ON THE ODD KUMARASWAMY INVERSE WEIBULL DISTRIBUTION WITH APPLICATION TO SURVIVAL DATA

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DECLARATION

This thesis is my original work and has not been presented for a degree award in any other university.

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DEDICATION

To my father Atem Manyuon Atem and mother Arok Akech Machar.

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ABBREVIATIONS AND ACRONYMS

AIC	Akaike Information Criterion
AICC	Corrected Akaike Information Criterion
BIC	Bayesian Information Criterion
CDF	Cumulative Distribution Function
EKIW	Exponentiated Kwamaraswamy Inverse Weibull
EPLG	Exponentiated power Lindley geometric
GE	Generalised Exponential
KIW	Kwamaraswamy Inverse Weibull
MGF	Moment Generating Function
MLEs	Maximum Likelihood Estimators/estimates
OGE	Odd Generalised Exponentiated family of distribution
OKIW	Odd Kwamaraswamy Inverse Weibull
PDF	Probability Distribution Function
RMSE	Root Mean Square Error

ABSTRACT

Probability distributions are very useful models for characterising inherent variability in lifetime data.The Weibull distribution is a widely used distribution in lifetime data analysis and hence has been modified many times to yield new distributions with greater flexibility. Modified forms of Weibull distribution are widely used in survival data analysis due to their versatility and relative simplicity. In this study, a new Odd Kumaraswamy inverse Weibull distribution is developed and its statistical properties are derived. The model contains several lifetime distributions as special submodels. The shapes of the probability density function and the hazard function are discussed. The model parameters are estimated using maximum likelihood method and a simulation to assess the performance of maximum likelihood estimators of the parameters is carried out.The average bias and root mean square error results from the simulation study decrease in terms of overall trend as the sample size increases indicating asymptotic consistency and unbiasedness of the estimators. The model is then applied to several survival data sets namely cancer patients data, guinea pigs data, glass fibres data, and Kelvar epoxy strand data to illustrate its flexibillity. Applications of the model to survival data empirically indicate its flexibility and usefulness in modeling various types of biomedical and reliability data and its superiority over three other lifetime distributions compared with the model in the study. The model may attract wider applications in survival analysis, reliability analysis, and insurance.

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Study of data is the most fundamental topic in statistics. Sometimes a simple exploratory study of data via descriptive measures such as graphical representation (histograms, bar plots, charts, etc.), measures of location and dispersion may seem adequate in helping us understand the sort of information the data conveys. However, one common and almost universal characteristic of data is their inherent variability. Probability distributions facilitate characterization of the variability and uncertainty prevailing in a data set by identifying the patterns of variation. Statistical probability distributions not only summarize the observations into a concise mathematical form containing a few parameters, but also provide means to analyze the basic structure that govern the data generating mechanism.

To describe (specify) the probability distribution of a random variable Y (say) we utilize the following concepts: cumulative distribution function ($F(y) = \Pr(Y \leq y)$), Probability density function ($f(y) = F'(y)$), quantile function ($Q(p) = F^{-1}(p)$), Quantile density function, Density Quantile ($f_p(p) = f_Y(Q(p))$. In classical statistics, cumulative distribution function (CDF) and probability density function (PDF) are the most popular techniques of defining most distributions in statistical theory and practice.

The objective of statistical modeling is finding appropriate probability distributions that adequately describe a data set generated by experiment, surveys, observational studies etc. In order to do this, there are two broad approaches viz deriving theoretical models from the basic assumptions and relations underlying the data and/or the data generating mechanism, and empirical modeling. The former makes assumptions about the physical characteristics governing data generating process and subsequently find a suitable model that satisfies such assumptions or by adapting existing models from other disciplines. The latter approach is data-dependent. It is suitable in situations where there is lack of understanding of the data generating process. The overarching goal of this approach is finding the best distributional approximation to the data by focusing attention on versatile families of distributions with enough parameters capable of producing different shapes and characteristics that match the features exhibited by the available observations.

In empirical statistical modeling, the challenge is finding a distribution function's parameter estimates that are as close as possible to the true values of the theoretical model parameters. Depending on the desired degree of accuracy several different modelling procedures that ensure proximity in the estimate and the true parameter value may be utilized. However, there is no single statistical distribution that is suitable for different data and so the need to extend existing distributions or develop new ones. Recent developments focus on defining new families of distributions that extend classical distributions and at the same time provide great versatility in modelling data. So, several techniques to generate new distributions by adding more parameters have been proposed. Most of the generalizations are developed for the following reasons: a physical or statistical theoretical argument to explain the mechanism of the generated data, an appropriate model that has previously been used successfully, and a model whose empirical fit is good to the data.

One of the most important distributions used in modelling lifetime data is the Weibull distribution whose cumulative distribution function (CDF) and probability density function (PDF) are respectively given by

$$F(x;\alpha,\beta) = 1 - \exp\left(-\frac{x}{\alpha}\right)^{\beta} \tag{1.1}$$

and

$$f(x;\alpha,\beta)=\frac{\beta}{\alpha^{\beta}}x^{\beta-1}\exp\left(-\frac{x}{\alpha}\right)^{\beta}, \tag{1.2}$$

where $x>0,\alpha>0$ and$\beta>0$, and α is a scale parameter and β is a shape parameter. Weibull distribution is an appropriate model for modelling failure times in instances where an item consists of numerous components and each component has an identical failure time distribution and the item fails when the weakest part fails (Liu, 1997). If X is a random variable following a Weibull distribution with parameters $\alpha>0,\beta>0$, under the transformation $\frac{1}{x}$, the inverse Weibull distribution proposed by Keller and Kamath [1982] is obtained. The CDF and PDF of the inverse Weibull distribution are respectively given by

$$G(x;\alpha,\beta)=\exp\left(-\frac{\alpha}{x}\right)^{\beta} \tag{1.3}$$

and

$$g(x;\alpha,\beta)=\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left(-\frac{\alpha}{x}\right)^{\beta} \tag{1.4}$$

where $x>0,\alpha>0$, and $\beta>0$. The inverse Weibull distribution has proven to be very useful in the modeling of lifetime data. Thus, it has been compounded with other continuous distributions to produce new more flexible lifetime distributions. For example, the Kumaraswamy inverse Weibull (KIW) distribution (Shahbaz et al., 2012). This is a generalisation of the inverse Weibull distribution based on the Kumaraswamy distribution. A random variable $X\sim KIW(\alpha,\beta,\lambda,\eta,)$ if its CDF is given by

$$F(x;\psi)=1-\left[1-\exp\left\{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right\}\right]^{\eta} \tag{1.5}$$

where $\psi=\{\alpha,\lambda,\beta,\eta\}$. Knowing the appropriate distribution a particular data set follow helps in making sound inference about the data. Limitations of classical statistical distributions in suitably modelling different data set motivated the need to extend existing distributions or develop new ones. Recent statistical advances focus on defining new families of models that extend well-known distributions and provide greater flexibility in modelling survival data arising in hordes of different field of survival analysis, insurance, medical and reliability engineering.

1.2 Statement of the Problem

Weibull distribution is popular in survival analysis due to its versatility to model lifetime data which exhibit monotone failure rates. However, in many practical situations, classical Weibull distribution fails to provide adequate fits to real life survival data such as machine life cycle, human mortality, and biomedical data which exhibit non-monotone failure rates. Recent statistical advances focus on defining new families of distributions that extend Weibull distribution to provide greater flexibility in modelling survival data arising in hordes of different fields of survival analysis, insurance, medical and reliability engineering. Existing techniques for generalising Weibull distributions such as Beta generators and Gamma generators involvle complexities of Beta-G distribution and Gamma-G distribution since they involve special functions such as Beta functions and incomplete Gamma. Recent alternative techniques deal with Kumaraswamy distribution which has similar properties as Beta-G but has advantage in terms of tractability. However, Kumaraswmay distribution is bounded on the interval $(0, 1)$ and consequently generalised forms of inverse Weibull distributions are also bounded on the unit interval hence limiting their applicability in survival data modelling. Furthermore, generalised distributions based on Gamma, Beta or Kumaraswamy generators are not flexible enough to adequately model survival data that exhibit non-monotone failure rates such as bathtub hazard shape and unimodality which are quite common in biological , human mortality, and reliability engineering studies. Proposing a new generalisation of Kumaraswamy inverse Weibull distribution is thus important in developing a model that is tractable and capable of modeling survival data with both monotone and non-monotone failure rates as well as enhanced flexibility of kurtosis and possibility of developing heavy-tailed distributions for modelling survival data.

1.3 Justification for the Study

Although the exisiting extended forms of Weibull distribution effectively provide better fits for unimodal survival data that exhibit monotone failure rates, they are not flexible enough to provide reasonable parametric fit for modelling phenomena with complex non-monotone failure rates such as bathtub and bimodal failure rates which are quite common in biological, medical, financial and reliability studies. Therefore, developing new highly versatile generalisation of the Weibull distribution with tractable cumulative distribution function is very important not only in statistical analysis of survival data characterised by both monotonic and non-monotonic failure rates but also in controlling skewness, kurtosis and tail variations of distributions common in insurance, finance, biomedical, and survival analysis applications.

1.4 General Objective

To propose and study the properties of odd Kumaraswamy inverse Weibull distribution and apply it to survival data.

1.5 Specific Objectives

1. To develop a new odd Kumaraswamy inverse Weibull distribution.
2. To derive the statistical properties of the new distribution.
3. To estimate the parameters of the new distribution using maximum likelihood method.
4. To assess the performance of the estimators using simulation.
5. To apply the new distribution to survival data sets.

1.6 Significance of the Study

Statistical lifetime distributions are widely employed in survival data modelling in a host of application areas such as reliability engineering, survival analysis, biomedical applications, insurance and social sciences. Particularly interesting are the applications of lifetime distributions in reliability engineering to estimate the survival time of electrical components, computation of time a patient takes to respond to a therapy, and modelling of the lifetime of marriages in social sciences. Numerous classical distributions have been extensively utilized over the past decades for modeling data in many areas of applied sciences. Essentially, there is a great need for extended forms of the classical distributions in many applied areas such as biology, insurance, finance, human mortality studies, medical, reliability and survival analysis owing to the fact that compounded distributions are more flexible to model real data. Generalized Weibull family of distributions such as OKIW distribution proposed in this study are more flexible and are capable of modeling real survival data that exhibit monotonic and non-monotonic hazard rates better than the classical Weibull distribution making them highly applicable in survival data analysis. Thus, it is expected that OKIW distribution may attract wider applications in the aforementioned areas.

1.7 Thesis Outline

The rest of the thesis is organised as follows. Chapter 2 presents literature review on modifications and extensions of Weibull, estimation techniques, and survival data. Chapter 3 presents the methodology for the study. Chapter 4 presents the study results and discussion. Finally, chapter 5 presents conclusions and recommendations of the study.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, a critical review of some of the techniques for developing extensions and modifications of distributions and parameters estimation based largely on some well-known generators is provided, clearly highlighting the limitations in existing modified forms of Weibull distribtuion and the need to adopt new extensions and generalisation.

2.2 Statistical Distributions and Survival Data Modelling

Statistical lifetime distributions are widely utilized in many different fields for modelling data sets. Some of the most common application areas include but not limited to: reliability engineering, survival analysis, duration analysis in economic, modelling of default rate for bank customers, and insurance such as the durations without claims of customers policies. Reliability is defined as the probability that a system or some process performs its prescribed duty without failure for some specified time if it is operated correctly in a given environment (Khan et al., 2008). In social sciences, a fascinating application of lifetime distributions is modelling the lifetime of marriages (Almalki and Nadarajah, 2014), that is, modelling time to divorce.

The hazard rate function plays a crucial role in lifetime modelling. A lifetime distribution is said to have a monotone hazard rate if its hazard function, $h(x)$, increases over time or decreases over time or remains approximately constant over time. Otherwise, a lifetime distribution is said to have a non-monotone hazard rate if $h(x)$ initially decreases, followed by an approximately constant period, then followed by an increasing period (bathtub failure rate) or if its hazard rate function has a unique mode (upside-down bathtub shape/unimodal). The different shapes of the hazard rate function can be examined using the first derivative of the hazard function.

Some of the most popular lifetime distributions include the exponential,Weibull, gamma, Pareto, and Rayleigh all of which have monotonic hazard rate functions (Lawless, 1982). However, certain lifetime data (e.g., human mortality, machine life cycles and data from some biological and medical studies) require non-monotonic shapes like the bathtub shape, the unimodal (upside-down bathtub) or modified unimodal shape.

2.3 Review of Modifications and Generalisation of Weibull Distribution

The Weibull distribution, introduced by Weibull [1951], is one of the most important lifetime distribution and has extensive appliciations in reliability, quality control and survival analysis. Its CDF is simple and so are the expressions for its survival and hazard functions. Its extended forms such as inverse Weibull distribution is commonly used to model a variety of failure characteristics: infant mortality, useful life and wear-out periods. Over the years, researchers have been developing various extensions and modified forms of the two-parameter inverse Weibull distribution that are more flexible than existing ones by adding more parameters to the distribution function in order to achieve non-monotonic hazard shapes and very versatile PDF (Bebbington et al., 2007; Mudholkar and Hutson, 1996; Zhang and Xie, 2011).

Khan [2010] introduced a new four-parameter reliability model of inverse Weibull distribution referred to as the beta inverse Weibull distribution generated from the logit of a beta random variable and Baharith et al. [2014] introduced an extension of the inverse Weibull distribution called the beta generalized inverse Weibull distribution by generalising beta inverse Weibull. These models have hazard rate that is monotonically decreasing and, for some of its special cases, an upside down bathtube shape. Models parameters were obtained via maximum likelihood estimation and finite sample performance of maximum likelihood estimators assessed by simulation.

Motivated by the intractability of cumulative distribution function of beta distribution due to the incomplete beta function ratio, Gusmao et al. [2011] defined a three-parameter generalized inverse Weibull distribution with rapidly decreasing and unimodal failure rate. While Khan and King [2012] proposed a generalised version of four-parameter modified inverse Weibull distribution with modified inverse exponential, modified inverse Rayleigh distribution as its special cases. The model has increasing and decreasing failure rate pattern for life time data making it more flexible in modelling survival data. Model parameters were obtained using maximum likelihood estimation and estimators' performance assessed via simulation. Shahbaz et al. [2012] introduced and studied a four-parameter inverse Weibull distribution with similar flexibility as Khan and King [2012] model but with fairly computational ease.

Owing to the difficulty that comes with distributions that involve the logit of the beta distribution, researchers have resorted to using other bounded distributions on the interval $(0, 1)$ to obtain the generalisation of any parent cumulative Weibull distribution function. Kumaraswamy [1980] is one such distribution, with distribution function given by

$$F(x; \alpha, \omega) = 1 - [1 - x^{\alpha}]^{\omega}, \tag{2.1}$$

and its pdf by,

$$f(x; \alpha, \omega) = \alpha\omega x^{\alpha-1}[1 - x^{\alpha}]^{\omega-1}, \tag{2.2}$$

where $\alpha > 0$ and $\omega > 0$. This density can be unimodal, increasing, decreasing or constant, thus making it a viable alternative to beta distribution.Cordeiro and Castro [2011] proposed to use the Kumaraswamy to generalize other distributions by considering a random variable X with a distribution G and applying the Kumarasawamy distribution to $G(x)$ to obtain a generalised Kumaraswamy$-G$ distribution. A similar idea is used to consider the distribution functions of Weibull and inverse Weibull distribution as candidates for G to obtain the Kumaraswamy-Weibull (Cordeiro et al., 2010) and Kumaraswamy inverse Weibull (Shahbaz et al., 2012) distributions. Besides, following a similar technique, Rodrigues et al. [2016] proposed a new distribution called the exponentiated Kumaraswamy inverse Weibull (EKIW), a generalisation of inverse Weibull distribution, which is more flexible than its predecessors and accommodate several special cases such as the inverse exponential, inverse Weibull, inverse Rayleigh and exponentiated Weibull distributions. The model parameters estimation was done via methods of moments and maximum likelihood estimation. However, generalisation via Kumaraswamy generator is limiting in that Kumaraswamy distribution is bounded on the interval $(0, 1)$ and that the consequent hazard functions does not exhibit highly desirable bathtub and modified bathtub failure rates in addition to limited flexibility of the resulting PDF.

2.4 Survival Data

Survival data consists of the time until an event of interest occurs and usually the censoring information for each individual or component. Thus, of great interest in this study will be characterising the distribution of "time to event" for a given population. Survival times of some individuals might not be fully observed due to different reasons: either the survival study stops before full survival times of all individuals can be observed; a subject drops out of a study, or a subject is lost to follow-up, subjects survives beyond the time of the study, etc. During a survival study either the individual is observed to fail at time X, or the observation on that individual ceases at time k. Then the observation is $\min(T, k)$ and an indicator variable I_k showing if the individual is censored or not. Therefore, estimators for hazard and survival functions must be adjusted to account for censoring (Lawless, 1982). This study intends to illustrate the applicability and flexibility of the proposed model using complete samples of survival data. Due to time constrained for the study, we wish to consider application of the proposed model to censored survival data as a subsequent separate work.

2.5 Summary of Literature Review

In this chapter, a review of various modified forms of inverse Weibull distribution is provided together with parameters estimation methods and versatility in terms of lifetime data modelling. The foregoing modified forms of inverse Weibull distribution effectively provide better fits for unimodal survival data that exhibit monotonically increasing, monotonically decreasing and constant failure rates, but they fail to provide reasonable parametric fit for phenomena with non-monotone failure rates such as bathtub and modified failure rates which are quite common in biological and reliability studies. Besides, Kumaraswmay distribution is bounded on the interval $(0, 1)$ and consequently generalised forms of inverse Weibull distributions are also bounded on the unit interval hence limiting their applicability in survival data modelling. There is a need for a new generalisation of the Kumaraswamy inverse Weibull distribution to enhance the versatility of its PDF and tractability of its CDF and also to construct heavy-tailed distributions for financial and insurance applictions since the research in this area has not thus far tackle this limitation. Proposing a new generalisation of Kumaraswamy inverse Weibull distribution is critical in providing a tractable model capable of modeling lifetime data with both monotone and non-monotone failure rates as well as enhanced flexibility of kurtosis and possibility of developing heavy-tailed distributions for modelling survival data.

CHAPTER 3

METHODOLOGY

3.1 Introduction

In this section, a technique for developing a new distribution is discussed in addition to candidate estimation procedure clearly highlighing the underlying philosophy in the estimation approach.The study introduces key concepts regarding survival data and provide a conceptual framework for survival data modelling, theoretical background and the methods behind the analysis of the survival data.

3.2 The Odd Generalized Exponential family of distributions

Some attempts have been made to define new classes of distributions to extend well-known families of distributions and at the same time provide great flexibility in modeling data in practice. Most of the generalizations are developed for one or more of the following reasons: a physical or statistical theoretical argument to explain the mechanism of the generated data, an appropriate model that has previously been used successfully, and a model whose empirical fit to the data is good. Most of the techniques for extending classical distributions involve addition of single shape parameter to the baseline distributions such as in exponentiated-G class of distributions, i.e., a two-parameter generalised-exponential (GE) distribution which is an extension of the exponential distribution.

The generalised-exponential family of distributions is widely applied in analysing lifetime data which exhibit monotonic failure rate but is limited if the failure rate is upside-down, J or reversed-J, bathtub or modified bathtub shapes.

Alternatively, more effective generalisation techniques involve argumenting a baseline distribution with multiple shape and scale parameters. Tahir et al. [2015] proposed a new family of continuous distributions called the odd generalized exponential family (OGE-G). This new family is very versatile and highly flexible because the hazard rate shapes could be increasing, decreasing, J, reversed-J, bathtub and upside down bathtub. A random variable X is said to have generalised exponential (GE) distribution with paramters α, λ if its CDF is given by

$$F(x;\alpha,\lambda) = \left(1 - \exp^{-\lambda x}\right)^{\alpha}$$

for $x > 0, \alpha > 0, \lambda > 0$.

The OGE-G family of distributions has found wider application in applied statistics owing to the basic motivations to make the kurtosis more flexible than is in the baseline model; produce a skewness for symmetrical distributions; construct heavy-tailed distributions; and to generate distributions with symmetric, left-skewed, right-skewed, and reversed-J shaped.

The OGE-G family is defined as follows. Let $G(x;\boldsymbol{\varphi})$ be the CDF of any distribution which depends on parameter(s) $\boldsymbol{\varphi}$ and thus the survival function is $\overline{G}(x;\boldsymbol{\varphi}) = 1 - G(x;\boldsymbol{\varphi})$, then the CDF of OGE-family is defined by

$$F(x;\alpha,\lambda,\boldsymbol{\varphi}) = \left(1 - \exp - \lambda \frac{G(x;\boldsymbol{\varphi})}{\overline{G}(x;\boldsymbol{\varphi})}\right)^{\alpha}, \ x > 0; \alpha, \boldsymbol{\varphi}, \lambda > 0. \tag{3.1}$$

Where $\lambda > 0, \alpha > 0$ are scale and shape paramters respectively. The pdf corresponding to (3.1) is given by

$$f(x;\alpha,\lambda,\boldsymbol{\varphi}) = \frac{\lambda\alpha g(x;\boldsymbol{\varphi})}{\overline{G}(x;\boldsymbol{\varphi})^2} \exp - \lambda \frac{G(x;\boldsymbol{\varphi})}{\overline{G}(x;\boldsymbol{\varphi})} \left(1 - \exp - \lambda \frac{G(x;\boldsymbol{\varphi})}{\overline{G}(x;\boldsymbol{\varphi})}\right)^{\alpha - 1}, \tag{3.2}$$

where $g(x;\boldsymbol{\varphi})$ is the baseline pdf.

The hazard function, $h(x)$, is the instantaneous rate at which events occur given no previous events (instantaneous failure rate), where

$$h(x) = \lim_{\Delta x \to 0} \frac{\Pr(x < X \leq x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{S(x)}.$$

Thus, the hazard function of X is given by

$$h(x; \alpha, \lambda, \boldsymbol{\varphi}) = \frac{\lambda \alpha g(x; \boldsymbol{\varphi}) \exp -\lambda \frac{G(x;\boldsymbol{\varphi})}{\overline{G}(x;\boldsymbol{\varphi})}}{\overline{G}(x; \boldsymbol{\varphi})^2 \left\{1 - \left(1 - \exp -\lambda \frac{G(x;\boldsymbol{\varphi})}{\overline{G}(x;\boldsymbol{\varphi})}\right)^{\alpha}\right\}} \left(1 - \exp -\lambda \frac{G(x; \boldsymbol{\varphi})}{\overline{G}(x; \boldsymbol{\varphi})}\right)^{\alpha - 1} \tag{3.3}$$

The OGE-G family of distributions can be illustrated as follows. Let Y be a lifetime random variable having a continuous CDF $G(x; \boldsymbol{\varphi}))$. The odds function/ ratio that a component following the lifetime Y will fail at time x is $\frac{G(x;\boldsymbol{\varphi}))}{\overline{G}(x;\boldsymbol{\varphi}))}$. As an illustration, let's consider the variability of this odd failure as represented by the random variable X having, say, a Weibull distribution with a scale σ and shape parameter ω. Then, we have

$$Pr(Y \leq x) = Pr\left(X \leq \frac{G(x; \boldsymbol{\varphi}))}{\overline{G}(x; \boldsymbol{\varphi}))}\right) = F(x; \sigma, \omega, \theta).$$

The Weibull-G density function becomes

$$f(x; \sigma, \omega, \theta) = \sigma \omega g(x; \theta) \left[\frac{G(x; \boldsymbol{\varphi}))^{\omega - 1}}{\overline{G}(x; \boldsymbol{\varphi}))^{\omega + 1}}\right] \exp\left\{-\sigma \left[\frac{G(x; \boldsymbol{\varphi}))}{\overline{G}(x; \boldsymbol{\varphi}))}\right]^{\omega}\right\}$$

That is, $X \sim Weibull - G(\sigma, \omega, \theta)$. If $\omega = 1$, it corresponds to the exponential generator.

3.3 Kumaraswamy Inverse Weibull Distribution

The Kumaraswamy inverse Weibull (KIW) distribution was proposed and studied by Shahbaz et al. [2012]. This is a generalisation of the inverse Weibull distribution based on the Kumaraswamy distribution. A random variable $X \sim$

$KIW(\alpha, \beta, \lambda, \eta,)$ if its CDF is given by

$$F(x; \psi) = 1 - \left[1 - \exp\left\{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right\}\right]^{\eta} \tag{3.4}$$

where $\psi = \{\alpha, \lambda, \beta, \eta\}$.

The study intends to propose a new distribution by generalising Kumaraswamy inverse Weibull distribution (3.4) dubbed a new Odd Kumaraswamy Inverse Weibull distribution via univariate distributions generator by Tahir et al. [2015] in order to obtain distributions with large class of sub-models and which show higher flexibility as well as wider applicability.

3.4 Maximum Likelihood Estimation

The Maximum Likelihood Estimation (MLE) is a method of estimating the parameters of a statistical model by selecting the set of values of the model parameters that maximizes the likelihood function. Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ be a vector of random variables from one of a class of distributions on $\mathbb{R}^n$ and indexed by a p-dimensional parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)^T$ where $\boldsymbol{\theta} \in \Omega \subset \mathbb{R}^p$ and $p \leq n$. Let $F(\boldsymbol{X}|\boldsymbol{\theta})$ be the distribution function of $\boldsymbol{X}$ and that the joint density function $f(x_1, x_2, ..., x_n|\boldsymbol{\theta})$ exists. Then the likelihood of $\boldsymbol{\theta}$ is the function

$$L(\boldsymbol{\theta}) = f(x_1, x_2, ..., x_n|\boldsymbol{\theta})$$

which is the probability of observing the given data as a function of $\boldsymbol{\theta}$. The maximum likelihood estimates (MLEs) of $\boldsymbol{\theta}$ are those values of $\boldsymbol{\theta}$ that maximise the likelihood function, i.e., the value(s) that make(s) the observed data the most probable. If the $\mathbf{X} = (X_1, X_2, ..., X_n)$ are iid, then the likelihood simplies to

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i|\boldsymbol{\theta}).$$

By MLEs, various statistics are built for assessing the goodness-of-fit of a model, such as: Akaike Information Criterion(AIC), Bayesian Information Criterion (BIC),

3.5 Methods of Evaluating Maximum Likelihood Estimators

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the sampling model $f(x/\theta)$, where θ is an unknown parameter. An estimator of θ obtained by method such as maximum likelihood estimation and method of moment, is a function of the sample, i.e., a statistic $\hat{\theta} = T(X_1, X_2, \cdots, X_n)$. To study the quality of an estimator or asymptotic properties of the estimator, Mean square error and bias (equivalently root mean square error and average bias).

3.5.1 Mean Square Error of an Estimator

Let $\hat{\theta}$ be the estimator of the unknown parameter θ from the random sample $X_1, X_2, \cdots, X_n$. Then the deviations from $\hat{\theta}$ to the true θ, $|\hat{\theta} - \theta|$, measures the quality or performance of the estimator. That is, the mean square error (MSE) of an estimator $\hat{\theta}$ of a parameter θ is the function of θ defined by

$$MSE_{\hat{\theta}} = \mathbb{E}(\hat{\theta} - \theta)^2 = Var(\hat{\theta}) + (\mathbb{E}(\hat{\theta}) - \theta)^2 = Var(\hat{\theta}) + (Bias(\hat{\theta}))^2 \tag{3.5}$$

. The expectation in (3.5) is with respect to the random variables $X_1, X_2, \cdots, X_n$ since they are the only random components in the expression. The sequence of estimators $\{\hat{\Theta_n}\}$ is weakly consistent or equivalently MSE consistent if $\hat{\Theta_n} \to \theta$ in probability as $n \to \infty$. That is, $\forall \epsilon > 0$, if $n \to \infty$

$$\mathbb{P}\left(|\hat{\Theta_n} - \theta| > \epsilon\right) \to 0. \tag{3.6}$$

Equivalently, anda sequence of estimators $\hat{\Theta_n}$ is weakly consistent if

$$\lim_{n\to\infty} MSE(\hat{\Theta_n}) = 0. \tag{3.7}$$

That is , if the number of observations increase the MSE descend to zero.

3.5.2 Bias of an Estimator

The bias of an estimator $\hat{\theta}$ of the parameter θ is the difference between the expected value of $\hat{\theta}$ and θ. That is,

$$Bias(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta \tag{3.8}$$

An estimator is unbiased if $\mathbb{E}(\hat{\theta}) = \theta, \forall\theta$. For an unbiased estimator $\hat{\theta}$,

$$MSE_{\hat{\theta}} = \mathbb{E}(\hat{\theta} - \theta)^2 = Var(\hat{\theta}) \tag{3.9}$$

and so, if an estimator is unbiased, its MSE is equal to its variance. The sequence of estimators $\{\hat{\Theta_n}\}$ is asymptotically unbiased if $\mathbb{E}(\hat{\Theta_n}) \to \theta$ as $n \to \infty$.

3.6 Model Comparison and Model Selection Criteria

To demonstrate the applicability and flexibility of our proposed model in modelling real data, we compare its performance with other existing competing models in terms of information lost. Essentially, a comparison of different model-selection approaches' ability to detect a true model involves a trade-off between goodness of fit and model's parsimony. So, we employ information criteria techniques and goodness-of-fit statistics that penalize model for complexity, to keep the model from overfitting, to assess the best model from a variety of alternative models which may have different number of parameters. The most commonly used information criteria are the Akaike information criterion (AIC), corrected Akaike information criterion (AICC) and the Bayesian information criterion (BIC). The information criterion selects model with smaller values of AIC, AICC, and BIC for a given set of candidate models and specified data set.

The Akaike information criterion(AIC) (Akaike, 1974) measures the quality of statistical models for a given data set. It quantifies information lost when the data generating process is represented by a statistical model by obtaining an equilibrium in the trade-off between goodness-of-fit of the model and its complexity. Suppose we have a statistical model of some data x. Let p be the number of estimated parameters in the model and $\hat{L}$ the maximum value of the model's likelihood function. That is, $\hat{L} = P(x/\hat{\boldsymbol{\theta}})$ where $\hat{\boldsymbol{\theta}}$ are the parameter values that maximise the likelihood function. Then the AIC is given by

$$AIC = 2p - 2\log(\hat{L}).$$

AIC rewards goodness of fit, but it also includes a penalty (to minimises overfitting) that is an increasing function of the number of estimated parameters.

AICC (Hurvich and Tsai, 1989) is AIC with a correction for finite sample sizes defined as follows:

$$AICC = AIC + \frac{2p(p+1)}{(n-p-1)}$$

where n is the number of observations, and p is the number of estimated parameters.That is, AICC is essentially AIC with a greater penalty for extra parameters. It is recommended to use AICC if the sampel size is not large or when the model has too many parameters (Anderson, 2002).

The **Bayesian information criterion** (BIC) (Schwarz, 1978) is a technique for model selection among a finite set of models. When fitting models, it is possible to increase the likelihood by adding parameters, but with trade-off for overfitting. Both BIC and AIC attempt to resolve this problem by introducing a penalty term for the number of parameters in the model; the penalty term is larger in BIC than in AIC. The BIC is defined as

$$BIC = \log(n)p - 2\log(\hat{L})$$

where $\hat{L}$ is the maximized value of the model likelihood function, n is the sample size, p is the number of parameters to be estimated.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 The Odd Kumaraswamy Inverse Weibul Distribution

Let $G(x;\boldsymbol{\zeta})$ be any baseline CDF of any distribution which depends on parameter(s) $\boldsymbol{\zeta}$, then the survival function is given by $\overline{G}(x;\boldsymbol{\zeta}) = 1 - G(x;\boldsymbol{\zeta})$. The CDF of OGE-family of distributions (Tahir et al., 2015) is defined by

$$F(x;\omega,\theta,\boldsymbol{\zeta}) = \left(1 - \exp -\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right)^{\omega},\ x>0;\omega,\boldsymbol{\zeta},\theta>0, \tag{4.1}$$

where $\theta > 0, \omega > 0$ are additional scale and shape parameters respectively. The PDF corresponding to (4.1) is given by

$$f(x;\omega,\theta,\boldsymbol{\zeta}) = \frac{\theta\omega g(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})^2}\exp -\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\left(1 - \exp -\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right)^{\omega-1}, \tag{4.2}$$

where $g(x;\boldsymbol{\zeta})$ is the corresponding baseline PDF. Thus, the hazard function of X is given by

$$h(x;\alpha,\theta,\boldsymbol{\zeta}) = \frac{\theta\alpha g(x;\boldsymbol{\zeta})\exp -\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}}{\overline{G}(x;\boldsymbol{\zeta})^2\left\{1 - \left(1 - \exp -\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right)^{\alpha}\right\}}\left(1 - \exp -\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right)^{\alpha-1} \tag{4.3}$$

We define a new five-parameter distribution dubbed Odd Generalised Exponentiated Kumaraswamy inverse Weibull distribution (henceforth Odd KIW or OKIW). The CDF of OKIW follows from (4.1) and (3.4) by taking $G(x;\boldsymbol{\zeta})$ to be Equation (3.4) and $g(x;\boldsymbol{\zeta})$ to be the PDF corresponding to (3.4) with $\boldsymbol{\zeta} = \{\alpha,\lambda,\beta,\eta,\}$ and also taking $\theta = 1$ in (4.1) so that we utilise a *standard* OGE-family generator.

Consequently, the CDF of OKIW becomes

$$F(x;\boldsymbol{\zeta},\omega) = \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega}, \quad x > 0 \tag{4.4}$$

where $\alpha > 0, \lambda > 0, \beta > 0, \eta > 0$, and $\omega > 0$. Here, λ, α are scale parameters and β, η, ω are shape parameters.

Proposition 4.1.1. *Equation (4.4) is a well-defined distribution function of the random variable X.*

Proof

If Equation (4.4) is a CDF of a random variable X, then the following conditions hold:

i). Limiting values:

a). $\lim_{x\to+\infty} F_X(x) = \lim_{x\to+\infty} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega} = 1$

b). $\lim_{x\to-\infty} F_X(x) = 0$, i.e, $\lim_{x\to 0} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega} = 0$

ii). Monotonicity: Suppose that $x_1 \le x_2$. Then, $\{X \le x_1\} \subset \{X \le x_2\}$, which implies that

$$F(x_1) = \mathbb{P}(X \le x_1) \le \mathbb{P}(X \le x_2) = F(x_2).$$

A particular case is confirmed in (i) above, i.e., $F_X(x)$ is monotonically increasing between 0 and 1.

iii). Right-continuity: For every x, we have $\lim_{x_2 \downarrow x_1} F_X(x_2) = F_X(x_1)$.
That is, for any x and any decreasing sequence $(x_n, n \ge 1)$, that converges to x, $\lim_{n\to\infty} F(x_n) = F(x)$. Consider a decreasing sequence x_n converging to x, then the sets $E_n = \{X \le x_n\}$ also form a decreasing sequence with

$$\{X \le x\} = \bigcap^{\infty} E_n.$$

Consequently,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mathbb{P}(E_n) = \lim_{n\to\infty} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_n}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega} = \mathbb{P}\{X \leq x\} = F(x)$$

by the continuity properties of the measures(propabilities). Since this is true for every such sequence $\{x_n\}$, we therefore conclude that $\lim\limits_{x_2 \downarrow x_1} F_X(x_2) = F_X(x_1)$.

This completes the proof. □

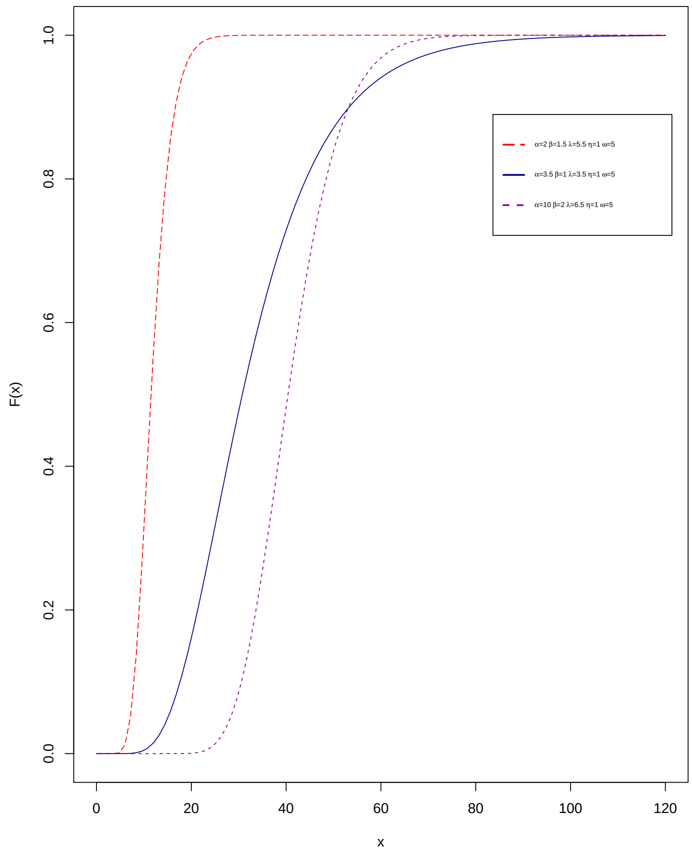

Figure 4.1: Plot of the OKIW CDF for some parameters values.

The plots of OKIW CDF (Figure 4.1) show a monotonic non-decreasing shape bounded between 0 and 1 which is typical of any CDF.

4.1.1 Quantile and Median of OKIW Distribution

For a random variable X with CDF F, the **inverse CDF** or **quantile function** is defined by

$$F^{-1}(q) = \inf\{x : F(x) > q\}, \qquad q \in [0,1].$$

If F is strictly increasing and continuous, then $F^{-1}(q)$ is the unique real number x such that $F(x) = q$. So, the quantile of (4.4) is given by solving $F(x_q) = q$, thus yielding

$$x_q = \alpha \left[\frac{-1}{\lambda} \left\{ \log \left(1 - \left[1 - \log \left(1 - q^{\frac{1}{\omega}} \right) \right]^{\frac{-1}{\eta}} \right) \right\} \right]^{\frac{-1}{\beta}} \tag{4.5}$$

From Equation (4.5) it is clear that the quantile is tractable(in closed form) and simulation can be performed fairly easily on this distribution since a random sample can be generated from (4.5) by using p as uniform random number.

In survival data analysis, the data are often skewed and oftentimes the median maybe preferable to mean as a measure of centrality. From (4.5) we can obtain the median of OKIW distribution as follows by substituting $q = \frac{1}{2}$ to get

$$Median = \alpha \left[\frac{-1}{\lambda} \left\{ \log \left(1 - \left[1 - \log \left(1 - \frac{1}{2}^{\frac{1}{\omega}} \right) \right]^{\frac{-1}{\eta}} \right) \right\} \right]^{\frac{-1}{\beta}} \tag{4.6}$$

4.1.2 Survival Function, PDF and Hazard Rate Function of OKIW

In biomedical applications and insurance problems, it is often common to use the survival function to describe the distribution of survival time. If the random variable X denotes survival time and $F_X(x)$ represents the CDF or the probability of failure by time x, then the *survival function* is defined as

$$S_X(x) = \mathbb{P}(X > x) = 1 - F_X(x).$$

That is, the survival function is the probability of survival beyond time x. Survival function is used to predict the quantiles of the survival time,e.g., the median survival time (say, x_{50}) or mean residual life time maybe of interest. The survival function of $X \sim OKIW$ is given by

$$S(x;\boldsymbol{\varphi}) = 1 - \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega} \tag{4.7}$$

and the PDF of OKIW follows from Equation (4.2) and (3.4) and is given by

$$\begin{aligned} f(x;\boldsymbol{\varphi}) &= \beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}\left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}} \\ &\times \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta-1} \end{aligned} \tag{4.8}$$

where $x > 0; \boldsymbol{\varphi} = \alpha, \lambda, \beta, \eta, \omega$; $\alpha > 0, \lambda > 0, \beta > 0, \eta > 0, \omega > 0$, and λ, α are scale parameters and β, η, ω are shape parameters. The graph of the density function for various values of the parameters is given in Figure 4.2.

The plots indicate that the OKIW PDF (Figure 4.2) can be decreasing or right skewed or symmetric(approximately), exponentially bounded tail, fat tail with highly flexible kurtosis, hence capable of handling variety of data from insurance and finance, survival analysis, biomedical data, reliability analysis.

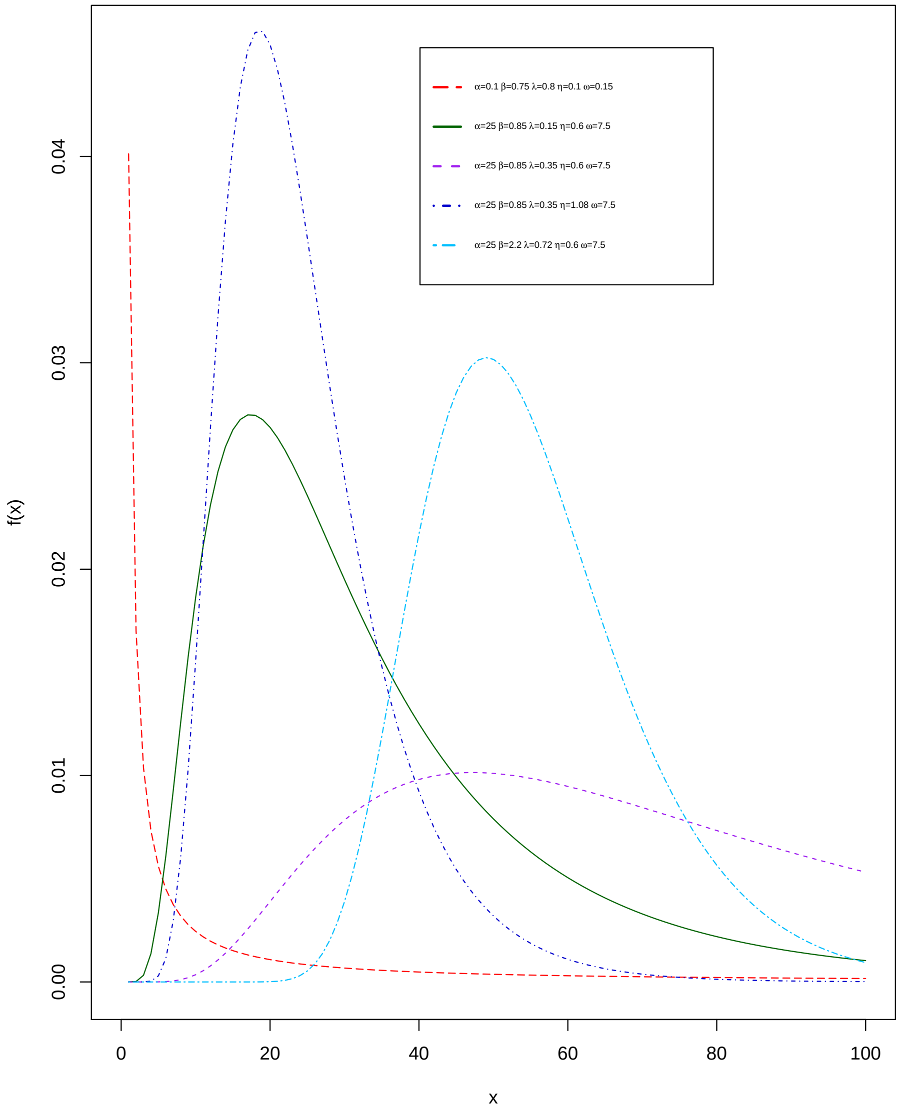

Figure 4.2: Plot of OKIW density for some parameters values.

The hazard function, $h(x)$, is the instantaneous rate at which events occur given no previous events (instantaneous failure rate), where

$$h(x) = \lim_{\Delta x \to 0} \frac{\Pr(x < X \leq x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{S(x)}.$$

The hazard rate function follows from Equation (4.3) and is thus given by

$$h(x;\boldsymbol{\varphi}) = \frac{\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}}{1-\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{(\eta+1)}} \tag{4.9}$$

where $x > 0$ and $\boldsymbol{\varphi} = \{\alpha, \lambda, \beta, \eta, \omega\}$.

The graphs of the hazard rate function for different values of the parameters (Figure 4.3) exhibits various shapes such as monotonically increasing, bathtub shape, constant and increasing-decreasing almost linearly, monotonically decreasing, constant and exponential increasing, and upside down bathtub shapes. These are very attractive features that render the OKIW distribution suitable for modelling monotonic and non-monotonic hazard behaviours which are more likely to be encountered in practical situations like reliability analysis, human mortality and biomedical applications, thus enhancing its adaptability to fit diverse survival data.

4.1.3 Special Models

Sub-models of OKIW distribution for selected values of the parameters are presented below.

1. Reducing IW we obtain Odd inverse exponential (for $\beta = 1$) with pdf given by

$$f_{OIE}(x;\varphi,\omega) = \eta\lambda\omega\alpha x^{-2}\left[1-e^{1-\left\{1-e^{-\lambda\left(\frac{\alpha}{x}\right)}\right\}^{-\eta}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)}\right]^{-\eta-1}$$

2. Reducing KW, we obtain:

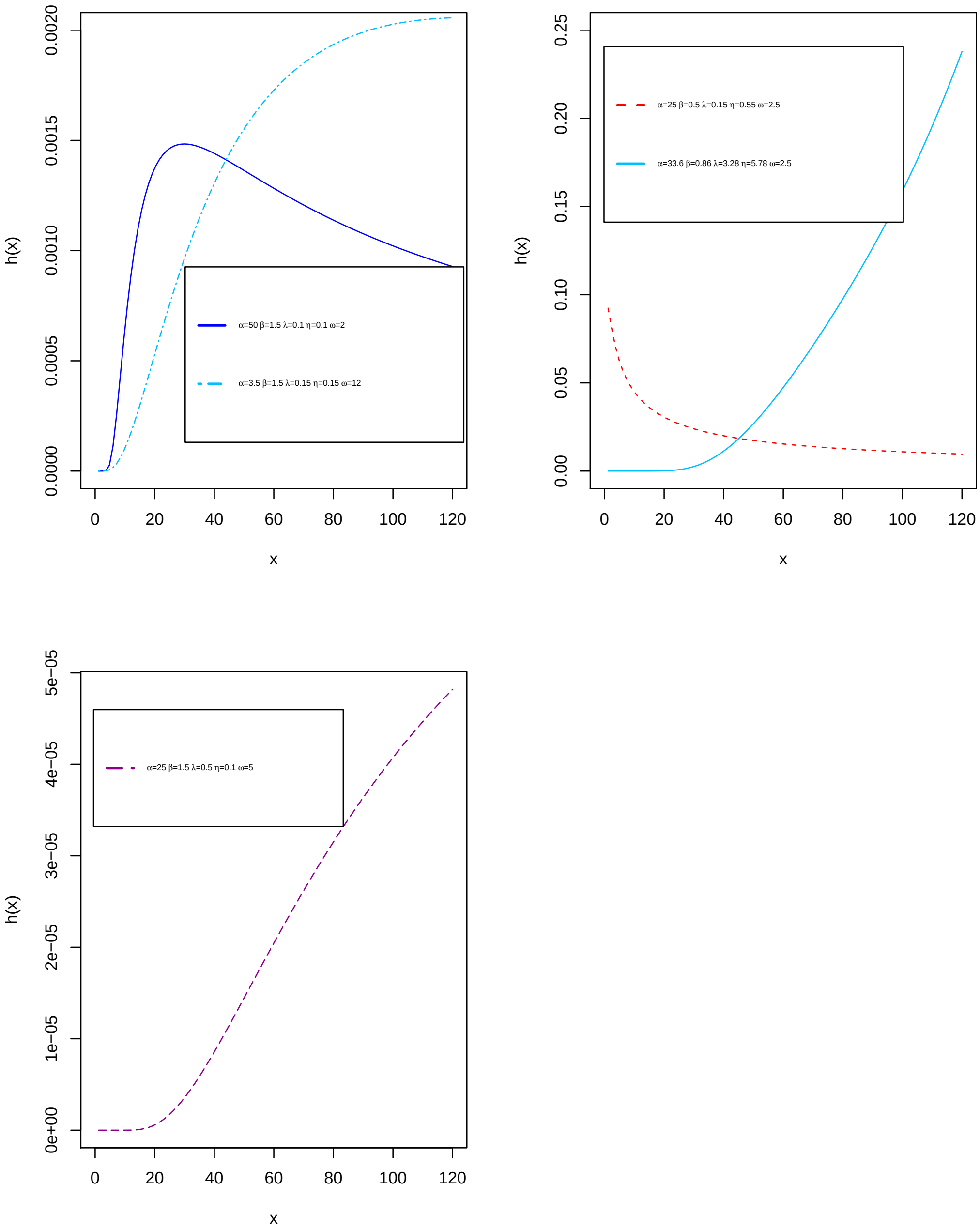

Figure 4.3: Plot of the hazard function for some parameters values.

(a) (for $\eta = 1$), Odd exponentiated inverse Weibull with pdf given by

$$\begin{aligned} f_{OEIW}(x;\varphi,\omega) &= \beta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}\left[1-e^{1-\left\{1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right\}^{-1}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-1}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}} \\ &\times \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-2} \end{aligned}$$

(b) for ($\beta = 1, \eta = 1$), Odd exponentiated inverse exponential with pdf given by

$$\begin{aligned} f_{OEIE}(x;\varphi,\omega) &= \lambda\omega\alpha x^{-2}\left[1-e^{1-\left\{1-e^{-\lambda\left(\frac{\alpha}{x}\right)}\right\}^{-1}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)}\right]^{-1}-\lambda\left(\frac{\alpha}{x}\right)} \\ &\times \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)}\right]^{-2}. \end{aligned}$$

4.2 Statistical Properties

4.2.1 Moments

Moments of a distribution are important in statistical inference. They are used to study the most important features and characteristics of a distribution (e.g., measures of central tendency, measures of dispersion, skewness and kurtosis). In this subsection, we derive the *rth* moments of the OKIW($\boldsymbol{\varphi}$) distribution.

Proposition 4.2.1. *If $X \sim OKIW(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \{\alpha, \beta, \lambda, \eta, \omega\}$, then the rth non-central moment of X is given by*

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)}\frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{(\frac{r}{\beta}-1)}\Gamma(k+j\eta+\eta+1)\Gamma(1-\frac{r}{\beta})}{j!k!\Gamma(j\eta+\eta+1)}.$$

Proof. The rth moment of a random variable X with pdf $f(x;\boldsymbol{\varphi})$ is defined by

$$\mu_r^{'} = \int_0^{\infty} x^r f(x;\boldsymbol{\varphi})dx. \tag{4.10}$$

substituting from (4.8) into (4.10), we get

$$\mu_r^{'} = \int_0^\infty x^r \beta\eta\lambda\omega\alpha^\beta x^{-(\beta+1)} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\right\}^{-\eta}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta}$$
$$\times \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta-1} dx. \tag{4.11}$$

Since $0 < 1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\right\}^{-\eta}} < 1$, we have by binomial expansion

$$\left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\right\}^{-\eta}}\right]^{\omega-1} = \sum_{i=0}^\infty \binom{\omega-1}{i} (-1)^i e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}}$$

Substituting back into the integral above, we have

$$\mu_r^{'} = \int_0^\infty \beta\eta\lambda\omega\alpha^\beta x^r x^{-(\beta+1)} \sum_{i=0}^\infty \binom{\omega-1}{i} (-1)^i e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}}$$
$$\times\ e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta} \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta-1} dx.$$

Grouping exponential terms and applying power series expansion yields

$$e^{-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}} e^{-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}} = e^{-(i+1)\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}} = \sum_{j=0}^\infty \frac{(-1)^j (i+1)^j}{j!}$$
$$\times \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-j\eta},$$

so,

$$\mu_r^{'} = \int_0^\infty \beta\eta\lambda\omega\alpha^\beta x^r x^{-(\beta+1)} \sum_{i=0}^\infty \binom{\omega-1}{i} (-1)^i e^i \sum_{j=0}^\infty \frac{(-1)^j (i+1)^j}{j!} \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-j\eta}$$
$$\times \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta-1} e^{1-\lambda\left(\frac{\alpha}{x}\right)^\beta} dx. \tag{4.12}$$

By generalised binomial expansion for negative powers, we have

$$\left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-j\eta} \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta-1} = \sum_{k=0}^\infty \frac{\Gamma(k+j\eta+\eta+1)}{k!\Gamma(j\eta+\eta+1)} e^{-\lambda k\left(\frac{\alpha}{x}\right)^\beta}.$$

Hence, the integral becomes

$$\mu_r^{'} = \int_0^\infty \beta\eta\lambda\omega\alpha^\beta x^r x^{-(\beta+1)} \sum_{i=0}^\infty \binom{\omega-1}{i}(-1)^i e^{-(i+1)} \sum_{j=0}^\infty \frac{(-1)^j(i+1)^j}{j!} \times \sum_{k=0}^\infty \frac{\Gamma(k+j\eta+\eta+1)}{k!\Gamma(j\eta+\eta+1)} e^{-\lambda(k+1)(\frac{\alpha}{x})^\beta} dx. \quad (4.13)$$

Setting $u = \lambda(k+1)\alpha^\beta x^{-\beta} \Rightarrow \frac{du}{dx} = (-\beta)\lambda\alpha^\beta(k+1)x^{-(\beta+1)}$ and $\lambda(k+1)\alpha^\beta x^{-\beta}|_0 = \infty$ and $\lambda(k+1)\alpha^\beta x^{-\beta}|_\infty = 0$ and $x = \left[\frac{u}{\lambda\alpha^\beta(k+1)}\right]^{\frac{-1}{\beta}}$, thus,

$$\mu_r^{'} = MD\int_\infty^0 \left[\frac{u}{\lambda\alpha^\beta(k+1)}\right]^{\frac{-r}{\beta}} x^{-(\beta+1)}e^{-u}\frac{du}{(-\beta)\lambda\alpha^\beta(k+1)x^{-(\beta+1)}} \Rightarrow$$

$$\mu_r^{'} = \frac{MD}{\beta\alpha^\beta\lambda(k+1)}\left[\alpha^\beta\lambda(k+1)\right]^{\frac{r}{\beta}}\int_0^\infty u^{\frac{-r}{\beta}}e^{-u}du = \frac{MD}{\beta\alpha^\beta\lambda(k+1)}\left[\alpha^\beta\lambda(k+1)\right]^{\frac{r}{\beta}}\Gamma(1-\frac{r}{\beta}), \quad r<\beta,$$

by the definition of gamma function in the form $\Gamma(\phi) = \int_0^\infty u^{\phi-1}e^{-u}du$ where $M = \beta\eta\lambda\omega\alpha^\beta$ and $D = \sum_{i=0}^\infty \binom{\omega-1}{i}(-1)^i e^{-(i+1)} \sum_{j=0}^\infty \frac{(-1)^j(i+1)^j}{j!} \sum_{k=0}^\infty \frac{\Gamma(k+j\eta+\eta+1)}{k!\Gamma(j\eta+\eta+1)}$.
Subsituting back M and D in the equation above and simplifying we have

$$\mu_r^{'} = \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty (-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)}\frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{(\frac{r}{\beta}-1)}\Gamma(k+j\eta+\eta+1)}{j!k!\Gamma(j\eta+\eta+1)} \times \frac{\Gamma(1-\frac{r}{\beta})}{j!k!\Gamma(j\eta+\eta+1)}.$$

This completes the proof .

□

4.2.2 Moment Generating Functions

The expected values $E(X), E(X^2), E(X^3), \cdots$, and $E(X^r)$ are called the moments. The mean $\mu = E(X)$ and the variance $\sigma^2 = Var(X) = E(X^2) - \mu^2$, which are functions of the moments, are sometimes difficult to find. **Moment-**

generating functions are special functions used to find the moments and functions of moments such as mean and variance of a random variable in a much simpler way and also aid in identifying which probability mass function or PDF a random variable X follows. Besides, they provide an easy way of characterizing the distribution of the sum of independent random variables and provide tools for dealing with the distribution of the sum of a random number of independent random variables.They also play a central role in the study of branching processes in stochastic processes and genetics.

Definition 4.2.1. The moment generating function(MGF) associated with a continuous random variable X, if it exists, is a function $M_X : \mathbb{R} \to [0, \infty]$ defined by

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \int_{-\infty}^{+\infty} e^{tx} f(x)dx,$$

where $\mathbb{D}_{M_X} = \{t : M_X(t) < \infty\}$ and the integral absolutely converges for some interval of t in the neighborhood of 0 for $-h < t < h$.

Proposition 4.2.2. *If $X \sim OKIW(\boldsymbol{\varphi}), where \quad \boldsymbol{\varphi} = \{\alpha, \beta, \lambda, \eta, \omega\}$, then the MGF of X is given by*

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} t^r(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)} \\ &\times \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{(\frac{r}{\beta}-1)}\Gamma(k+j\eta+\eta+1)\Gamma(1-\frac{r}{\beta})}{j!k!r!\Gamma(j\eta+\eta+1)}. \end{aligned}$$

Proof. From the foregoing definition, we have

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \int_{-\infty}^{+\infty} e^{tx} f(x)dx.$$

That is,

$$
\begin{aligned}
M_X(t) &= \int_0^{+\infty} e^{tx}\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} \\
&\times\; e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta-1}dx.
\end{aligned}
$$

By power series expansion of MGF, we have

$$
\begin{aligned}
M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{'} &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)} \\
&\times\; \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)\Gamma(1-\frac{r}{\beta})}{j!k!\Gamma(j\eta+\eta+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
\therefore M_X(t) &= \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}t^r(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)} \\
&\times\; \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)\Gamma(1-\frac{r}{\beta})}{j!k!r!\Gamma(j\eta+\eta+1)}, \\
r<\beta.
\end{aligned}
$$

This completes the proof. □

4.2.3 Distribution of Order Statistics

Order statistics are key tools in non-parametric statistics and inference. They result from transformation that involves the *ordering* of an entire set of observations on a random variable. Since order statistics have wide applications in many areas of statistics, it is important to derive some commonly required order statistics distributions for the OKIW distribution.

Let $X_1, X_2, \cdots, X_n$ be *iid* forming a simple random sample of size n from $OKIW(\boldsymbol{\varphi})$ distribution with cumulative distribution function $F(x;\boldsymbol{\varphi})$ and PDF $f(x;\boldsymbol{\varphi})$. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics obtained from the sample. The PDF of s^{th} order statistic, for $s = 1, \cdots, n$, is given by

$$f_{s:n}(x;\boldsymbol{\varphi}) = \frac{1}{\mathrm{B}(s, n-s+1)} \left[F(x;\boldsymbol{\varphi})\right]^{s-1} \left[1 - F(x;\boldsymbol{\varphi})\right]^{n-s} f(x;\boldsymbol{\varphi}) \qquad (4.14)$$

where B(., .) denotes a beta function. Since $0 < F(x;\boldsymbol{\varphi}) < 1$ for $x > 0$, we have

$$[1 - F(x;\boldsymbol{\varphi})]^{n-s} = \sum_{m=0}^{n-s} \binom{n-s}{m} (-1)^m [F(x;\boldsymbol{\varphi})]^m \qquad (4.15)$$

Thus, substituting Equation (4.16) into Equation (4.15), we obtain

$$f_{s:n}(x;\boldsymbol{\varphi}) = \frac{1}{\mathrm{B}(s, n-s+1)} f(x;\boldsymbol{\varphi}) \sum_{m=0}^{n-s} \binom{n-s}{m} (-1)^m [F(x;\boldsymbol{\varphi})]^{m+s-1}. \qquad (4.16)$$

And finally substituting Equation (4.4) and (4.8) into (4.17), we obtain

$$\begin{aligned} f_{s:n}(x;\boldsymbol{\varphi}) &= \frac{\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}}{\mathrm{B}(s, n-s+1)} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} \\ &\times e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}} \left[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta-1} \\ &\times \sum_{m=0}^{n-s} \binom{n-s}{m} (-1)^m \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega(m+s-1)}. \end{aligned}$$

4.2.4 Distribution of Extreme Order Statistics

Let $X_1, X_2, \cdots, X_n$ be *iid* forming a simple random sample of size n from $OKIW(\boldsymbol{\varphi})$ distribution with cumulative distribution function $F(x;\boldsymbol{\varphi})$ and PDF $f(x;\boldsymbol{\varphi})$. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics obtained from the sample. The PDF of the *largest* order statistic is

$$f_{X_n}(x) = n\left[F_X(x;\boldsymbol{\varphi})\right]^{n-1} f_X(x;\boldsymbol{\varphi}). \qquad (4.17)$$

Utilising Equation (4.4) and (4.8) in (4.18) and simplifying, we obtain the PDF of the *largest* order statistic

$$\begin{aligned} f_{X_n}(x) &= n\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega n-1} \\ &\times \; e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta-1}. \end{aligned}$$

The PDF of the smallest order statistic is defined by

$$f_{X_1}(x) = n\left[1-F_X(x;\boldsymbol{\varphi})\right]^{n-1} f_X(x;\boldsymbol{\varphi}). \tag{4.18}$$

Since $0 < F(x;\boldsymbol{\varphi}) < 1$ for $x > 0$, we have by binomial expansion

$$\left[1-F_X(x;\boldsymbol{\varphi})\right]^{n-1} = \sum_{t=0}^{n-1}\binom{n-1}{t}(-1)^t[F(x;\boldsymbol{\varphi})]^t.$$

And so

$$f_{X_1}(x) = n\sum_{t=0}^{n-1}\binom{n-1}{t}(-1)^t[F(x;\boldsymbol{\varphi})]^t f_X(x;\boldsymbol{\varphi}). \tag{4.19}$$

Utilising Equation (4.4) and (4.8) in (4.20) and simplifying, we obtain the PDF of the *smallest* order statistic

$$\begin{aligned} f_{X_1}(x) &= n\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}\sum_{t=0}^{n-1}\binom{n-1}{t}(-1)^t\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega t} \\ &\times \left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta-1}. \end{aligned}$$

4.2.5 Entropy

Entropy is an index for measuring variation or uncertainty of a random variable. It is an important concept in many fields of science especially theory of communication, physics and reliability. Two popular entropy measures are Rényi entropy

(Rényi, 1961) and Shannon entropy (Shannon, 1951). A large value of the entropy indicates a greater uncertainty in the data. Rényi entropy is defined as follows . If a random variable X has a pdf $f(.)$, then the v order Renyi entropy is defined by

$$E_R(v) = \frac{1}{1-v}\log\left[\int_0^\infty f^v(x)dx\right] \tag{4.20}$$

where $v > 0$ and $v \neq 1$. The Shanon entrophy is a special case of the Rényi entrophy when $v \to 1$ and is given by $\mathbb{E}[-\log(f(x))]$.

Proposition 4.2.3. *If $X \sim OKIW(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \{\alpha, \beta, \lambda, \eta, \omega\}$, then its Rényi entropy, $E_R(v)$, is given by*

$$\begin{aligned} E_R(v) &= \frac{1}{1-v}\left(\ln\left(\frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}\left(\lambda\alpha^\beta(v+k)\right)^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\right)\right) \\ &+ \frac{1}{1-v}\left(\ln\left(\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma\left(k+v(\eta+1)+j\eta\right)}{k!\Gamma\left(v(\eta+1)+j\eta\right)}\right)\right) \\ &+ \frac{1}{1-v}\left(\ln\left(\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)\right)\right). \end{aligned}$$

Proof. From Equation(4.21), we have that

$$\begin{aligned} \int_0^\infty f^v(x)dx &= \int_0^\infty (\beta\eta\lambda\omega)^v\alpha^{v\beta}x^{-v(\beta+1)}\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\right\}^{-\eta}}\right]^{v(\omega-1)} \\ &\times e^{v\left(1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-v(\eta+1)}dx. \end{aligned} \tag{4.21}$$

Since $0 < 1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\right\}^{-\eta}} < 1$, we have by binomial expansion for positive integer powers

$$\left[1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\right\}^{-\eta}}\right]^{v(\omega-1)} = \sum_{i=0}^\infty\binom{v(\omega-1)}{i}(-1)^i e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}},$$

So substituting back into the integral above, we have

$$\int_0^\infty f^\gamma(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} x^{-v(\beta+1)} \sum_{i=0}^\infty \binom{v(\omega-1)}{i} (-1)^i e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}} \times e^{v\left(1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)} \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-v(\eta+1)} dx. \quad (4.22)$$

Grouping the exponent terms and applying power series expansion, we obtain

$$e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}} e^{v\left(1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)} = e^{(v+i)} e^{-v\lambda\left(\frac{\alpha}{x}\right)^\beta} e^{-(v+i)\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}}$$
$$= e^{(v+i)} e^{-v\lambda\left(\frac{\alpha}{x}\right)^\beta} \times \sum_{j=0}^\infty \frac{(-1)^j (v+i)^j}{j!} \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-j\eta} \quad (4{,}23)$$

So,

$$\int_0^\infty f^\gamma(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} x^{-v(\beta+1)} \sum_{i=0}^\infty \binom{v(\omega-1)}{i} (-1)^i e^{(v+i)} e^{-v\lambda\left(\frac{\alpha}{x}\right)^\beta} \times \sum_{j=0}^\infty \frac{(-1)^j (v+i)^j}{j!} \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-j\eta} \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-v(\eta+1)} dx \quad (4.24)$$

Now, by generalised binomial expansion for negative powers, we have

$$\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-j\eta} \left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-v(\eta+1)} = \sum_{k=0}^\infty \frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)} e^{-k\lambda\left(\frac{\alpha}{x}\right)^\beta} \quad (4.25)$$

Hence, substituting back and regrouping exponent terms,the integral becomes

$$\int_0^\infty f^\gamma(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} \sum_{i=0}^\infty \binom{v(\omega-1)}{i} (-1)^i e^{(v+i)} \sum_{j=0}^\infty \frac{(-1)^j (v+i)^j}{j!} \times \sum_{k=0}^\infty \frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)} x^{-v(\beta+1)} e^{-\lambda(v+k)\left(\frac{\alpha}{x}\right)^\beta} dx. \quad (4.26)$$

Letting $u = \lambda(v+k)\alpha^\beta x^{-\beta} \Rightarrow du = (-\beta)\lambda\alpha^\beta(v+k)x^{-(\beta+1)}dx$ and

$\lambda(v+k)\alpha^{\beta}x^{-\beta}|_0 = \infty$ and $\lambda(v+k)\alpha^{\beta}x^{-\beta}|_\infty = 0$ and $x = \left[\frac{u}{\lambda\alpha^{\beta}(v+k)}\right]^{\frac{-1}{\beta}}$, thus,

$$\begin{aligned}
\int f^v(x)dx &= MD\int_{\infty}^{0} x^{-v(\beta+1)}e^{-u}\frac{du}{(-\beta)\lambda\alpha^{\beta}(v+k)x^{-(\beta+1)}}\\
\Rightarrow \int f^v(x)dx &= \frac{MD^-}{(v+k)}\int_0^{\infty} x^{-v(\beta+1)+\beta+1}e^{-u}du\\
\Rightarrow \int f^v(x)dx &= \frac{MD^-}{(v+k)}\int_0^{\infty}\left[\frac{u}{\lambda\alpha^{\beta}(v+k)}\right]^{\frac{-1}{\beta}\{-v(\beta+1)+\beta+1\}} e^{-u}du\\
\Rightarrow \int f^v(x)dx &= \frac{MD^-\left(\lambda\alpha^{\beta}(v+k)\right)^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\int_0^{\infty} u^{\frac{-1}{\beta}\{-v(\beta+1)+\beta+1\}}e^{-u}du
\end{aligned}$$

where $D = (\beta\eta\lambda\omega)^v\alpha^{v\beta}$ and $D^- = \frac{D}{\beta\lambda\alpha^{\beta}} = (\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}$ and

$$M = \sum_{i=0}^{\infty}\binom{v(\omega-1)}{i}(-1)^i e^{(v+i)}\sum_{j=0}^{\infty}\frac{(-1)^j(v+i)^j}{j!}\sum_{k=0}^{\infty}\frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)}.$$

So, invoking the definition of gamma function in the form $\Gamma(\phi) = \int_0^{\infty} u^{\phi-1}e^{-u}du$, we obtain

$$\int f^v(x)dx = \frac{MD^-\left(\lambda\alpha^{\beta}(v+k)\right)^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)$$

or

$$\begin{aligned}
\int f^v(x)dx &= \frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}\left(\lambda\alpha^{\beta}(v+k)\right)^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\\
&\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma\left(k+v(\eta+1)+j\eta\right)}{k!\Gamma\left(v(\eta+1)+j\eta\right)}\\
&\times \Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right),\\
&\quad \frac{1}{\beta}\{-v(\beta+1)+\beta+1\} < 1.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
\ln\left(\int f^v(x)dx\right) &= \ln\left(\frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}\left(\lambda\alpha^{\beta}(v+k)\right)^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\right)\\
&+ \ln\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma\left(k+v(\eta+1)+j\eta\right)}{k!\Gamma\left(v(\eta+1)+j\eta\right)}\right)\\
&+ \ln\left(\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\therefore E_R(v) &= \frac{1}{1-v}\left(\ln\left(\int f^v(x)dx\right)\right)\\
&= \frac{1}{1-v}\left(\ln\left(\frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}\left(\lambda\alpha^{\beta}(v+k)\right)^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\right)\right)\\
&+ \frac{1}{1-v}\left(\ln\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma\left(k+v(\eta+1)+j\eta\right)}{k!\Gamma\left(v(\eta+1)+j\eta\right)}\right)\right)\\
&+ \frac{1}{1-v}\left(\ln\left(\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)\right)\right).
\end{aligned}
$$

This completes the proof.

□

4.3 Estimation of Model Parameters

In this subsection, we present estimates of the parameters of the model via method of maximum likelihood estimation. The elements of the score function are presented. There are no closed form solutions to the nonlinear equations obtained by setting the elements of the score function to zero.

4.3.1 The Maximum Likelihood Estimators

Let $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)^T$ be a random sample from OKIW distribution with unknown parameter vector $\Theta = (\alpha, \beta, \lambda, \eta, \omega)^T$, then the likelihood function $L(\boldsymbol{X}, \Theta)$ is defined as

$$L(\boldsymbol{X}; \Theta) = \prod_{i=1}^{n} f(x_i; \Theta).$$

Substituting from (4.8), we obtain

$$
\begin{aligned}
L(\boldsymbol{X};\Theta) &= \prod_{i=1}^{n} \beta\eta\lambda\omega\alpha^{\beta} x_i^{-(\beta+1)} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} \\
&\times \; e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left[1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right]^{-\eta-1}, \qquad (4.27)
\end{aligned}
$$

or

$$
\begin{aligned}
L(\boldsymbol{X};\Theta) &= (\beta\eta\lambda\omega\alpha^{\beta})^{n} \prod_{i=1}^{n} x_i^{-(\beta+1)} \left[1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right]^{\omega-1} \\
&\times \; e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left[1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right]^{-\eta-1}.
\end{aligned}
$$

The log-likelihood function for Θ is

$$
\begin{aligned}
\ln(L(\boldsymbol{X};\Theta)) &= n\ln(\beta\eta\lambda\omega\alpha^{\beta}) - (\beta+1)\sum_{i=1}^{n}\ln x_i - (\eta+1)\sum_{i=1}^{n}\ln\left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right) \\
&+ \; (\omega-1)\sum_{i=1}^{n}\ln\left(1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right) \\
&+ \; \sum_{i=1}^{n}\left(1 - \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)., \qquad (4.28)
\end{aligned}
$$

By maximising the log-likelihood function above, we obtain the components of the score function vector $U(\Theta) = \frac{\partial \ln L}{\partial \alpha}, \frac{\partial \ln L}{\partial \beta}, \frac{\partial \ln L}{\partial \lambda}, \frac{\partial \ln L}{\partial \eta}, \frac{\partial \ln L}{\partial \omega}$ which are given by:

$$\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \frac{n\beta}{\alpha} + \beta\lambda \sum_{i=1}^{n} \frac{\eta e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta-1} - \left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i \left(1 - \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} - \lambda \left(\frac{\alpha}{x_i}\right)^{\beta}\right)} \\
&- \beta\eta\lambda(\omega - 1) \sum_{i=1}^{n} \frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i \left(1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right)} \\
&- \beta\lambda(\eta + 1) \sum_{i=1}^{n} \frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)},
\end{aligned} \tag{4.29}$$

$$\frac{\partial \ln L}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n} \ln \left(1 - e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right), \tag{4.30}$$

$$\begin{aligned}
\frac{\partial \ln L}{\partial \eta} &= \frac{n}{\eta} - \sum_{i=1}^{n} \ln \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right) + \sum_{i=1}^{n} \frac{\left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} \ln \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)}{1 - \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} - \lambda \left(\frac{\alpha}{x_i}\right)^{\beta}} \\
&- (\omega - 1) \sum_{i=1}^{n} \frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}} \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} \ln \left(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)}{1 - e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}}},
\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{\eta e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta} - \left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \\
&- (\eta+1)\sum_{i=1}^{n} \frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}} \\
&- \eta(\omega-1)\sum_{i=1}^{n} \frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}}}, \qquad (4.32)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} + n\ln(\alpha) - \sum_{i=1}^{n} \ln x_i - \lambda(\eta+1)\sum_{i=1}^{n} \left(\frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right) \\
&+ \sum_{i=1}^{n} \left(\frac{-\lambda\ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta} + \eta\lambda e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)} ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right) \\
&- \lambda\eta(\omega-1)\sum_{i=1}^{n} \left(\frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)} \ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}}\right), \qquad (4.33)
\end{aligned}
$$

The normal equations whose simultaneously solutions give the MLEs are:

$$\frac{n\beta}{\alpha}+\beta\lambda\sum_{i=1}^{n}\frac{\eta e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)}\left(\frac{\alpha}{x_i}\right)^{\beta-1}-\left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i\left(1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}-$$

$$\beta\eta\lambda(\omega-1)\sum_{i=1}^{n}\frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)}\left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i\left(1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right)}-$$

$$\beta\lambda(\eta+1)\sum_{i=1}^{n}\frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)}=0, \tag{4.34}$$

$$\frac{n}{\omega}+\sum_{i=1}^{n}\ln\left(1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}\right)=0, \tag{4.35}$$

$$\frac{n}{\eta}-\sum_{i=1}^{n}\ln\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)+\sum_{i=1}^{n}\frac{\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}\ln\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)}{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}-$$

$$(\omega-1)\sum_{i=1}^{n}\frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}\ln\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)}{1-e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}}}=0, \tag{4.36}$$

$$\frac{n}{\lambda}+\sum_{i=1}^{n}\frac{\eta e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)}\left(\frac{\alpha}{x_i}\right)^{\beta}-\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}-$$

$$(\eta+1)\sum_{i=1}^{n}\frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}}-$$

$$\eta(\omega-1)\sum_{i=1}^{n}\frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)}\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}}}=0, \qquad (4.37)$$

$$\frac{n}{\beta}+n\ln(\alpha)-\sum_{i=1}^{n}\ln x_i-\lambda(\eta+1)\sum_{i=1}^{n}\left(\frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}}\right)+$$

$$\sum_{i=1}^{n}\left(\frac{-\lambda\ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}+\eta\lambda e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)}ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)-$$

$$\lambda\eta(\omega-1)\sum_{i=1}^{n}\left(\frac{e^{1-\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-\eta}-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\left(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)^{-(\eta+1)}\ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-e^{1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\right\}^{-\eta}}}\right)=0, (4.38)$$

The MLE of $(\alpha, \beta, \lambda, \eta)$, and ω can be obtained by solving numerically(via iterative methods as is demonstrated in the application to the data sets) the normal equations

$$\frac{\partial \ln L}{\partial \alpha}=0, \frac{\partial \ln L}{\partial \beta}=0, \frac{\partial \ln L}{\partial \lambda}=0, \frac{\partial \ln L}{\partial \eta}=0, \frac{\partial \ln L}{\partial \omega}=0$$

, thus yielding the ML estimate: $\hat{\mathbf{\Theta}}=\{\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\eta}, \hat{\omega}\}$.

4.4 Monte Carlo Simulation Study

In this section, a simulation study is conducted to assess the performance (stability of point estimates) of OKIW distribution by examining the average bias and root mean square error of the maximum likelihood estimates for each parameter. Various simulations are conducted for different sample sizes and different parameter values. Equation (4.5) (**Quantile**) is used to generate random data from the OKIW distribution. That is, if $Q \sim Unif(0,1)$, then

$$X_i = \alpha \left[\frac{-1}{\lambda} \left\{ \log \left(1 - \left[1 - \log \left(1 - Q_i^{\frac{1}{\omega}} \right) \right]^{\frac{-1}{\eta}} \right) \right\} \right]^{\frac{-1}{\beta}}.$$

The following steps were followed:

(1) Specify the sample size(s) n and the values of the parameters $\beta, \lambda, \omega, \alpha, \eta$;

(2) Generate $Q_i \sim Unif(0,1), i = 1, 2, \cdots, n$;

(3) Set
$$X_i = \alpha \left[\frac{-1}{\lambda} \left\{ \log \left(1 - \left[1 - \log \left(1 - Q_i^{\frac{1}{\omega}} \right) \right]^{\frac{-1}{\eta}} \right) \right\} \right]^{\frac{-1}{\beta}};$$

(4) Compute the MLEs of the five parameters;

(5) Repeat steps 2-3, N times

(6) Compute the mean square error (MSE) for each parameter.

The simulation study is repeated for $N = 1500$ iterations each with sample size $n = 50, 150, 300, 500, 600$ and parameter values in set $I : \beta = 2.5, \lambda = 1, \omega = 5, \alpha = 15, \eta = 0.5$ and $II : \beta = 0.25, \lambda = 1, \omega = 8, \alpha = 20, \eta = 0.5$. Two quantities are computed in the study namely average bias and root mean square error (RMSE) as follows:

(a) Average bias of the MLE $\hat{\mathbf{\Theta}}$ of the parameter $\mathbf{\Theta} = \{\beta, \lambda, \omega, \alpha, \eta\}$:

$$\frac{1}{N} \sum_{i=1}^{N} (\hat{\mathbf{\Theta}} - \mathbf{\Theta}).$$

(b) Root mean squared error (RMSE) of the MLE $\hat{\boldsymbol{\Theta}}$ of the parameter $\boldsymbol{\Theta} = \{\beta, \lambda, \omega, \alpha, \eta\}$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})^2}.$$

Table 4.1: Monte Carlo simulation study results

		I		II	
Paramter	**n**	**Average Bias**	**RMSE**	**Average Bias**	**RMSE**
β	50	0.00234923	1.254459	0.5164315	0.248549
	150	0.223251	1.649059	0.1151803	0.235575
	300	0.3311303	1.26418	0.1193271	0.207821
	500	0.3507965	1.190819	0.1143479	0.196312
	600	0.3075173	1.158524	0.1133413	0.180956
η	50	1.09536	1.979973	0.8198106	1.726021
	150	0.5398698	1.141238	0.1854606	0.644092
	300	0.2940589	0.7577926	0.05933698	0.310956
	500	0.1550613	0.45176	0.00615733	0.300049
	600	0.2240075	0.70721827	-0.0176586	0.207272
λ	50	2.957456	5.444899	8.967035	13.286160
	150	0.6784766	1.903525	4.117874	6.093303
	300	0.4062774	1.219227	2.915989	4.751907
	500	0.2391422	1.212782	1.735717	2.900594
	600	0.2367967	1.08309	1.531613	2.506491
ω	50	-1.179363	6.627073	-4.178481	8.678087
	150	0.7065778	7.084176	-0.9368763	8.474572
	300	-0.2239991	5.670298	-1.718957	8.244601
	500	-0.0842186	5.030306	-2.183191	7.488534
	600	-0.3626967	4.456074	-2.334901	6.110951
α	50	9.743556	21.3734	32.72563	233.448000
	150	10.00698	18.98498	59.83609	157.671000
	300	7.357111	14.15756	55.73701	83.912370
	500	5.017567	9.498253	36.87842	69.012460
	600	5.977159	12.36388	33.28421	63.288320

The Average Bias and RMSE values of the parameters $\beta, \lambda, \omega, \alpha$ and η for different sample sizes are presented in **Table 4.1**. From the results, it is clear that as the sample size n increases, the RMSEs, on average, decreases. It is also observed that for all the parametric values, the average biases decrease with increasing sample size n (overall trend).

4.5 Applications to Survival Data

In this section, four real different data sets are used to illustrate the flexibility of the model in the modelling of survival data as well as compare it with competing models namely EKIW (exponentiated KIW(Rodrigues et al., 2016) and EPLG (exponentiated power Lindley geometric (Alizadeh et al., 2016) distributions. We fit the density functions of the OKIW distribution and the EKIW. The pdf of EKIW and EPLG distributions are given by

$$f_{EKIW}(x) = \beta\eta\lambda\theta\alpha^{\beta}x^{-(\beta+1)}e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{\eta-1}\left[1-\left\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right\}^{\eta}\right]^{\theta-1}$$

and

$$f_{EPLG}(x) = \frac{\frac{\alpha\beta(1-\theta)\lambda^{2}x^{\beta-1}}{\lambda+1}(1+x^{\beta})e^{-\lambda x^{\beta}}\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha-1}}{\left(1-\theta\left[1-\left(1+\frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right)^{2}},$$

respectively. For each data set, the estimates of the parameters of the OKIW and EKIW distributions and information criterion statistics are computed. The maximum likelihood estimates of the OKIW and EKIW models' parameters are computed using the nonlinear optimisation function in **R** known as the Limited-Memory Quasi-Newton Code for Bound-Constrained Optimization (L-BFGS-B) and the log-likelihood function evaluated at the MLEs. The technique maximizes the log-likelihood function via the subroutine mle2 using the bbmle package in **R** (Bolker, 2014) and uses a wide range of initial values. The process often leads to more than one maximum, thus in such cases, the largest maxima is chosen as the maximum likelihood estimates. In cases where no maximum is identified for the chosen initial values, a new set of initial values are used and the optimisation is repeated untill a maximum is obtained.

Finally, we plot the histogram of the data sets and estimate of probability densities of OKIW and EKIW distributions.

4.5.1 Kevlar 49/Epoxy Strands Failure Times Data

This data set consists of 101 observations corresponding to the failure times (in hours) (i.e.,time until rupture of 49) of Kevlar 49/epoxy strands with pressure at 90%. Theses data were originally given in Barlow et al. [1984], and were analysed by Cooray and Ananda [2008].The summary of key descriptive statistics of the data is given in **Table 4.2**. The maximum likelihood estimates of the parameters of OKIW and EKIW distributions are given in **Table 4.3** along with the corresponding standard errors, p-values, −2log-likelihod statistics, Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC). The results based on the smaller values of the statistics: AIC, AICC, and BIC show that the OKIW distribution provides a significantly better fit than the EKIW model.

Table 4.2: Descriptive statistics for the Kelvar Data

min.	**max.**	**median**	**mean**	**var.**	**sd**	**CV**	**skewness**	**kurtosis**
0.01	7.890	0.8	1.025	1.25299	1.11937	1.09223	2.95725	13.3798

Table 4.3: Table for MLEs of OKIW and EKIW Models.

	MLEs Estimates of the parameters						**Statistics**		
Distribution	β	η	λ	φ	α	**-2log L**	AIC	AICC	BIC
O-KIW	0.14300	45.95447	2.95200	1.07503	10.88135	206.00350	216.00350	216.63508	229.07910
Std. errors	0.03691	0.00340	0.09331	0.50853	0.00367				
p-values	0.00011	0.00000	0.00000	0.03452	0.00000				
	β	η	λ	θ	α				
EKIW	0.28257	132.73000	2.71820	0.30503	17.20000	209.48030	219.48030	220.11188	232.55590
Std. errors	0.05396	0.00114	0.19954	0.12880	0.00891				
p-values	0.00000	0.00000	0.00000	0.01787	0.00000				

Plots of the estimated probability density functions for OKIW and EKIW and histogram for the kelvar data are given in Figure 4.4. The plots further indicate that the OKIW distribution is superior to EKIW distribution in terms of emprical model fitting to survival data.

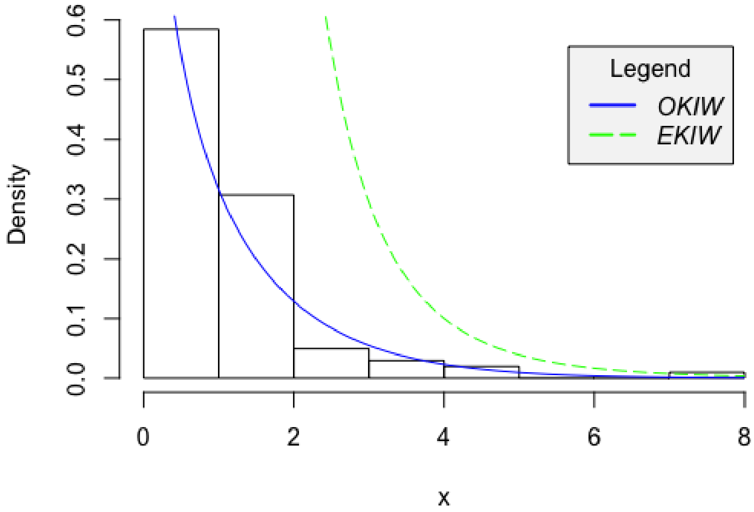

Figure 4.4: Histogram and estimated densities for Kelvar data.

4.5.2 Strength of the Glass Fibres Data

These data represent the strength of 1.5cm glass fibers, measured at National physical laboratory, England (Smith and Naylor, 1987). The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

These data are analysed in Alizadeh et al. [2016]. The summary of key descriptive statistics of the data is given in **Table 4.4** below.

Table 4.4: Descriptive statistics for strength of glass fibres data

min.	**max.**	**median**	**mean**	**var.**	**sd.**	**CV**	**skewness**	**kurtosis**
0.55	2.24	1.59	1.507	0.10506	0.32413	0.06972	-0.87858	0.80019

The maximum likelihood estimates of the parameters of OKIW and EKIW distributions are given in **Table 4.5** along with the corresponding standard errors, p-values, -2log-likelihod statistics, Akaike Information Criterion(AIC), corrected Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC).

The results based on the smaller values of the statistics: AIC, AICC, and BIC show that the OKIW distribution provides a significantly better fit than the EKIW but not as good as EPLG model.

Table 4.5: Table for MLEs of OKIW and EKIW Models.

	MLEs Estimates of the parameters						**Statistics**		
Distribution	β	η	λ	φ	α	**-2log L**	AIC	AICC	BIC
O-KIW	0.73504	74.65615	1.04537	1.17434	12.14228	33.04560	43.00456	44.05719	53.76127
Std. errors	0.17243	0.31757	0.73751	0.50496	10.75254				
p-values	0.00002	0.00000	0.15636	0.02004	0.25879				
	β	η	λ	θ	α				
EKIW	1.47210	19.13500	0.01645	0.71991	57.89700	61.97205	63.02468	72.68772	61.97205
Std. errors	0.30800	0.00139	0.01603	0.40493	0.00006				
p-values	0.00000	0.00000	0.30477	0.07543	0.00000				
	β	-	λ	θ	α				
EPLG*	0.9173	-	3.08735	0.94201	0.69931	23.88	31.88	-	40.45
Note: * MLEs estimates as in Alizadeh et al.(2016)									

Plots of the estimated probability density functions for OKIW and EKIW and histogram for the strength of glass fibres data are given in Figure 4.5. The plots further indicate that the OKIW distribution is superior to EKIW distributions in term of empirical model fitting to glass fibres data.

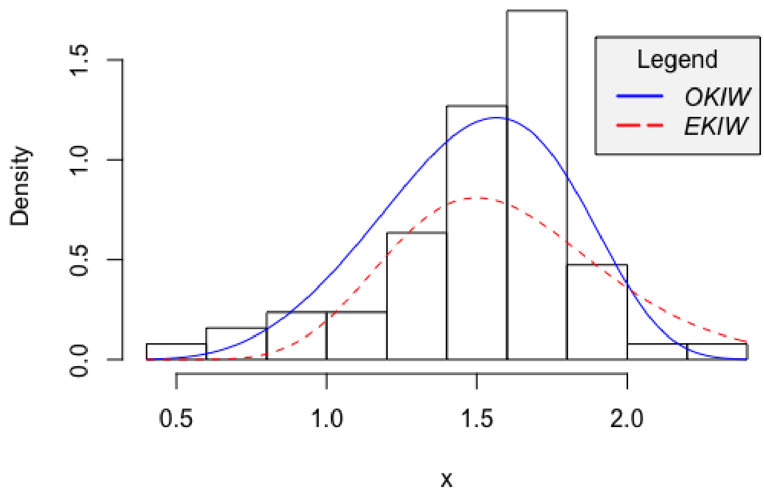

Figure 4.5: Histogram and estimated densities for strength of glass fibres data.

Consequently, from the values of the statistics in Table 4.5 and from **Figures 4.5**, we conclude that OKIW distribution gives the better fit for the glass fibre data than EKIW.

4.5.3 Guinea Pigs Data

These data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [1960] and analysed by Alizadeh et al. [2016]. The data are:

$10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.$

The summary of key descriptive statistics of the data is given in **Table 4.6**. The maximum likelihood estimates of the parameters of OKIW and EKIW distributions are given in **Table 4.7** along with the corresponding standard errors, p-values, -2log-likelihod statistics, Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC). The results based on the smaller values of the statistics: AIC, AICC, and BIC show that the OKIW distribution provides a significantly better fit than the EKIW and EPLG models.

Table 4.6: Descriptive statistics for the guinea pigs data

min.	**max.**	**median**	**mean**	**var.**	**sd.**	**CV**	**skewness**	**kurtosis**
0.1	5.55	1.495	1.768	1.07029	1.03455	0.58509	1.31401	1.85338

Table 4.7: Table for MLEs of OKIW and EKIW Models.

	MLEs Estimates of the parameters						**Statistics**		
Distribution	β	η	λ	φ	α	**-2log L**	AIC	AICC	BIC
O-KIW	0.18836	22.74434	1.69122	3.88360	40.12850	189.054	198.9712	199.88029	210.43733
Std. errors	0.06235	0.18773	0.20959	2.93846	0.02005				
p-values	0.00252	0.00000	0.00000	0.18629	0.00000				
	β	η	λ	θ	α				
EKIW	0.40794	84.10154	1.19373	0.90356	49.01261	190.8867	200.66930	201.57839	212.27003
Std. errors	0.10228	0.00248	0.26919	0.56982	0.00274				
p-values	0.00007	0.00000	0.00001	0.01128	0.00000				
	β	-	λ	θ	α				
EPLG*	4.34313	-	0.23122	0.99998	6.7385	849.25	857.2500	-	866.3500

Note: * MLEs estimates as in Alizadeh et al.(2016)

Plots of the estimated probability density functions for OKIW and EKIW and histogram for the strength of glass fibres data are given in Figure 4.6. The plots further indicate that the OKIW distribution is superior to EKIW distribution in terms of empirical model fitting to survival data.

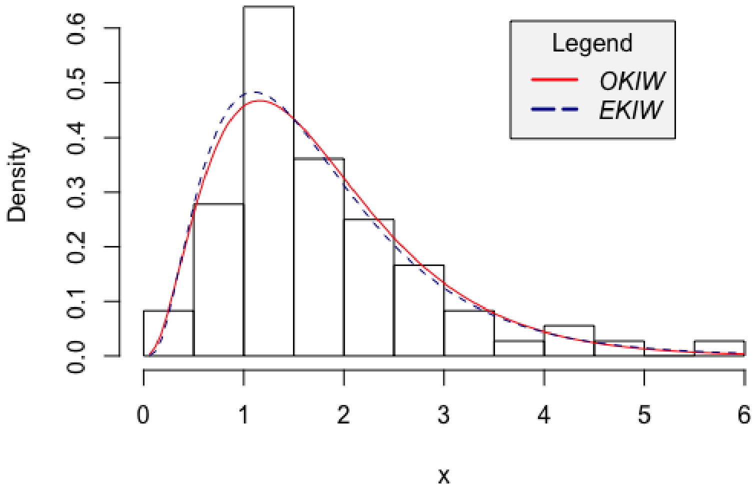

Figure 4.6: Histogram and estimated densities for guinea pigs data.

Consequently, from the values of the statistics in Table 4.7 and from **Figures 4.6**, we conclude that the OKIW distribution gives the better fit for the guinea pigs data than EPLG and EKIW.

4.5.4 Bladder Cancer Patients Data

This data set consists of data of cancer patients. The data represents the remission times (in months) of a sample of 128 bladder cancer patients obtained from Lee and Wang [2003]. The summary of key descriptive statistics of the data is given in **Table 4.8**. The maximum likelihood estimates of the parameters of OKIW

Table 4.8: Descriptive statistics for Blader Cancer Patients data

min.	**max.**	**median**	**mean**	**var.**	**sd.**	**CV**	**skewness**	**kurtosis**
0.08	79.05	6.395	9.365	110.43220	10.50867	1.12210	1.31401	1.85338

and EKIW distributions are given in **Table 4.9** along with the corresponding standard errors, p-values, −2log-likelihod statistics, Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC). The results based on the smaller values of the statistics: AIC, AICC, and BIC show that the OKIW distribution provides a better fit than the EKIW model.

Table 4.9: Table for MLEs of OKIW and EKIW Models.

	MLEs Estimates of the parameters						**Statistics**		
Distribution	β	η	λ	φ	α	**-2log L**	AIC	AICC	BIC
O-KIW	0.11728	16.46529	2.42920	4.09863	19.95129	822.37200	832.37200	832.86380	846.63215
Std. errors	0.09432	32.30712	1.82307	2.63592	0.67726				
p-values	0.21370	0.61030	0.18270	0.12000	0.00000				
	β	η	λ	θ	α				
EKIW	0.22833	96.82200	3.16200	0.92664	46.28500	823.26720	833.26720	833.75900	847.52735
Std. errors	0.04542	0.00977	0.11671	0.45050	0.00180				
p-values	0.00000	0.00000	0.00000	0.03969	0.00000				

Plots of the estimated probability density functions for OKIW and EKIW and histogram for the strength of glass fibres data are given in Figure 4.7. The plots further indicate that the OKIW distribution is superior to EKIW distribution in terms of empirical model fitting to bladder cancer patients data.

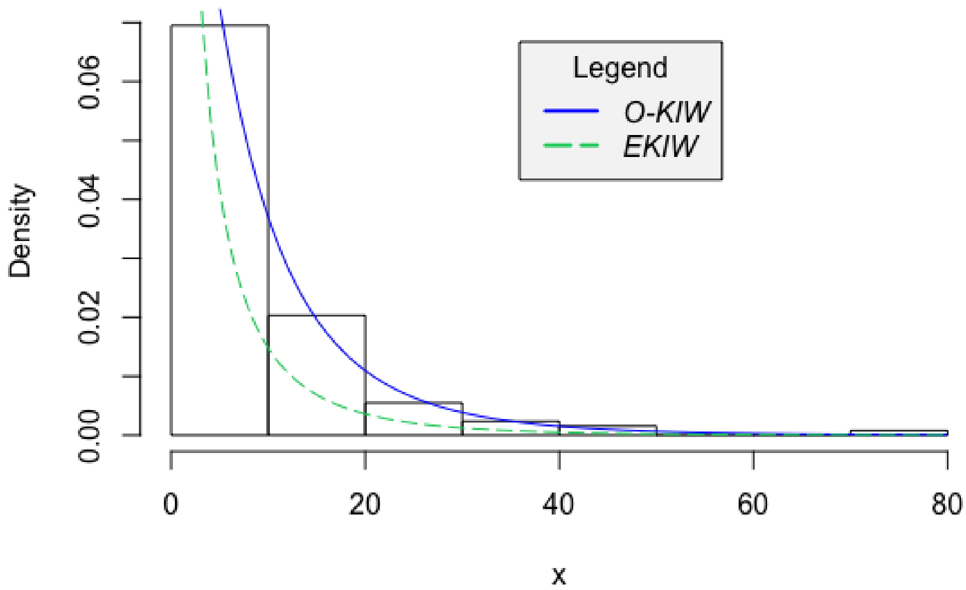

Figure 4.7: Histogram and estimated densities for Bladder Cancer Patients data.

From the values of the statistics in Table 4.9 and from **Figure 4.7**, we conclude that the OKIW distribution gives a better fit for the Bladder Cancer Patients data than EKIW.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

Statistical lifetime probability distributions are fundamental concepts in data modelling in a host of application areas such as reliability engineering, survival analysis, biomedical research, insurance and social sciences. Data arise from dynamic proceses. This means that new statistical models with greater capability and versatility are often required in order to appropriately characterise given data.

This study proposes a new five-parameter lifetime model, called the Odd Kumaraswamy inverse Weibull distribution (OKIW), and study its mathematical and statistical properties. The model hazard function exhibits versatile behaviours: increasing, decreasing, J-shaped, reversed-J shaped, unimodal and upside-down bathtub. These are very attractive features that render the OKIW distribution suitable for modelling monotonic and nonmonotonic hazard behaviours. Special models, sub-models of the proposed model, are introduced namely Odd inverse exponential, Odd exponentiated inverse Weibull, and Odd exponentiated inverse exponential. The PDF also has varied shapes suitable for modelling right-skewed, left-skewed, and approximately symmetric survival data and also survival data with highly varied kurtosis.

We obtain point estimates of the parameters using maximum likelihood estimation. A Monte Carlo simulation study is carried out to examine the stability of the maximum likelihood estimators (MLEs) of the parameters in terms of the average biases and root mean square errors. The study finds that MLEs are asymptotically consistent and unbiased. Applications of the model to real survival data show emprically its flexibility and usefulness in modeling various types of biomedical and reliability engineering data and that the model offers a more superior fit than the competing exponentiated Kumaraswamy inverse Weibull distribution. Hence, it is expected that OKIW distribution may attract wider applications in survival analysis, reliability analysis, insurance, among others.

5.2 Recommendations

In this thesis, we illustrate the applicability and flexibility of the proposed model using complete samples of survival data in the estimation of model parameters. However, survival times of some individuals of interest might not be fully observed due to different reasons: either the survival study stops before full survival times of all individuals can be observed; a subject drops out of a study, or a subject is lost to follow-up, subjects survives beyond the time of the study, etc. These phenomena are pervasive in biomedical research particularly in clinical trials and generate censored survival data. Therefore, subsequent further research should consider application of the proposed model to censored survival data and carry out model parameters estimation using maximum likelihood method implemented via expectation-maximization or estimate model parameters by Bayesian method and assess estimators' stability by simulation.

Moreover, in myraid of applications in biomedical research, the lifetimes of items of interest are affected by covariates such as cholesterol level, weight, blood pressure among others. Parametric regression models to estimate univariate survival functions for censored data are commonly utilized to yield estimates of quantities of interest. Hence, based on the OKIW density function, a linear regression model for censored data linking the response variable of interest and the covariates may be proposed.

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A.0.1 R Codes for Optimisation

```
library(AdequacyModel)
library(bbmle)
x<-data
logf<-function(beta,eta,lambda,omega,alpha){-sum(log(beta*eta*
   lambda*omega*(alpha^beta)*x^(-beta-1)*
(1-exp(1-(1-exp(-lambda*(alpha/x)^beta))^(-eta)))^(omega-1)*
exp((1-(1-exp(-lambda*(alpha/x)^beta))^(-eta))-(lambda*(alpha
   /x)^beta))*
(1-exp(-lambda*(alpha/x)^beta))^(-eta-1)))
}
goodnessOfFit<-mle2(logf,start=list(beta=beta,eta=eta,lambda=
   lambda,omega=omega,alpha=alpha), method="BFGS") # # list
   contains initialising optim. values
summary(goodnessOfFit)
vcov(goodnessOfFit) ## checking the existence of variance-covariance
   matrix
AIC(goodnessOfFit) ## Computing AIC statistics
###
###
## Plotting for estimated pdfs and histogram for each given data set
Data<-data
hist(Data,probability = T, main="",xlab="x",ylab="Density")
library(AdequacyModel)
library(bbmle)
x<-data
logf<-function(x,beta,eta,lambda,omega,alpha){
f11=beta*eta*lambda*omega*(alpha^beta)*x^(-beta-1)
```

```
f22=(1-exp(1-(1-exp(-lambda*(alpha/x)^beta))^(-eta)))^(omega
    -1)
f33=exp((1-(1-exp(-lambda*(alpha/x)^beta))^(-eta))-(lambda*(
    alpha/x)^beta))
f44=(1-exp(-lambda*(alpha/x)^beta))^(-eta-1)
y=f11*f22*f33*f44
}
## parameters in function argument represent MLEs for parameters for
    OKIW
curve(logf,add=T,col="red",lty=1)
# Plotting  estimate of EKIW PDF on same histogram
library(AdequacyModel)
library(bbmle)
x<-data
logg<-function(beta,eta,lambda,omega,alpha){
-sum(log(beta*eta*lambda*omega*(alpha^beta)*x^(-beta-1)*
    exp(-lambda*(alpha/x)^beta)*
(1-exp(-lambda*(alpha/x)^beta))^(eta-1)*(1-(1-exp(-lambda*(
    alpha/x)^beta))^eta)^(omega-1)))
}
## #parameters in function argument represent MLEs for parameters
    for EKIW
curve(logg,add=T,col="blue4",lty=2)
legend(locator(1),
inset=.05,
cex = 1,
title="Legend",
c("OKIW","EKIW"),
horiz=F,
```

```
lty=c(1,5),
lwd=c(2,2),
col=c("red","blue4"), #for chosen colors of lines  of estimated
    desnities
bg="grey96",
text.font=3)
###
```

A.0.2 R Codes for Monte Carlo Simulation

```
#### define quantile and negative log−likelihood
quantile=function(beta,eta,lambda,omega,alpha,u){
result<−alpha/((((−1)/lambda)*(log(1−(1−log(1−u^(1/omega)))
    ^(−(1/eta)))))^(1/beta))
return(result)
}
OKIW<−function(par){
−sum(log(par[1]*par[2]*par[3]*par[4]*(par[5]^par[1])*x^(−par
    [1]−1)*
(1−exp(1−(1−exp(−par[3]*(par[5]/x)^par[1]))^(−par[2])))^(par
    [4]−1)*
exp((1−(1−exp(−par[3]*(par[5]/x)^par[1]))^(−par[2]))−(par[3]*(
    par[5]/x)^par[1]))*
(1−exp(−par[3]*(par[5]/x)^par[1]))^(−par[2]−1)))
}
######### Algorithm for the Monte−Carlo simulation study
library(numDeriv)
library(Matrix)
beta=beta ## Choices for initial values
eta=eta
lambda=lambda
omega=omega
alpha=alpha
n1=c(n1,n2,n3,n4,n5)
for (j in 1:length(n1)){
n=n1[j]
N=N0   # number of iterations
```

```
mle_lambda<-c(rep(0,N))
mle_omega<-c(rep(0,N))
mle_beta<-c(rep(0,N))
mle_eta<-c(rep(0,N))
mle_alpha<-c(rep(0,N))
LC_lambda<-c(rep(0,N))
UC_lambda<-c(rep(0,N))
LC_omega<-c(rep(0,N))
UC_omega<-c(rep(0,N))
LC_beta<-c(rep(0,N))
UC_beta<-c(rep(0,N))
LC_eta<-c(rep(0,N))
UC_eta<-c(rep(0,N))
LC_alpha<-c(rep(0,N))
UC_alpha<-c(rep(0,N))
count_lambda=0
count_omega=0
count_beta=0
count_eta=0
count_alpha=0
temp=1
HH1<-matrix(c(rep(2,25)),nrow=5,ncol=5)
HH2<-matrix(c(rep(2,25)),nrow=5,ncol=5)
for (i in 1:N)
{
print(i)
flush.console()
repeat{
x<-c(rep(0,n))
```

```
#Generate a random variable from uniform distribution
u<-0
u<-runif(n,min=0,max=1)
for (k in 1:n){
x[k]<-quantile(beta,eta,lambda,omega,alpha,u[k])
}
#Maximum likelihood estimation
mle.result<-nlminb(c(beta,eta,lambda,omega,alpha),OKIW,lower=0,
    upper=Inf)
temp=mle.result$convergence
if(temp==0){
temp_beta<-mle.result$par[1]
temp_eta<-mle.result$par[2]
temp_lambda<-mle.result$par[3]
temp_omega<-mle.result$par[4]
temp_alpha<-mle.result$par[5]
HH1<-hessian(OKIW,c(temp_beta,temp_eta,temp_lambda,temp_
    omega,temp_alpha))
if( sum(is.nan(HH1))==0 & (diag(HH1)[1]>0) &
(diag(HH1)[2]>0) & (diag(HH1)[3]>0) & (diag(HH1)[4]>0)
& (diag(HH1)[5]>0) ){
HH2<-solve(HH1)
#print(det(HH1))
}
else{
temp=1}
}
if ((temp==0) & (diag(HH2)[1]>0) & (diag(HH2)[2]>0)
& (diag(HH2)[3]>0) & (diag(HH2)[4]>0) &
```

```
(diag(HH2)[5]>0) & (sum(is.nan(HH2))==0)){
break
}
else{
temp=1}
}
temp=1
mle_beta[i]<-mle.result$par[1]
mle_eta[i]<-mle.result$par[2]
mle_lambda[i]<-mle.result$par[3]
mle_omega[i]<-mle.result$par[4]
mle_alpha[i]<-mle.result$par[5]
HH<-hessian(OKIW,c(mle_beta[i],mle_eta[i],mle_lambda[i],mle_
    omega[i],mle_alpha[i]))
H<-solve(HH)
LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[1])
if( (LC_beta[i]<=beta) & (beta<=UC_beta[i])){
count_beta=count_beta+1
}
LC_eta[i]<-mle_eta[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_eta[i]<-mle_eta[i]+qnorm(0.975)*sqrt(diag(H)[2])
if( (LC_eta[i]<=eta) & (eta<=UC_eta[i])){
count_eta=count_eta+1
}
LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[3])
if( (LC_lambda[i]<=lambda) & (lambda<=UC_alpha[i])){
count_lambda=count_lambda+1
```

```
}
LC_omega[i]<-mle_omega[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_omega[i]<-mle_omega[i]+qnorm(0.975)*sqrt(diag(H)[4])
if( (LC_omega[i]<=omega) & (omega<=UC_omega[i])){
count_omega=count_omega+1
}
LC_alpha[i]<-mle_alpha[i]-qnorm(0.975)*sqrt(diag(H)[5])
UC_alpha[i]<-mle_alpha[i]+qnorm(0.975)*sqrt(diag(H)[5])
if( (LC_alpha[i]<=alpha) & (alpha<=UC_alpha[i])){
count_alpha=count_alpha+1
}
}
#Calculate Average Bias
ABias_beta<-sum(mle_beta-beta)/N
ABias_eta<-sum(mle_eta-eta)/N
ABias_lambda<-sum(mle_lambda-lambda)/N
ABias_omega<-sum(mle_omega-omega)/N
ABias_alpha<-sum(mle_alpha-alpha)/N
print(cbind(ABias_beta,ABias_eta,ABias_lambda,ABias_omega,
   ABias_alpha))
#Calculate RMSE
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
RMSE_omega<-sqrt(sum((omega-mle_omega)^2)/N)
RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)
RMSE_eta<-sqrt(sum((eta-mle_eta)^2)/N)
RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/N)
print(cbind(RMSE_beta,RMSE_eta,RMSE_lambda,RMSE_omega,
   RMSE_alpha))
#Converge Probability
```

```
CP_lambda<-count_lambda/N
CP_omega<-count_omega/N
CP_beta<-count_beta/N
CP_eta<-count_eta/N
CP_alpha<-count_alpha/N
print(cbind(CP_beta,CP_eta,CP_lambda,CP_omega,CP_alpha))
#Average Width
AW_lambda<-sum(abs(UC_lambda-LC_lambda))/N
AW_omega<-sum(abs(UC_omega-LC_omega))/N
AW_beta<-sum(abs(UC_beta-LC_beta))/N
AW_eta<-sum(abs(UC_eta-LC_eta))/N
AW_alpha<-sum(abs(UC_alpha-LC_alpha))/N
print(cbind(AW_beta,AW_eta,AW_lambda,AW_omega,AW_alpha
   ))
}
## End
```

A.0.3 R Codes for Plots of CDF and PDF and Hazard Function

```
#### CDF OF OKIW
F=(1−exp(1−(1−exp(−lambda*(alpha/x)^beta))^(−eta)))^omega
#### PDF of OKIW
f1=function(x,beta,eta,lambda,omega,alpha){
        f11=beta*eta*lambda*omega*(alpha^beta)*x^(−beta−1)
        f22=(1−exp(1−(1−exp(−lambda*(alpha/x)^beta))^(−eta)))
            ^(omega−1)
        f33=exp((1−(1−exp(−lambda*(alpha/x)^beta))^(−eta))−(
            lambda*(alpha/x)^beta))
        f44=(1−exp(−lambda*(alpha/x)^beta))^(−eta−1)
        y=f11*f22*f33*f44
        return(y)
}
f1 (1.5,75,1.45,4.50,7.5,25)
#
#
##### OKIW CDF PLOTS ######
F1=(1−exp(1−(1−exp(−5.5*(2/x)^1.5))^(−1)))^5
curve((1−exp(1−(1−exp(−5.5*(2/x)^1.5))^(−1)))^5, from=0,to
    =120,xlab="x",ylab="F(x)",col="red",lty=5)
F2=(1−exp(1−(1−exp(−3.5*(3.5/x)^1))^(−1)))^5
curve((1−exp(1−(1−exp(−3.5*(3.5/x)^1))^(−1)))^5, from=0,to
    =120,xlab="x",ylab="F(x)",col="blue4",lty=1, add=T)
F3=(1−exp(1−(1−exp(−6.5*(10/x)^2))^(−1)))^5
curve((1−exp(1−(1−exp(−6.5*(10/x)^2))^(−1)))^5, from=0,to=120,
    xlab="x",ylab="F(x)",col="magenta4",lty=2,add=T)
```

```
#
legend(locator(1),
inset=.05,
cex = 0.5,
c(expression(paste(alpha,"=",2,~beta,"=",1.5,~lambda,"=",5.5,~eta
    ,"=",1,~omega,"=",5)), expression(paste(alpha,"=",3.5,~beta,"=
    ",1,~lambda,"=",3.5,~eta,"=",1,~omega,"=",5)),expression(paste
    (alpha,"=",10,~beta,"=",2,~lambda,"=",6.5,~eta,"=",1,~omega,"
    =",5))),
horiz=F,
lty=c(5,1,2),
lwd=c(2,2,2),
col=c("red","blue4","magenta4"), #for chosen colors of lines of
    estimated desnities
bg="white",
text.font=5)
#
#
###### OKIW PDF PLOTS ######
f1=function(x,beta,eta,lambda,omega,alpha){
        f11=beta*eta*lambda*omega*(alpha^beta)*x^(-beta-1)
        f22=(1-exp(1-(1-exp(-lambda*(alpha/x)^beta))^(-eta)))
            ^(omega-1)
        f33=exp((1-(1-exp(-lambda*(alpha/x)^beta))^(-eta))-(
            lambda*(alpha/x)^beta))
        f44=(1-exp(-lambda*(alpha/x)^beta))^(-eta-1)
        y=f11*f22*f33*f44
        return(y)
}
```

```
f1 (26,75,65,4.50,70.5,25)
#
#
f1=function(x){
        f11=0.75*0.1*0.8*0.15*(0.1^0.75)*x^(-0.75-1)
        f22=(1-exp(1-(1-exp(-0.8*(0.1/x)^0.75))^(-0.1)))
            ^(0.15-1)
        f33=exp((1-(1-exp(-0.8*(0.1/x)^0.75))^(-0.1))-(0.8*(0.1/
            x)^0.75))
        f44=(1-exp(-0.8*(0.1/x)^0.75))^(-0.1-1)
        y=f11*f22*f33*f44
}
curve(f1,from=0,to=100,xlab="x",ylab="f(x)",col="red",lty=5,ylim=
    c(0,0.0455))
#
f2=function(x){
        g2=0.85*0.6*0.15*7.5*(25^0.85)*x^(-0.85-1)
        g3=(1-exp(1-(1-exp(-0.15*(25/x)^0.85))^(-0.6)))^(7.5-1)
        g4=exp((1-(1-exp(-0.15*(25/x)^0.85))^(-0.6))-(0.15*(25/
            x)^0.85))
        g5=(1-exp(-0.15*(25/x)^0.85))^(-0.6-1)
        y=g2*g3*g4*g5
}
curve(f2,from=0,to=100,xlab="x",ylab="f(x)",col="darkgreen",lty=1,
     add=T,ylim=c(0,0.0455))
#
#
f3=function(x){
        h1=0.85*0.6*0.35*7.5*(25^0.85)*x^(-0.85-1)
```

```
        h2=(1-exp(1-(1-exp(-0.35*(25/x)^0.85))^(-0.6)))^(7.5-1)
        h3=exp((1-(1-exp(-0.35*(25/x)^.85))^(-0.6))-(0.35*(25/x
            )^0.85))
        h4=(1-exp(-0.35*(25/x)^0.85))^(-0.6-1)
        y=h1*h2*h3*h4
}
curve(f3,from=0,to=100,xlab="x",ylab="f(x)",col="purple",lty=2,
    add=T,ylim=c(0,0.0455))
#
#
f4=function(x){
        h11=0.85*1.08*0.35*7.5*(25^0.85)*x^(-0.85-1)
        h22=(1-exp(1-(1-exp(-0.35*(25/x)^0.85))^(-1.08)))
            ^(7.5-1)
        h33=exp((1-(1-exp(-0.35*(25/x)^.85))^(-1.08))-(0.35*(25
            /x)^0.85))
        h44=(1-exp(-0.35*(25/x)^0.85))^(-1.08-1)
        y=h11*h22*h33*h44
}
curve(f4,from=0,to=100,xlab="x",ylab="f(x)",col="mediumblue",lty
    =4, add=T,ylim=c(0,0.0455))
#
#
f5=function(x){
        k1=2.2*0.6*0.72*7.5*(25^2.2)*x^(-2.2-1)
        k2=(1-exp(1-(1-exp(-0.72*(25/x)^2.2))^(-0.6)))^(7.5-1)
        k3=exp((1-(1-exp(-0.72*(25/x)^2.2))^(-0.6))-(0.72*(25/x
            )^2.2))
        k4=(1-exp(-0.72*(25/x)^2.2))^(-0.6-1)
```

```
        y=k1*k2*k3*k4
}
curve(f5,from=0,to=100,xlab="x",ylab="f(x)",col="deepskyblue",lty
    =6, add=T,ylim=c(0,0.0455))
legend(locator(1),
inset=.05,
cex = 0.5,
c(expression(paste(alpha,"=",0.1,~beta,"=",0.75,~lambda,"=",0.8,~
    eta,"=",0.1,~omega,"=",0.15)), expression(paste(alpha,"=",25,~
    beta,"=",0.85,~lambda,"=",0.15,~eta,"=",0.6,~omega,"=",7.5)),
    expression(paste(alpha,"=",25,~beta,"=",0.85,~lambda,"="
    ,0.35,~eta,"=",0.6,~omega,"=",7.5)),expression(paste(alpha,"="
    ,25,~beta,"=",0.85,~lambda,"=",0.35,~eta,"=",1.08,~omega,"="
    ,7.5)),expression(paste(alpha,"=",25,~beta,"=",2.2,~lambda,"="
    ,0.72,~eta,"=",0.6,~omega,"=",7.5))),
horiz=F,
lty=c(5,1,2,4,6) ,
lwd=c(2,2,2,2,2),
col=c("red","darkgreen","purple","mediumblue","deepskyblue"), #for
    chosen colors of lines of estimated desnities
bg="white",
text.font=5)
#
# #########OKIW HAZARD PLOTS ########
H=function(x,beta,eta,lambda,omega,alpha){
        f11=beta*eta*lambda*omega*(alpha^beta)*x^(-beta-1)
        f22=(1-exp(1-(1-exp(-lambda*(alpha/x)^beta))^(-eta)))
            ^(omega-1)
        f33=exp((1-(1-exp(-lambda*(alpha/x)^beta))^(-eta))-(
```

```
        lambda*(alpha/x)^beta))
    f44=(1-exp(-lambda*(alpha/x)^beta))^(-eta-1)
    F=(1-exp(1-(1-exp(-lambda*(alpha/x)^beta))^(-eta)))^
        omega # F1 is CDF
    y=(f11*f22*f33*f44)/(1-F) # 1-F1=Survival


}
#
par(mfrow=c(2,2)) # To plot them bundled together
H1=function(x){
    f11=1.5*0.1*0.1*2.0*(50^1.5)*x^(-1.5-1)
    f22=(1-exp(1-(1-exp(-0.1*(50/x)^1.5))^(-0.1)))^(2.0-1)
    f33=exp((1-(1-exp(-0.1*(50/x)^1.5))^(-0.1))-(0.1*(50/x)
        ^1.5))
    f44=(1-exp(-0.1*(50/x)^1.5))^(-0.1-1)
    F1=(1-exp(1-(1-exp(-0.1*(50/x)^1.5))^(-0.1)))^2.0 # F1
        is CDF
    y=(f11*f22*f33*f44)/(1-F1) # 1-F1=Survival
}
curve(H1,from=0,to=120,xlab="x",ylab="h(x)",col="blue",lty=1,ylim
    =c(0,0.0020))
#
#
H2=function(x){
    g1=1.5*0.15*0.15*12*(3.5^1.5)*x^(-1.5-1)
    g2=(1-exp(1-(1-exp(-0.15*(3.5/x)^1.5))^(-0.15)))^(12-1)
    g3=exp((1-(1-exp(-0.15*(3.5/x)^1.5))^(-0.15))-(0.15*(3.5
        /x)^1.5))
    g4=(1-exp(-0.15*(3.5/x)^1.5))^(-0.15-1)
```

```
        F2=(1−exp(1−(1−exp(−0.15*(3.5/x)^1.5))^(−0.15)))^12 #
            F2 is CDF
        y=(g1*g2*g3*g4)/(1−F2) # 1−F2=Survival

}
curve(H2,from=0,to=120,xlab="x",ylab="h(x)",col="deepskyblue",lty
    =6, ylim=c(0,0.0020),add=T)
#
legend(locator(1),
inset=.05,
cex = 0.5,
c(expression(paste(alpha,"=",50,~beta,"=",1.5,~lambda,"=",0.1,~
    eta,"=",0.1,~omega,"=",2.0)),expression(paste(alpha,"=",3.5,~
    beta,"=",1.5,~lambda,"=",0.15,~eta,"=",0.15,~omega,"=",12))),
horiz=F,
lty=c(1,6),
lwd=c(2,2),
col=c("blue","deepskyblue"), #for chosen colors of lines of estimated
    desnities
bg="white",
text.font=5)
#
#
H3=function(x){
        g11=0.5*0.55*0.15*2.5*(25^0.5)*x^(−0.5−1)
        g22=(1−exp(1−(1−exp(−0.15*(25/x)^0.5))^(−0.55)))
            ^(2.5−1)
        g33=exp((1−(1−exp(−0.15*(25/x)^0.5))^(−0.55))−(0.15*(25
            /x)^0.5))
```

```
        g44=(1−exp(−0.15∗(25/x)^0.5))^(−0.55−1)
        F3=(1−exp(1−(1−exp(−0.15∗(25/x)^0.5))^(−0.55)))^2.5 #
            F3 is CDF
        y=(g11∗g22∗g33∗g44)/(1−F3) # 1−F3=Survival

}
curve(H3,from=0,to=120,xlab="x",ylab="h(x)",col="red",lty=2,ylim
    =c(0,0.25))
#
#
H4=function(x){
        k11=0.86∗5.78∗3.28∗2.5∗(33.6^0.86)∗x^(−0.86−1)
        k22=(1−exp(1−(1−exp(−3.28∗(33.6/x)^0.86))^(−5.78)))
            ^(2.5−1)
        k33=exp((1−(1−exp(−3.28∗(33.6/x)^0.86))^(−5.78))−(3.28∗
            (33.6/x)^0.86))
        k44=(1−exp(−3.28∗(33.6/x)^0.86))^(−5.78−1)
        F4=(1−exp(1−(1−exp(−3.28∗(33.6/x)^0.86))^(−5.78)))^2.5
            # F4 is CDF
        y=(k11∗k22∗k33∗k44)/(1−F4) # 1−F4=Survival

}
curve(H4,from=0,to=120,xlab="x",ylab="h(x)",col="deepskyblue",lty
    =1,ylim=c(0,0.25), add=T)
#
legend(locator(1),
inset=.05,
cex = 0.5,
c(expression(paste(alpha,"=",25,~beta,"=",0.5,~lambda,"=",0.15,~
```

```
    eta,"=",0.55,~omega,"=",2.5)),expression(paste(alpha,"=",33.6,~
    beta,"=",0.86,~lambda,"=",3.28,~eta,"=",5.78,~omega,"=",2.5))),
horiz=F,
lty=c(2,1),
lwd=c(2,2),
col=c("red","deepskyblue"), #for chosen colors of lines of estimated
     desnities
bg="white",
text.font=5)
#
H5=function(x){
        k1=1.5*0.1*0.5*5*(25^1.5)*x^(-1.5-1)
        k2=(1-exp(1-(1-exp(-0.5*(25/x)^1.5))^(-0.1)))^(5-1)
        k3=exp((1-(1-exp(-0.5*(25/x)^1.5))^(-0.1))-(0.5*(25/x)
            ^1.5))
        k4=(1-exp(-0.5*(25/x)^1.5))^(-0.1-1)
        F5=(1-exp(1-(1-exp(-0.5*(25/x)^1.5))^(-0.1)))^5 # F5 is
            CDF
        y=(k1*k2*k3*k4)/(1-F5) # 1-F5=Survival


}
curve(H5,from=0,to=120,xlab="x",ylab="h(x)",col="magenta4",lty
    =5)
#
legend(locator(1),
inset=.05,
cex = 0.5,
c(expression(paste(alpha,"=",25,~beta,"=",1.5,~lambda,"=",0.5,~
    eta,"=",0.1,~omega,"=",5))),
```

```
horiz=F,
lty=c(5),
lwd=c(2),
col=c("magenta4"), #for chosen colors of lines of estimated densities
bg="white",
text.font=5) ## Bol Atem ## End
```