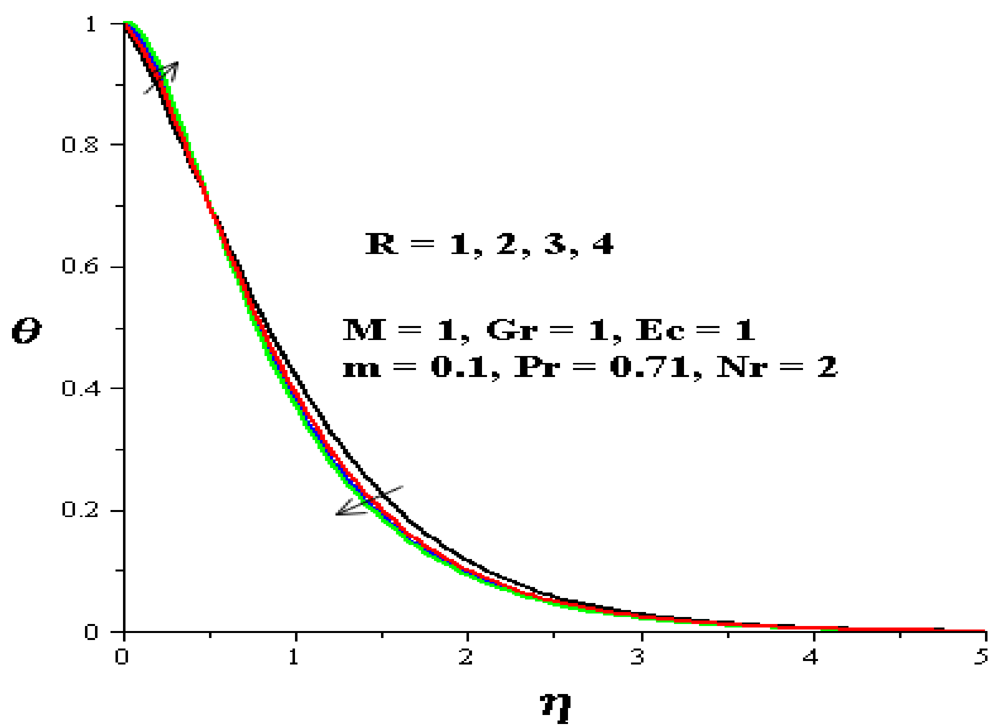

Figure 5. 15: Temperature profiles with increasing R.

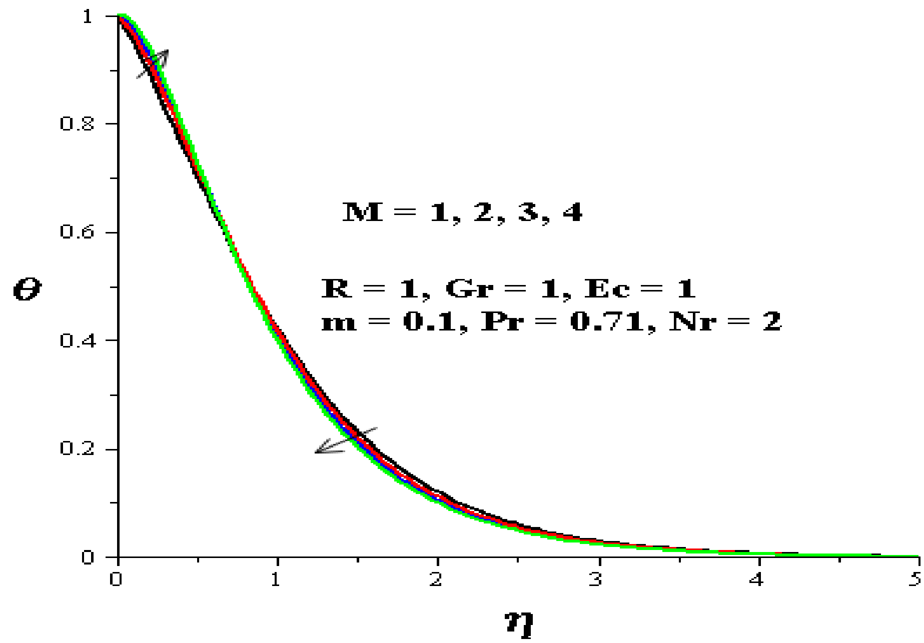

Figure 5. 16: Temperature profiles with increasing M.

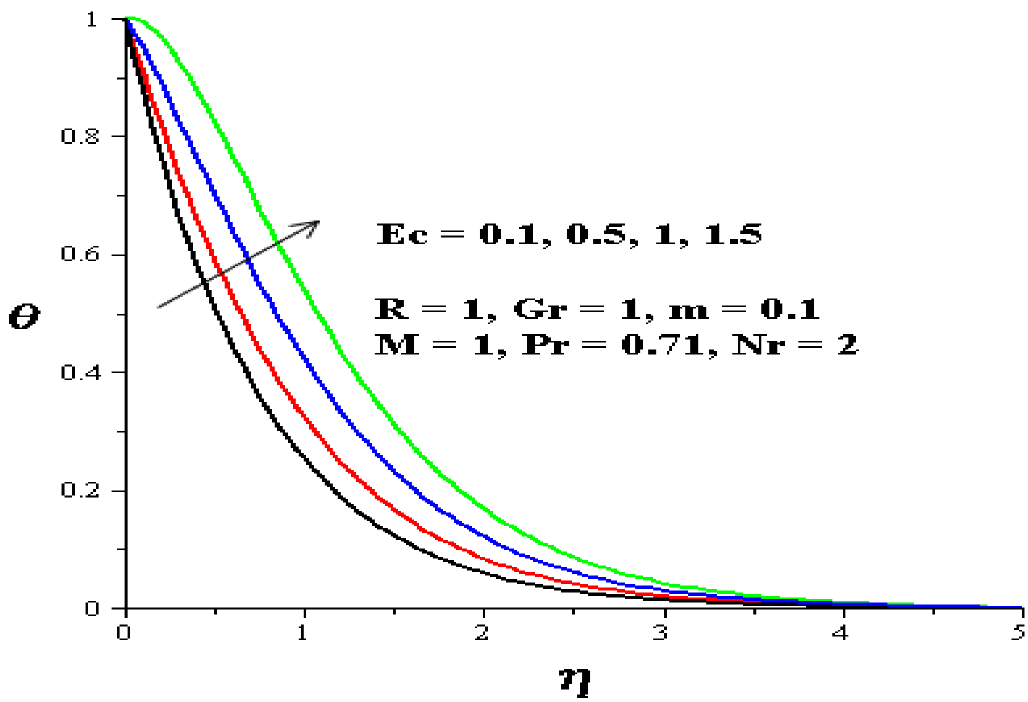

Figure 5. 17: Temperature profiles with increasing Ec.

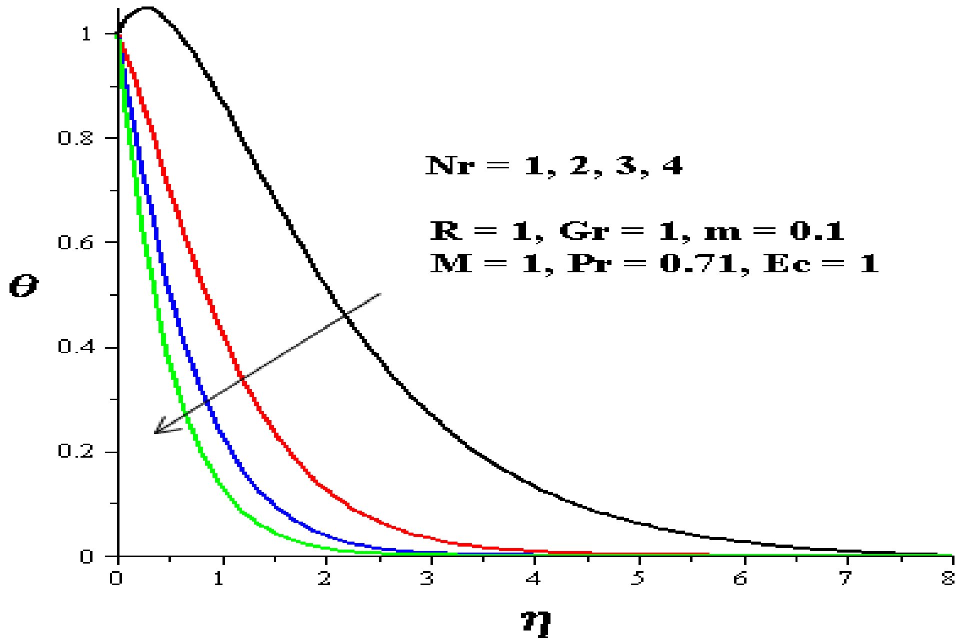

Figure 5. 18: Temperature profiles with increasing Ec.

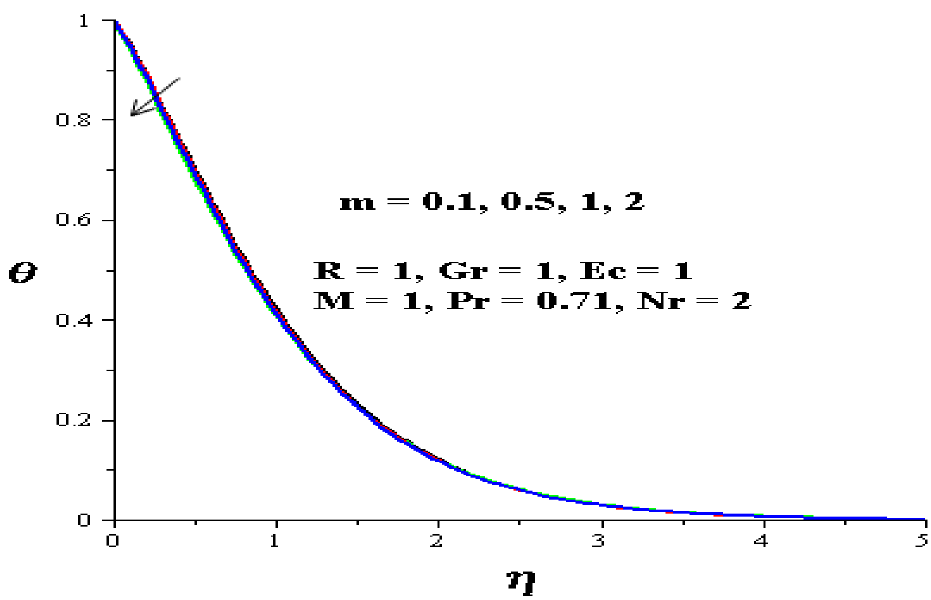

Figure 5. 19: Temperature profiles with increasing m.

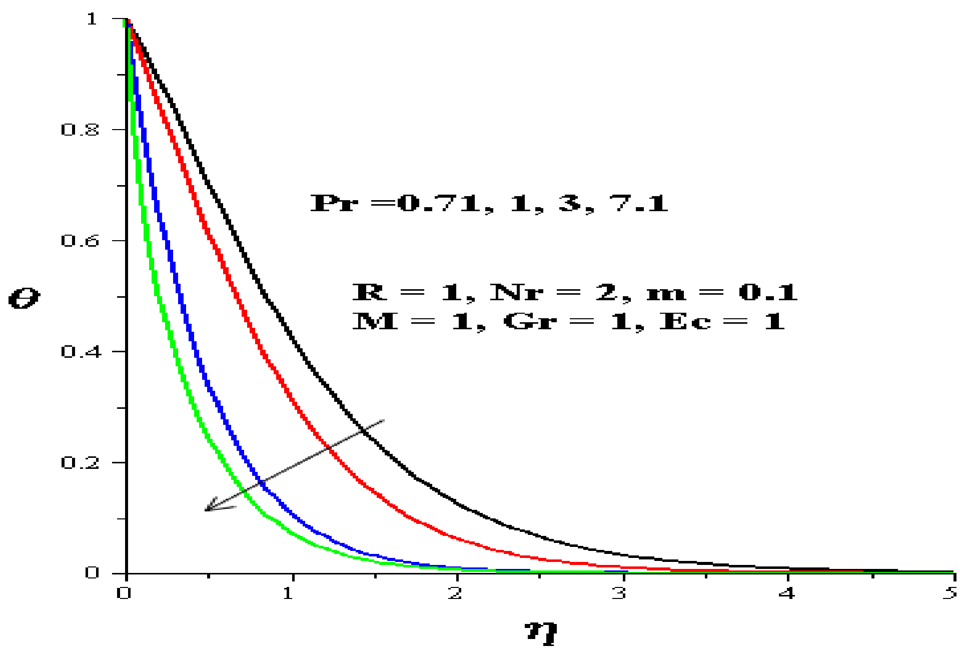

Figure 5. 20: Temperature profiles with increasing Pr.

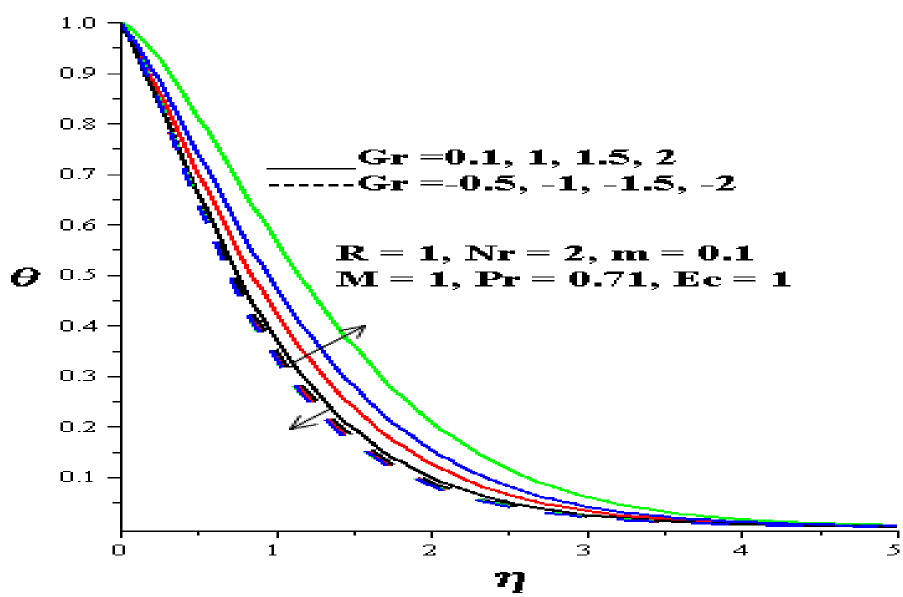

Figure 5. 21: Temperature profiles with increasing Gr.

5.4 Effects of Parameter Variation on Skin Friction and Nusselt number

Figures 5.22-5.24 demonstrate the effects of increasing each thermophysical parameter on the primary skin friction. We observed that the primary skin friction increases with an increase in magnetic field intensity, fluid rotation, thermal radiation, Prandtl number and plate heating by convectional current. This can be attributed to a rise in mean velocity gradient at the plate surface. Meanwhile a fall in the primary skin friction is noticed with increasing intensity of Hall current, viscous heating and plate cooling due to buoyancy force. Figures 25-27 depict the effects of parameter variation on secondary skin friction. A combined increase in the fluid rotation, Hall current, viscous heating and plate cooling due to buoyancy force increases the secondary skin friction while an increase in the Ohmic heating; thermal radiation and Prandtl number decreases the secondary skin friction at the plate surface. In figures 28-30, the variation in Nusselt number is illustrated with different parameters. An increase in Hall current, thermal radiation and Prandtl number enhances the Nusselt number due to a rise in the temperature gradient at the plate surface. However, the Nusselt number decreases with a combined increase in Ohmic and viscous heating, fluid rotation and buoyancy force.

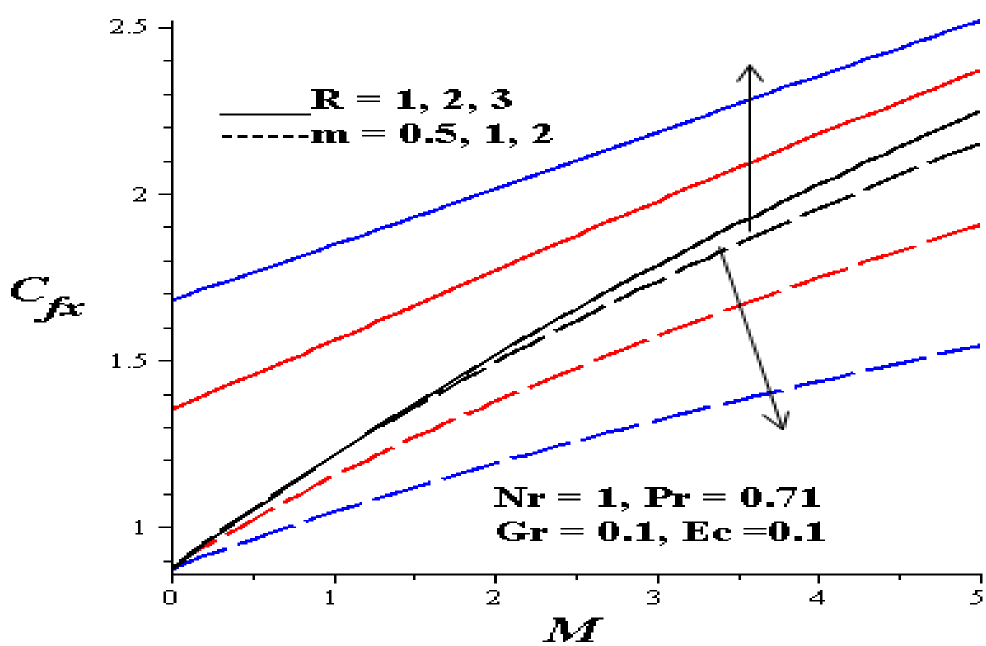

Figure 5. 22: Primary skin friction with increasing M, R and m.

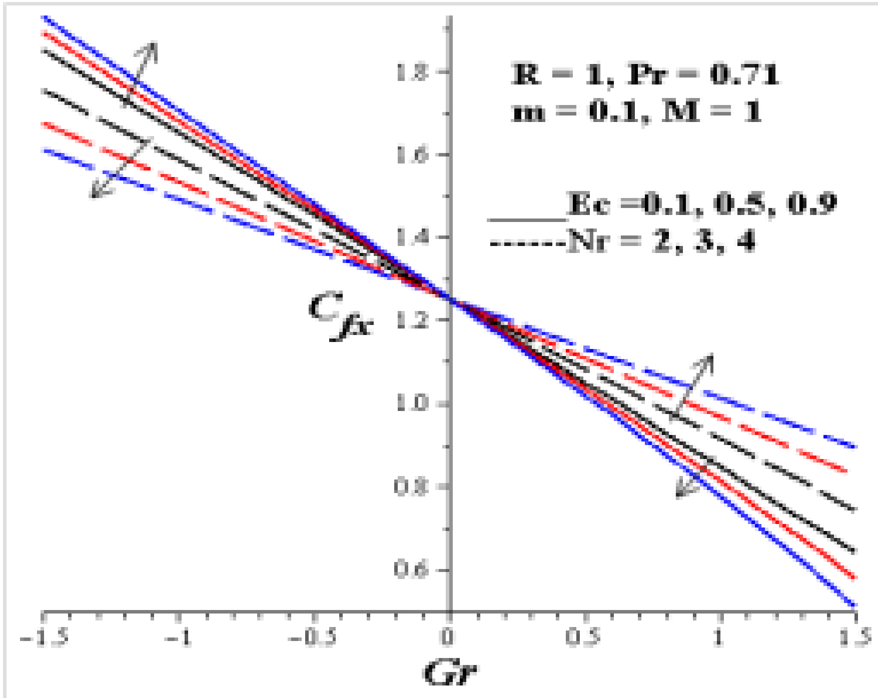

Figure 5. 23: Primary skin friction with increasing Ec, Nr and Gr.

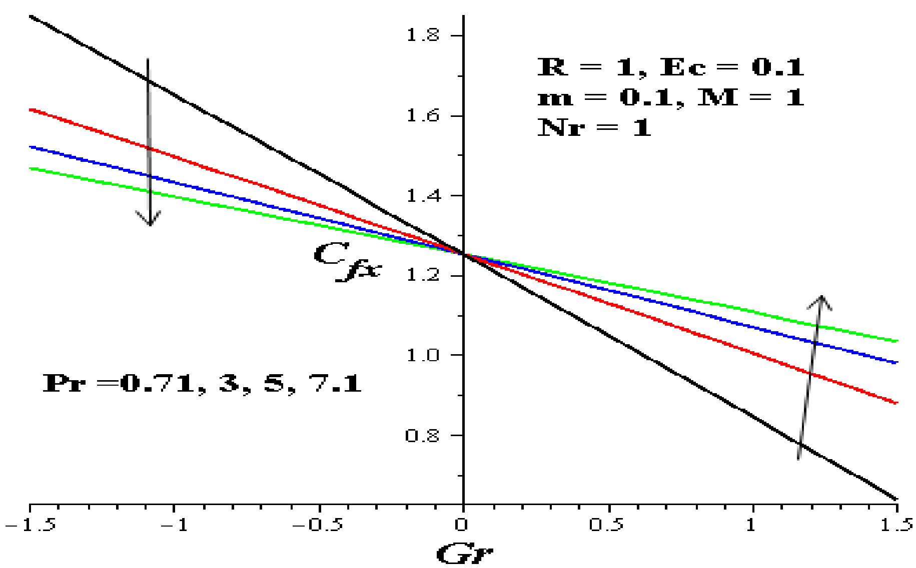

Figure 5. 24: Primary skin friction with increasing Pr.

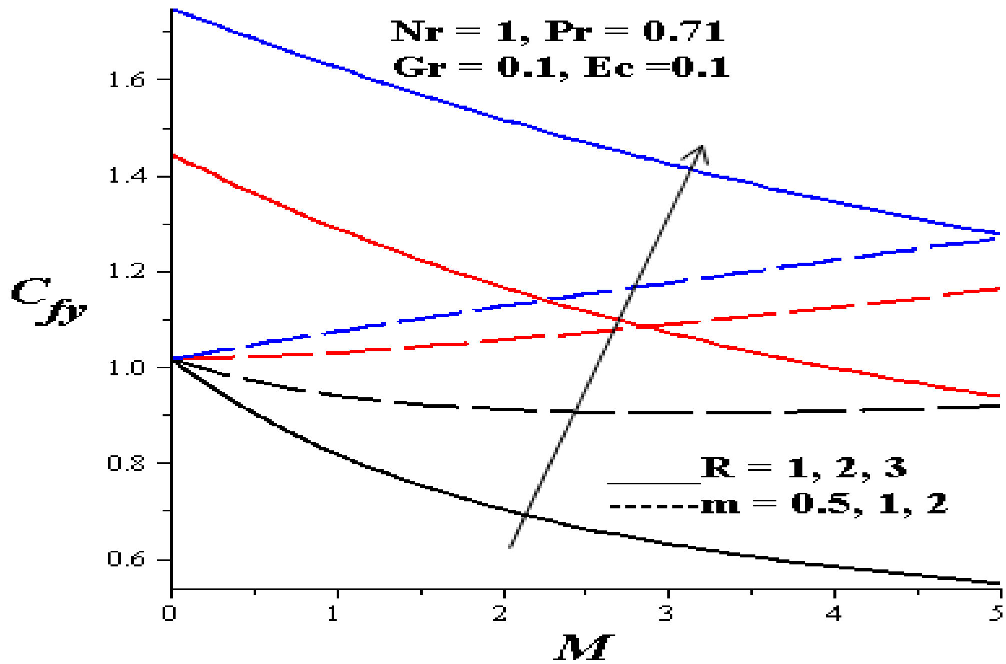

Figure 5. 25: Secondary skin friction with increasing M, R and m.

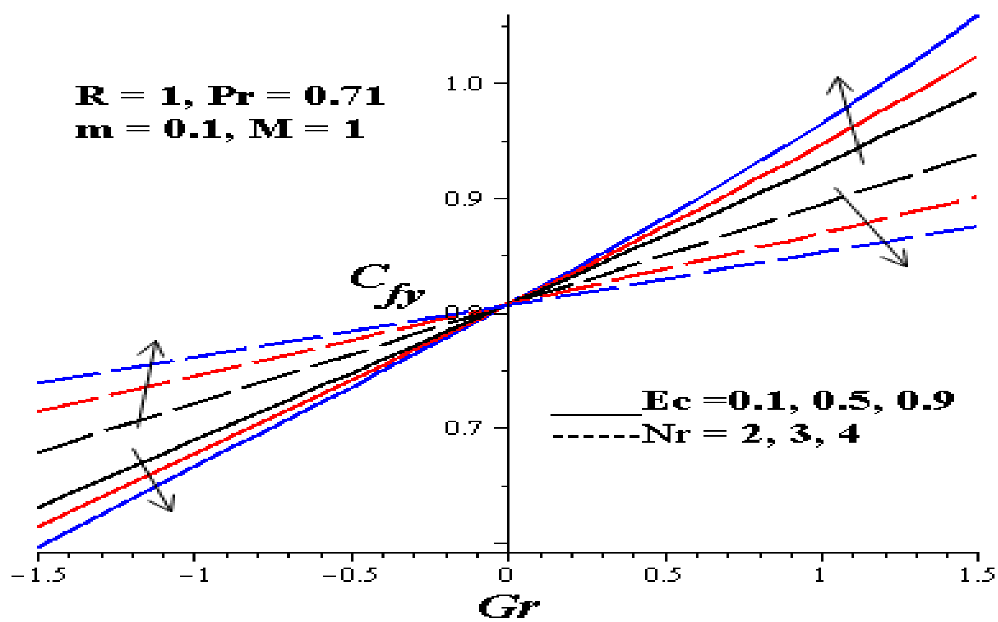

Figure 5. 26: Secondary skin friction with increasing Ec, Nr and Gr.

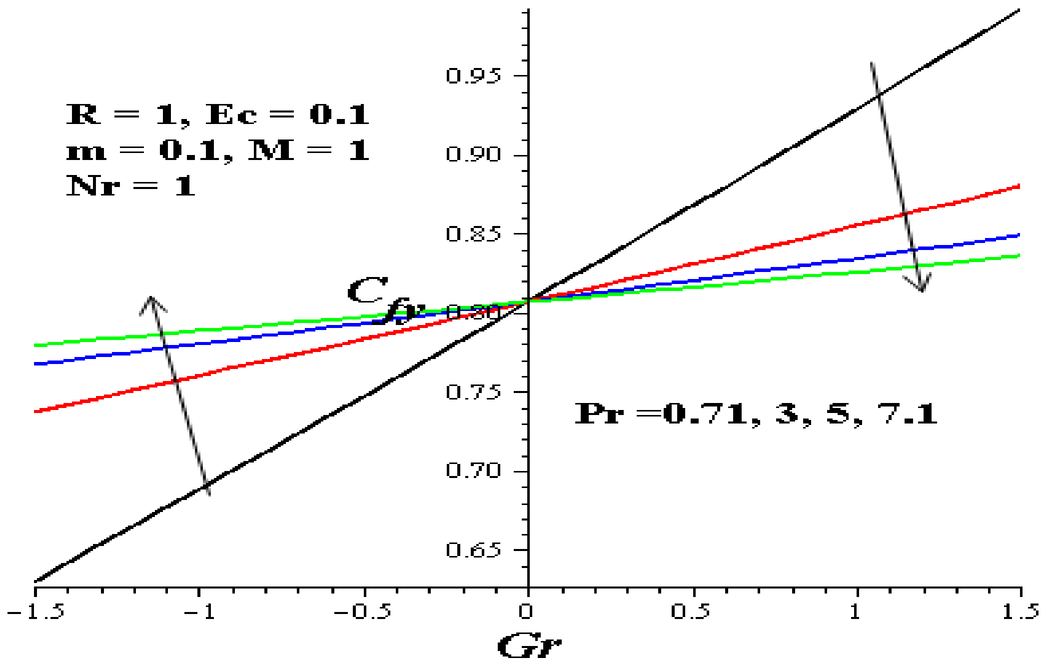

Figure 5. 27: Secondary skin friction with increasing Pr.

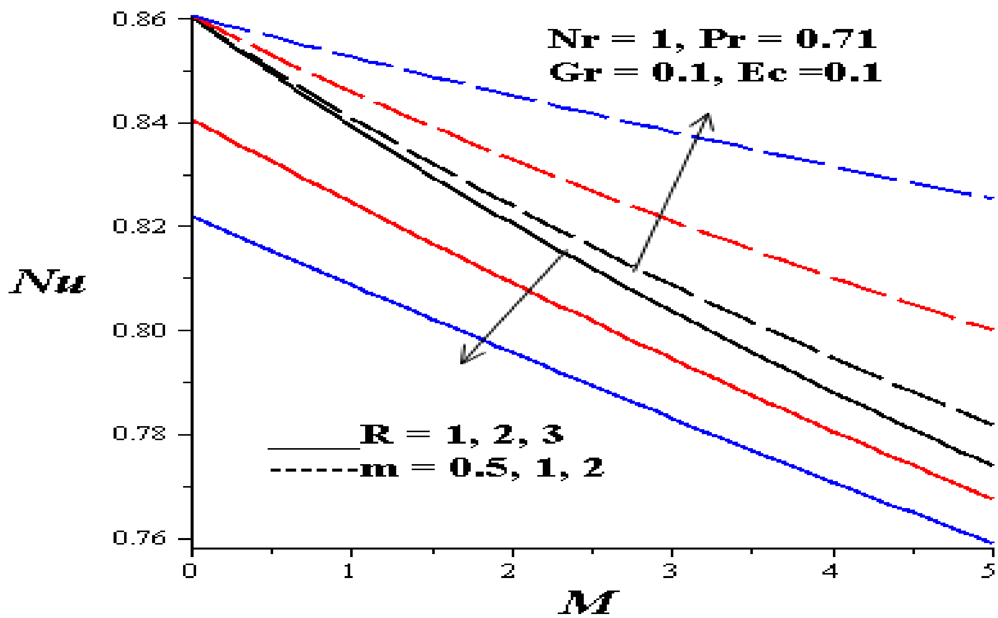

Figure 5. 28: Nusselt number with increasing M, R and m.

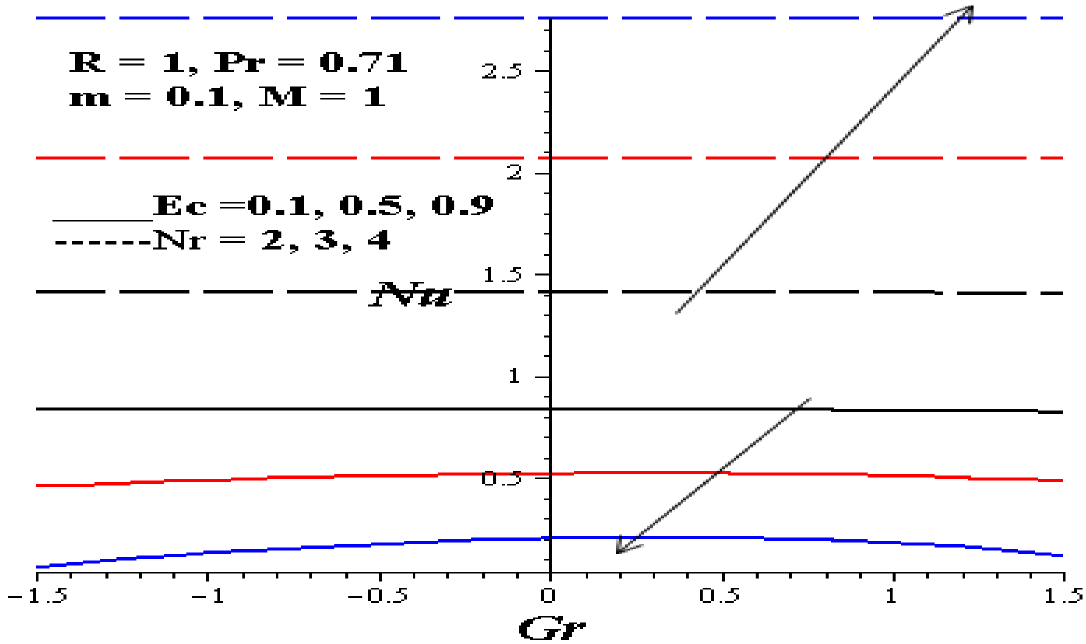

Figure 5. 29: Nusselt number with increasing Ec, Nr and Gr.

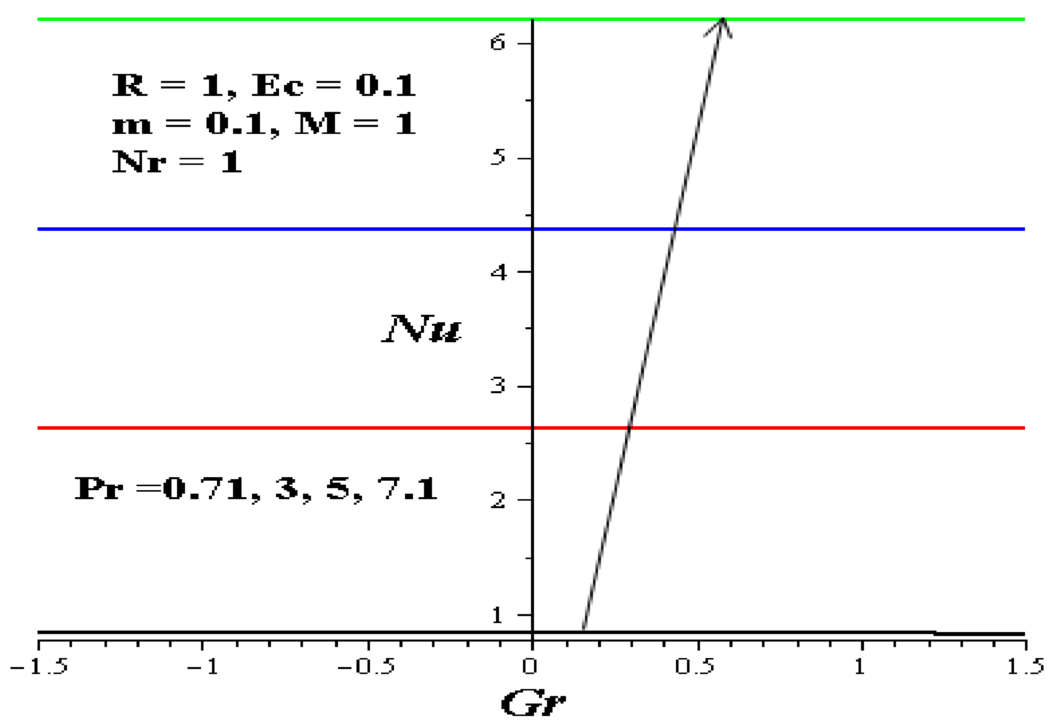

Figure 5. 30: Nusselt number with increasing Pr.

CHAPTER SIX

Conclusion and Recommendation

In this chapter, a conclusion is given with references to the results obtained in the preceding chapter. Recommendations are also given for further areas of study as well as published work.

6.1 Conclusions

We have analyzed the effect of various parameters on the velocity and temperature; moreover, the hydromagnetic turbulent flow of a conducting fluid over a moving plate in a rotating system with thermal radiation, viscous and Ohmic heating is numerically investigated. The effects of various thermophysical parameters on the mean velocity, mean temperature, skin friction and Nusselt number were obtained. Our results can be summarized as follows:

The primary mean velocity and momentum boundary layer increases with $Gr > 0$, Ec and decreases with R, M, m, Nr, Pr, $Gr < 0$. The secondary mean velocity shows flow reversal and momentum boundary layer increases with m, $Gr > 0$, Ec and decreases with R, Nr, M, Pr, $Gr < 0$. The mean temperature and thermal boundary layer increases with R, M, Ec, $Gr > 0$ and decreases with Nr, Pr, m, $Gr < 0$. The primary skin friction increases with M, R, Nr, Pr, $Gr < 0$ and decreases with m, Ec and $Gr > 0$. The secondary skin friction increases m, R, Ec, $Gr > 0$ and decreases M, Nr, Pr, $Gr < 0$. The Nusselt number increases with m, Nr, Pr increases and decreases with M, R, Ec, Gr.

It is evident that the results reported in the present model are in a good agreement with general trends. This follows the physical expectation of the effect of various parameters. As discussed in chapter five physical trends were eminent in the results.

6.2 Recommendation

Our research work is a significant contribution to the study of turbulance MHD and heat transfer. However its deemed there is much more that is yet to be studied and researched. We therefore recommended further analysis and development in

- A similar problem embedded in porous medium on it and fluid flow in three dimensions.
- Study of turbulent fluid flow where one or both plates are rotating in rotating system.
- Given an applied magnetic filed is inclined study both laminar and turbulent fluid flow.
- The study of MHD and heat transfer for a Coutte flow.

APPENDICES

Appendix I: Publications

[1]. Dawit H. Gebre, O. D. Makinde, M. Kinyanjui, Analysis of Turbulent Hydromagnetic Flow with Radiative Heat over a Moving Vertical Plate in a Rotating System, Applied and Computational Mathematics. Vol. 3, No. 3, 2014, pp. 100-109. doi: 10.11648/j.acm.20140303.15.

http://www.sciencepublishinggroup.com/journal/paperinfo.aspx?journalid=147&doi=10.11648/j.acm.20140303.15

[2]. RANS – Modeling of MHD Flow Over Infinite Vertical Plate in Rotating System Under the Effect of Viscous Dissipation, Joule Heating and Radiation.

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