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THE ODD KUMARASWAMY INVERSE WEIBULL DISTRIBUTION WITH APPLICATION TO SURVIVAL DATA

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Abstract

Probability distributions are very useful models for characterising inherent variability in lifetime data. Modified forms of Weibull distribution are widely used in survival data analysis due to their versatility and relative simplicity. In this study, a new odd Kumaraswamy inverse Weibull distribution is developed and its mathematical properties are derived. The model parameters are estimated using maximum likelihood estimation and a simulation to assess the performance of maximum likelihood estimators of the

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parameters is carried out. The model is then applied to several survival data sets to illustrate its flexibility. Applications of the model to survival data empirically prove its flexibility and usefulness in modeling various types of biomedical and reliability data and its superiority over other lifetime distributions. Thus, the model may attract wider applications in survival analysis, reliability analysis, and insurance.

1. Introduction

Probability distributions are very useful models for characterising variability in lifetime data. Weibull distribution is popular in survival analysis due to its versatility to model lifetime data which exhibit monotone hazard rates (increasing, decreasing or constant hazard rate). But, in many practical situations, classical Weibull distribution fails to provide adequate fits to real life survival data such as machine life cycle, human mortality, and biomedical data which exhibit non-monotonic hazard rates. Thus, more flexible forms of Weibull distribution such as the inverse Weibull distribution have been proposed and applied to unimodal survival data, and consequently several new techniques for generating new versatile modified Weibull distributions by adding more parameters have been proposed to achieve non-monotonic shapes.

Some recent generalisations include four-parameter beta inverse Weibull distribution [6] and modified inverse Weibull distribution [7]. Baharith et al. [2] extended the inverse Weibull distribution to beta generalised inverse Weibull distribution. Pararai et al. [8] proposed a new generalisation of inverse Weibull distribution to obtain a three-parameter gamma inverse Weibull distribution via the gamma-exponentiated exponential generator [10]. However, due to complexities of beta-*G* distribution and gamma-*G* since they involve special functions such as beta functions and incomplete gamma, researchers prefer to deal with Kumaraswamy distribution which has similar properties as beta-*G* but has advantage in terms of tractability. Generalised distributions from Kumaraswamy generator include Kumaraswamy inverse Weibull (KIW) distribution [12] and a five-parameter exponentiated Kumaraswamy inverse Weibull (EKIW) distribution [11].

Motivated by the advantages of compounded Weibull distributions with respect to having hazard functions characterised by monotonic and non-monotonic shapes such as bathtub and unimodality, we propose and study a new distribution called odd KIW distribution, a generalisation of KIW. The model inherits desirable properties from Kumaraswamy distribution and the odd generalised exponential (OGE) family of distributions [14] such as monotonic and non-monotonic shapes as well as enhanced flexibility of kurtosis and possibility of developing heavy-tailed distributions for modelling survival data.

The rest of the paper is organised as follows. Section 2 presents the odd KIW distribution, its density, survival and hazard functions as well as quantiles and plots of these quantities. Section 3 presents derivations of odd KIW mathematical properties. Model parameters estimation and simulation are presented in Sections 4 and 5, respectively. Finally, Section 6 deals with application of the model to survival data while Section 7 concludes the study.

2. The Odd Kumaraswamy Inverse Weibull Distribution

If a random variable $X \sim Weibull(\alpha, \beta)$, then the CDF and PDF of the inverse Weibull distribution are, respectively, given by

$$G(x; \alpha, \beta) = \exp\left(-\frac{\alpha}{x}\right)^{\beta}, \quad x > 0, \alpha > 0, \beta > 0 \tag{1}$$

and

$$g(x; \alpha, \beta) = \beta\alpha^{\beta} x^{-(\beta+1)} \exp\left(-\frac{\alpha}{x}\right)^{\beta}, \quad x > 0, \alpha > 0, \beta > 0. \tag{2}$$

The CDF of Kumaraswamy inverse Weibull (KIW) distribution [12], a generalisation of (1), is given by

$$F(x; \psi) = 1 - \left[1 - \exp\left\{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right\}\right]^{\eta}, \tag{3}$$

where $\psi = \{\alpha, \lambda, \beta, \eta\}$. Let $G(x; \zeta)$ be any baseline CDF of any distribution which depends on parameter(s) ζ, then the survival function is given by $\overline{G}(x; \zeta) = 1 - G(x; \zeta)$. The CDF of OGE-family of distributions [14] is defined by

$$F(x; \omega, \theta, \zeta) = \left(1 - \exp -\theta \frac{G(x; \zeta)}{\overline{G}(x; \zeta)}\right)^{\omega}, \quad x > 0; \omega, \zeta, \theta > 0, \tag{4}$$

where $\theta > 0$, $\omega > 0$ are additional scale and shape parameters, respectively. The PDF corresponding to (4) is given by

$$f(x; \omega, \theta, \zeta) = \frac{\theta\omega g(x; \zeta)}{\overline{G}(x; \zeta)^2} \exp -\theta \frac{G(x; \zeta)}{\overline{G}(x; \zeta)} \left(1 - \exp -\theta \frac{G(x; \zeta)}{\overline{G}(x; \zeta)}\right)^{\omega-1}, \tag{5}$$

where $g(x; \zeta)$ is the corresponding baseline PDF.

So, we define a new five-parameter distribution dubbed odd generalised exponentiated KIW distribution (henceforth odd KIW or OKIW). The CDF of OKIW follows from (4) and (3) by taking $G(x; \zeta)$ to be equation (3) and $g(x; \zeta)$ to be the PDF corresponding to (3) with $\zeta = \{\alpha, \lambda, \beta, \eta\}$ and also taking $\theta = 1$ in (4) so that we utilise a *standard* OGE-family generator. Consequently, the CDF of OKIW becomes

$$F(x; \zeta, \omega) = [1 - e^{1-\{1-e^{(-\lambda(\frac{\alpha}{x})^{\beta})}\}^{-\eta}}]^{\omega}, \quad x > 0, \tag{6}$$

where $\alpha > 0$, $\lambda > 0$, $\beta > 0$, $\eta > 0$, and $\omega > 0$. Here, λ, α are scale parameters and β, η, ω are shape parameters.

2.1. Quantile and median of OKIW distribution

The OKIW quantile function is obtained by solving $F(x_q) = q$, thus, from (6), yielding

$$x_q = \alpha\left[\frac{-1}{\lambda}\left\{\log\left(1 - \left[1 - \log\left(1 - q^{\frac{1}{\omega}}\right)\right]^{\frac{-1}{\eta}}\right)\right\}\right]^{\frac{-1}{\beta}}. \tag{7}$$

From (7) we can obtain the median of OKIW distribution by substituting $q = \frac{1}{2}$ to get

$$Median = \alpha\left[\frac{-1}{\lambda}\left\{\log\left(1-\left[1-\log\left(1-\frac{1}{2}^{\frac{1}{\omega}}\right)\right]^{\frac{-1}{\eta}}\right)\right\}\right]^{\frac{-1}{\beta}}. \tag{8}$$

2.2. Survival function, PDF and hazard rate function of OKIW

The survival function of $X \sim OKIW(\varphi)$ is given by

$$S(x;\ \varphi) = 1 - [1 - e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega} \tag{9}$$

and the PDF of OKIW follows from equations (5) and (3) and is given by

$$f(x;\ \varphi) = \beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}[1 - e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega-1}$$

$$\times\, e^{1-[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}[1 - e^{-\lambda(\frac{a}{x})^{\beta}}]^{-\eta-1} \tag{10}$$

and the hazard rate function is thus given by

$$h(x;\varphi) = \frac{\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega-1}e^{1-[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}}{1-[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{(\eta+1)}}, \tag{11}$$

where $x > 0$ and $\varphi = \{\alpha,\ \lambda,\ \beta,\ \eta,\ \omega\}$.

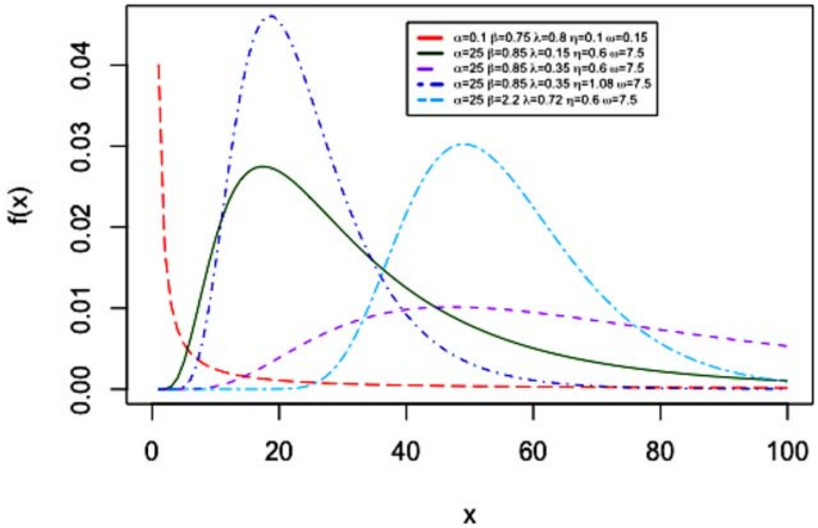

Figure 1. Plot of the OKIW PDF for some parameters values.

The graphs of hazard rate function for different values of the parameters exhibit various shapes such as monotone, non-monotone, unimodality and upside down bathtub shapes. These are very attractive features that render the OKIW distribution suitable for modelling monotonic and non-monotonic hazard behaviours which are more likely to be encountered in practical situations like reliability analysis, human mortality and biomedical applications, thus enhancing its adaptability to fit diverse survival data.

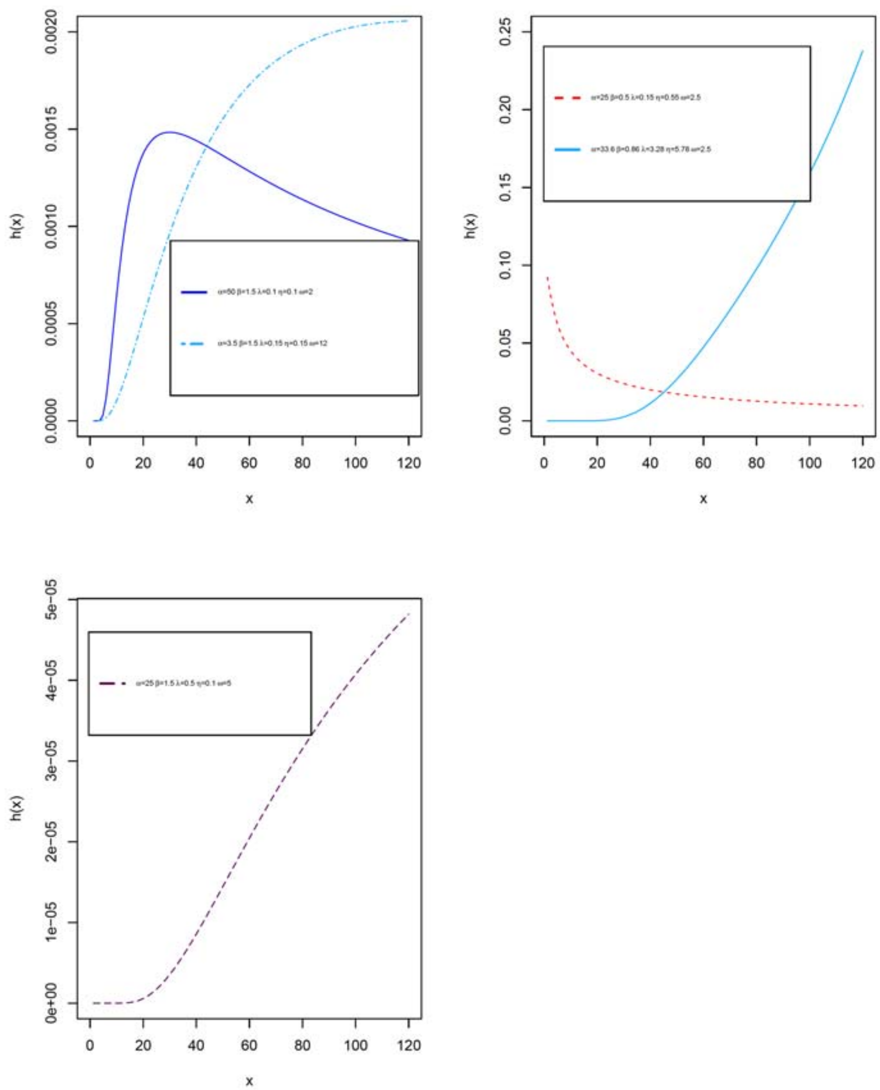

Figure 2. Plot of the OKIW hazard rate for some parameters values.

3. Mathematical Properties

3.1. Moments

Moments of a statistical distribution are critical in any statistical analysis since they are used to study characteristics of a distribution which include measures of location and dispersion, skewness and kurtosis. The rth moment of the OKIW distribution is derived.

Proposition 3.1. *If* $X \sim OKIW(\varphi)$, *where* $\varphi = \{\alpha, \beta, \lambda, \eta, \omega\}$, *then the rth non-central moment is given by*

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)}$$

$$\times \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)\Gamma\left(1-\frac{r}{\beta}\right)}{j!k!\Gamma(j\eta+\eta+1)}.$$

Proof. The rth moment of a random variable X with PDF $f(x; \varphi)$ is defined by

$$\mu'_r = \int_0^{\infty} x^r f(x; \varphi)dx. \tag{12}$$

Substituting from (10) into (12), we get

$$\mu'_r = \int_0^{\infty} x^r\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega-1}$$

$$\times e^{1-|1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}|^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta-1}dx. \tag{13}$$

Since $0 < 1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}} < 1,$ we have by binomial expansion

$$\mu'_r = \int_0^{\infty}\beta\eta\lambda\omega\alpha^{\beta}x^r x^{-(\beta+1)}\sum_{i=0}^{\infty}\binom{\omega-1}{i}(-1)^i e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}}$$

$$\times e^{1-\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta-1}dx.$$

Grouping exponential terms and applying power series expansion yields

$$e^{-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-\eta}} e^{-[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta}} = \sum_{j=0}^{\infty} \frac{(-1)^{j}(i+1)^{j}}{j!}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-j\eta},$$

so,

$$\mu_r' = \int_0^{\infty} \beta\eta\lambda\omega\alpha^{\beta} x^{r} x^{-(\beta+1)} \sum_{i=0}^{\infty} \binom{\omega-1}{i} (-1)^{i} e^{i} \sum_{j=0}^{\infty} \frac{(-1)^{j}(i+1)^{j}}{j!}$$

$$\times [1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-j\eta}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta-1} e^{1-\lambda\left(\frac{\alpha}{x}\right)^{\beta}} dx. \tag{14}$$

By generalised binomial expansion for negative powers, we have

$$[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-j\eta}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta-1} = \sum_{k=0}^{\infty} \frac{\Gamma(k+j\eta+\eta+1)}{k!\,\Gamma(j\eta+\eta+1)} e^{-\lambda k\left(\frac{\alpha}{x}\right)^{\beta}}.$$

Hence, the integral becomes

$$\mu_r' = \int_0^{\infty} \beta\eta\lambda\omega\alpha^{\beta} x^{r} x^{-(\beta+1)} \sum_{i=0}^{\infty} \binom{\omega-1}{i} (-1)^{i} e^{-(i+1)} \sum_{j=0}^{\infty} \frac{(-1)^{j}(i+1)^{j}}{j!}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(k+j\eta+\eta+1)}{k!\,\Gamma(j\eta+\eta+1)} e^{-\lambda(k+1)\left(\frac{\alpha}{x}\right)^{\beta}} dx. \tag{15}$$

Setting

$$u = \lambda(k+1)\alpha^{\beta} x^{-\beta} \Rightarrow \frac{du}{dx} = (-\beta)\lambda\alpha^{\beta}(k+1)x^{-(\beta+1)}$$

and

$$x = \left[\frac{u}{\lambda\alpha^{\beta}(k+1)}\right]^{\frac{-1}{\beta}},$$

thus,

$$\mu'_r = \frac{MD}{\beta\alpha^{\beta}\lambda(k+1)}[\alpha^{\beta}\lambda(k+1)]^{\frac{r}{\beta}}\int_0^{\infty} u^{\frac{-r}{\beta}} e^{-u} du$$

$$= \frac{MD}{\beta\alpha^{\beta}\lambda(k+1)}[\alpha^{\beta}\lambda(k+1)]^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right), \quad r < \beta,$$

by the definition of gamma function in the form $\Gamma(\phi) = \int_0^{\infty} u^{\phi-1}e^{-u}du$, where

$M = \beta\eta\lambda\omega\alpha^{\beta}$ and

$$D = \sum_{i=0}^{\infty}\binom{\omega-1}{i}(-1)^i e^{-(i+1)}\sum_{j=0}^{\infty}\frac{(-1)^j(i+1)^j}{j!}\sum_{k=0}^{\infty}\frac{\Gamma(k+j\eta+\eta+1)}{k!\Gamma(j\eta+\eta+1)}.$$

Substituting back M and D in the equation above and simplifying, we have

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)}$$

$$\times \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r(k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)}{j!k!\Gamma(j\eta+\eta+1)}\frac{\Gamma\left(1-\frac{r}{\beta}\right)}{j!k!\Gamma(j\eta+\eta+1)}.$$

This completes the proof. □

3.2. Moment generating functions

Moment generating functions (MGFs) are special functions used to find moments and functions of moments of a random variable and also in identifying its distribution function by invoking the uniqueness of MGFs.

Proposition 3.2. *If* $X \sim OKIW(\varphi)$, *where* $\varphi = \{\alpha, \beta, \lambda, \eta, \omega\}$, *then the MGF of X is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} t^r(-1)^{i+j}(i+1)^j\binom{\omega-1}{i}e^{-(i+1)}$$

$$\times \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r (k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)\Gamma\left(1-\frac{r}{\beta}\right)}{j!k!r!\Gamma(j\eta+\eta+1)}.$$

Proof. By the definition of MGF, we have

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{+\infty} e^{tX} f(x) dx.$$

Invoking the power series expansion of MGF, we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{i+j}(i+1)^j \binom{\omega-1}{i} e^{-(i+1)}$$

$$\times \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r (k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)\Gamma\left(1-\frac{r}{\beta}\right)}{j!k!\Gamma(j\eta+\eta+1)},$$

and hence

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} t^r (-1)^{i+j}(i+1)^j \binom{\omega-1}{i} e^{-(i+1)}$$

$$\times \frac{\eta\omega\lambda^{\frac{r}{\beta}}\alpha^r (k+1)^{\left(\frac{r}{\beta}-1\right)}\Gamma(k+j\eta+\eta+1)\Gamma\left(1-\frac{r}{\beta}\right)}{j!k!r!\Gamma(j\eta+\eta+1)},$$

where $r < \beta$. This completes the proof. □

3.3. Distribution of order statistics

Order statistics are fundamental tools in non-parametric statistics and inference. Let $X_1, X_2, ..., X_n$ be *iid* forming a simple random sample of size *n* from $OKIW(\varphi)$ distribution with CDF $F(x; \varphi)$ and PDF $f(x; \varphi)$. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics obtained from the sample. The PDF of *s*th order statistic, for $s = 1, ..., n,$ is given by

$$f_{s:n}(x;\,\varphi) = \frac{1}{B(s,\, n-s+1)}[F(x;\,\varphi)]^{s-1}[1-F(x;\,\varphi)]^{n-s} f(x;\,\varphi), \quad (16)$$

where $B(.,\,.)$ denotes a beta function. Since $0 < F(x;\,\varphi) < 1$ for $x > 0,$ we have

$$[1-F(x;\,\varphi)]^{n-s} = \sum_{m=0}^{n-s}\binom{n-s}{m}(-1)^m[F(x;\,\varphi)]^m. \quad (17)$$

Thus, substituting equation (17) into equation (16), we obtain

$$f_{s:n}(x;\,\varphi) = \frac{1}{B(s,\, n-s+1)} f(x;\,\varphi)\sum_{m=0}^{n-s}\binom{n-s}{m}(-1)^m[F(x;\,\varphi)]^{m+s-1}. \quad (18)$$

Finally substituting equations (6) and (10) into (18), we obtain

$$f_{s:n}(x;\,\varphi) = \frac{\beta\eta\lambda\omega\alpha^{\beta}x^{-(\beta+1)}}{B(s,\, n-s+1)}[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega-1}$$

$$\times e^{1-[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{-\eta-1}$$

$$\times \sum_{m=0}^{n-s}\binom{n-s}{m}(-1)^m[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{-\eta}}]^{\omega(m+s-1)}.$$

3.4. Entropy

An entropy is a measure of variation or lack of predictability of a random variable X. The most common entropy measures are Shannon and Rényi [9]. If X has a pdf $f(.),$ then the v order Rényi entropy is defined by

$$E_R(v) = \frac{1}{1-v}\ln\left[\int_0^{\infty} f^{v}(x)dx\right], \quad (19)$$

where $v > 0$ and $v \neq 1.$ The Shannon entropy is given by $\mathbb{E}[-\ln(f(x))].$ It is a special case of Rényi entropy when $v \to 1.$

Proposition 3.3. *If* $X \sim OKIW(\varphi)$, *where* $\varphi = \{\alpha, \beta, \lambda, \eta, \omega\}$, *then its Rényi entropy,* $E_R(v)$, *is given by*

$$E_R(v) = \frac{1}{1-v}\left(\ln\left(\frac{(\eta\omega)^v \beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}(\lambda\alpha^\beta(v+k))^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\right)\right)$$

$$+\frac{1}{1-\nu}\left(\ln\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)}\right)\right)$$

$$+\frac{1}{1-v}\left(\ln\left(\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)\right)\right).$$

Proof. From equation (19), we have

$$\int_0^\infty f^v(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} x^{-v(\beta+1)}[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\}^{-\eta}}]^{v(\omega-1)}$$

$$\times e^{v(1-[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta)}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-v(\eta+1)}dx. \quad (20)$$

Since $0 < 1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\}^{-\eta}} < 1$, applying binomial expansion on $[1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^\beta\right)}\}^{-\eta}}]^{v(\omega-1)}$ and substituting into the integral, we get

$$\int_0^\infty f^v(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} x^{-v(\beta+1)} \sum_{i=0}^{\infty}\binom{v(\omega-1)}{i}(-1)^i e^{i-i\left[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}\right]^{-\eta}}$$

$$\times e^{v(1-[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-\eta}-\lambda\left(\frac{\alpha}{x}\right)^\beta)}[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-v(\eta+1)}dx. \quad (21)$$

Grouping the exponent terms and applying power series expansion, then substituting back into the integral, we obtain

$$\int_0^\infty f^v(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} x^{-v(\beta+1)} \sum_{i=0}^\infty \binom{v(\omega-1)}{i} (-1)^i e^{(v+i)} e^{-v\lambda\left(\frac{\alpha}{x}\right)^\beta}$$

$$\times \sum_{j=0}^\infty \frac{(-1)^j (v+i)^j}{j!} [1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-j\eta} [1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-v(\eta+1)} dx. \quad (22)$$

Now, by generalised binomial expansion for negative powers, we have

$$[1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-j\eta} [1-e^{-\lambda\left(\frac{\alpha}{x}\right)^\beta}]^{-v(\eta+1)} = \sum_{k=0}^\infty \frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)} e^{-k\lambda\left(\frac{\alpha}{x}\right)^\beta}. \quad (23)$$

Hence, substituting back and regrouping exponent terms, the integral becomes

$$\int_0^\infty f^\gamma(x)dx = \int_0^\infty (\beta\eta\lambda\omega)^v \alpha^{v\beta} \sum_{i=0}^\infty \binom{v(\omega-1)}{i} (-1)^i e^{(v+i)} \sum_{j=0}^\infty \frac{(-1)^j (v+i)^j}{j!}$$

$$\times \sum_{k=0}^\infty \frac{\Gamma(k+v(\eta+1)+jn)}{k!\Gamma(v(\eta+1)+j\eta)} x^{-v(\beta+1)} e^{-\lambda(v+k)\left(\frac{\alpha}{x}\right)^\beta} dx. \quad (24)$$

Letting $u = \lambda(v+k)\alpha^\beta x^{-\beta} \Rightarrow du = (-\beta)\lambda\alpha^\beta (v+k) x^{-(\beta+1)}dx$ and $x = \left[\frac{u}{\lambda\alpha^\beta(v+k)}\right]^{\frac{-1}{\beta}}$, thus,

$$\int f^v(x)dx = \frac{MD^-(\lambda\alpha^\beta(v+k))^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)} \int_0^\infty u^{\frac{-1}{\beta}\{-v(\beta+1)+\beta+1\}} e^{-u} du,$$

where $D = (\beta\eta\lambda\omega)^v \alpha^{v\beta}$ and $D^- = \frac{D}{\beta\lambda\alpha^\beta} = (\eta\omega)^v \beta^{v-1} \lambda^{v-1} \alpha^{v\beta-1}$ and

$$M = \sum_{i=0}^{\infty}\binom{v(\omega-1)}{i}(-1)^i e^{(v+i)}\sum_{j=0}^{\infty}\frac{(-1)^j(v+i)^j}{j!}\sum_{k=0}^{\infty}\frac{\Gamma(k+v(n+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)}.$$

So, invoking the definition of gamma function in the form $\Gamma(\phi) = \int_0^\infty u^{\phi-1}e^{-u}du$, we obtain

$$\int f^v(x)dx = \frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}(\lambda\alpha^\beta(v+k))^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)}$$

$$\times\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right),\quad \frac{1}{\beta}\{-v(\beta+1)+\beta+1\}<1.$$

Consequently,

$$\ln\left(\int f^v(x)dx\right) = \ln\left(\frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}(\lambda\alpha^\beta(v+k))^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\right)$$

$$+\ln\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)}\right)$$

$$+\ln\left(\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)\right).$$

Therefore,

$$E_R(v) = \frac{1}{1-v}\left(\ln\left(\frac{(\eta\omega)^v\beta^{v-1}\lambda^{v-1}\alpha^{v\beta-1}(\lambda\alpha^\beta(v+k))^{\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}}}{(v+k)}\right)\right)$$

$$+\frac{1}{1-v}\left(\ln\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j}\binom{v(\omega-1)}{i}\frac{(v+i)^j}{j!}e^{(v+i)}\frac{\Gamma(k+v(\eta+1)+j\eta)}{k!\Gamma(v(\eta+1)+j\eta)}\right)\right)$$

$$+\frac{1}{1-v}\left(\ln\left(\Gamma\left(1-\frac{1}{\beta}\{-v(\beta+1)+\beta+1\}\right)\right)\right).$$

This completes the proof. □

4. Estimation of Model Parameters

In this section, we present estimates of the parameters of OKIW distribution using maximum likelihood estimation. The elements of the score function are presented.

The maximum likelihood estimators

MLEs are important point estimators in statistical inference. We estimate the MLEs of the model parameters from complete samples.

Let $\boldsymbol{X} = (X_1, X_2, ..., X_n)^T$ be a random sample from OKIW distribution with unknown parameter vector $\boldsymbol{\Theta} = (\alpha, \beta, \lambda, \eta, \omega)^T$, then the likelihood function $L(\boldsymbol{X}, \boldsymbol{\Theta})$ is defined as

$$L(\boldsymbol{X}; \boldsymbol{\Theta}) = \prod_{i=1}^{n} f(x_i; \boldsymbol{\Theta}).$$

The log-likelihood function for $\boldsymbol{\Theta}$ is

$$\ln(L(\boldsymbol{X}, \boldsymbol{\Theta})) = n\ln(\beta\eta\lambda\omega\alpha^{\beta}) - (\beta+1)\sum_{i=1}^{n}\ln x_i - (\eta+1)\sum_{i=1}^{n}\ln(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})$$

$$+ (\omega-1)\sum_{i=1}^{n}\ln(1-e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\}^{-\eta}})$$

$$+ \sum_{i=1}^{n}(1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}). \tag{25}$$

By maximising the log-likelihood function, we obtain the components of the score function vector $U(\boldsymbol{\Theta}) = \frac{\partial \ln L}{\partial \alpha}, \frac{\partial \ln L}{\partial \beta}, \frac{\partial \ln L}{\partial \lambda}, \frac{\partial \ln L}{\partial \eta}, \frac{\partial \ln L}{\partial \omega}$, which are given by:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n\beta}{\alpha} + \beta\lambda \sum_{i=1}^{n} \frac{\eta e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta-1} - \left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i \left(1 - (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}$$

$$- \beta\eta\lambda(\omega - 1) \sum_{i=1}^{n} \frac{e^{1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta+1} \left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i (1 - e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\}^{-\eta}})}$$

$$- \beta\lambda(\eta + 1) \sum_{i=1}^{n} \frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(\frac{\alpha}{x_i}\right)^{\beta-1}}{x_i (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})}, \tag{26}$$

$$\frac{\partial \ln L}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n} \ln(1 - e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\}^{-\eta}}), \tag{27}$$

$$\frac{\partial \ln L}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n} \ln(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}) + \sum_{i=1}^{n} \frac{(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} \ln(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})}{1 - (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}$$

$$- (\omega - 1) \sum_{i=1}^{n} \frac{e^{1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} \ln(1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})}{1 - e^{1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta}}}, \tag{28}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{\eta e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta} - \left(\frac{\alpha}{x_i}\right)^{\beta}}{1 - (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}$$

$$- (\eta + 1)\sum_{i=1}^{n} \frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \left(\frac{\alpha}{x_i}\right)^{\beta}}{1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}}$$

$$- \eta(\omega - 1)\sum_{i=1}^{n} \frac{e^{1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-(\eta+1)} \left(\frac{\alpha}{x_i}\right)^{\beta}}{1 - e^{1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta}}}, \quad (29)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + n\ln(\alpha) - \sum_{i=1}^{n} x_i - \lambda(\eta + 1)\sum_{i=1}^{n} \left(\frac{e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}}} \right)$$

$$+ \sum_{i=1}^{n} \left(\frac{-\lambda \ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta} + \eta\lambda e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-(\eta+1)} \ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1 - (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} \right)$$

$$- \lambda\eta(\omega - 1)\sum_{i=1}^{n} \left(\frac{e^{1-(1-e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-\eta} - \lambda\left(\frac{\alpha}{x_i}\right)^{\beta}} (1 - e^{-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}})^{-(\eta+1)} \ln\left[\frac{\alpha}{x_i}\right]\left(\frac{\alpha}{x_i}\right)^{\beta}}{1 - e^{1-\{1-e^{\left(-\lambda\left(\frac{\alpha}{x_i}\right)^{\beta}\right)}\}^{-\eta}}} \right). \quad (30)$$

Since there are no closed form solutions to the nonlinear equations obtained by setting the score function elements to zero, the MLEs of α, β, λ, η, and ω can be obtained by solving numerically (via iterative methods such as a Newton-Raphson algorithm) the normal equations

$$\frac{\partial \ln L}{\partial \alpha} = 0, \frac{\partial \ln L}{\partial \beta} = 0, \frac{\partial \ln L}{\partial \lambda} = 0, \frac{\partial \ln L}{\partial \eta} = 0, \frac{\partial \ln L}{\partial \omega} = 0,$$

thus yielding the ML estimate: $\hat{\mathbf{\Theta}} = \{\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\eta}, \hat{\omega}\}$.

5. Simulation Study

In this section, a simulation study is conducted to assess the performance of OKIW distribution by examining the average bias and root mean square error of the maximum likelihood estimates for each parameter. We conduct various simulations for different sample sizes and different parameter values. Equation (7) is used to generate random data from the OKIW distribution. That is, if $Q \sim Unif(0, 1)$, then

$$X_i = \alpha\left[\frac{-1}{\lambda}\{\log(1 - [1 - \log(1 - Q_i^{\frac{1}{\omega}})]^{\frac{-1}{\eta}})\}\right]^{\frac{-1}{\beta}}.$$

The simulation study is repeated for $N = 1500$ times each with sample size $n = 50, 150, 300, 500$ and parameter values in set $I : \beta = 2.5, \lambda = 1, \omega = 5, \alpha = 15, \eta = 0.5$ and $II : \beta = 0.25, \lambda = 1, \omega = 8, \alpha = 20, \eta = 0.5$. We compute:

(a) Average bias of the MLE $\hat{\mathbf{\Theta}}$ of the parameter $\mathbf{\Theta} = \{\beta, \lambda, \omega, \alpha, \eta\}$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}).$$

(b) Root mean squared error (RMSE) of the MLE $\hat{\mathbf{\Theta}}$ of the parameter $\mathbf{\Theta} = \{\beta, \lambda, \omega, \alpha, \eta\}$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{\Theta}} - \mathbf{\Theta})^2}.$$

The average bias and RMSE values of the parameters β, λ, ω, α and η for different sample sizes are presented in Table 1. From the results, it can be seen that as the sample size *n* increases, the RMSEs decrease and also that for all the parametric values, average biases decrease with increasing sample size *n*. Thus, the MLEs together with their asymptotic results can be utilized in constructing confidence intervals even for fairly small sample sizes.

Table 1. Monte Carlo simulation study results

		I		**II**	
Parameter	***n***	**Average Bias**	**RMSE**	**Average Bias**	**RMSE**
β	50	0.00234923	1.254459	0.05164315	0.248549
	150	0.223251	1.649059	0.1151803	0.235575
	300	0.3311303	1.26418	0.1193271	0.207821
	500	0.3507965	1.190819	0.1143479	0.196312
η	50	1.09536	1.979973	0.8198106	1.726021
	150	0.5398698	1.141238	0.1854606	0.644092
	300	0.2940589	0.7577926	0.05933698	0.310956
	500	0.1550613	0.45176	0.00615733	0.300049
λ	50	2.957456	5.444899	8.967035	13.286160
	150	0.6784766	1.903525	4.117874	6.093303
	300	0.4062774	1.219227	2.915989	4.751907
	500	0.2391422	1.212782	1.735717	2.900594
ω	50	–1.179363	6.627073	–4.178481	8.678087
	150	0.7065778	7.084176	–2.183191	8.474572
	300	–0.2239991	5.670298	–1.718957	8.244601
	500	–0.0842186	5.030306	–0.9368763	7.488534
α	50	9.743556	21.3734	32.72563	233.448000
	150	10.00698	18.98498	36.87842	157.671000
	300	7.357111	14.15756	55.73701	83.912370
	500	5.017567	9.498253	59.83609	69.012460

6. Applications to Survival Data

In this section, we use three real different data sets to illustrate the flexibility of OKIW distribution in the modelling of survival data as well as compare it with EKIW (exponentiated Kumaraswamy inverse Weibull) and EPLG (exponentiated power Lindley geometric [1]) distributions. The PDFs of EKIW and EPLG distributions are given by

$$f_{EKIW}(x) = \beta\eta\lambda\theta\alpha^{\beta}x^{-(\beta+1)}e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}[1 - e^{-\lambda\left(\frac{\alpha}{x}\right)^{\beta}}]^{\eta-1}$$

$$\times [1 - \{1 - e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\}^{\eta}]^{\theta-1}$$

and

$$f_{EPLG}(x) = \frac{\frac{\alpha\beta(1-\theta)\lambda^{2}x^{\beta-1}}{\lambda+1}(1+x^{\beta})e^{-\lambda x^{\beta}}\left[1 - \left(1 + \frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha-1}}{\left(1 - \theta\left[1 - \left(1 + \frac{\lambda x^{\beta}}{\lambda+1}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right)^{2}},$$

respectively.

For each data set, we compute the estimates of the parameters of OKIW and EKIW distributions. The MLEs of the OKIW and EKIW parameters are computed by maximising the log-likelihood function via the nonlinear optimisation function `nlm` in **R**. After estimating models parameters, we also compute the information-criterion statistics: Akaike information criterion $(AIC = 2p - 2\ln(\hat{L}))$, corrected Akaike information criterion $(AICC = AIC + \frac{2p(p+1)}{(n-p-1)})$, and Bayesian information criterion $(BIC = p\ln(n) - 2\ln(\hat{L}))$, where $\hat{L} = L(\hat{\boldsymbol{\Theta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. The standard errors for parameters are useful in constructing confidence intervals for the parameters. When comparing models, the model with the smallest AIC is considered to be the best fit model for a given data set. We then plot the histogram of the data sets and estimated probability density functions of OKIW and EKIW distributions.

6.1. Kevlar 49/epoxy strands failure times data

This data set consists of 101 observations corresponding to the failure times (in hours) (time until rupture) of Kevlar 49/epoxy strands with pressure at 90%. Theses data were originally given in [3], and analysed in [5]. The maximum likelihood estimates of the parameters of OKIW and EKIW distributions are given in Table 2 along with standard errors, –2 log-likelihood, AIC, AICC and BIC. The results show that OKIW provides a better fit than EKIW model.

Table 2. MLEs estimates of OKIW and EKIW for Kevlar data

	MLEs estimates of the parameters						Statistics		
Distribution	β	η	λ	φ	α	**–2 log *L***	AIC	AICC	BIC
OKIW	0.14300	45.95447	2.95200	1.07503	10.88135	206.00350	216.00350	216.63508	229.07910
Std. errors	0.03691	0.00340	0.09331	0.50853	0.00367				
	β	η	λ	θ	α				
EKIW	0.28257	132.73000	2.71820	0.30503	17.20000	209.48030	219.48030	220.11188	232.55590
Std. errors	0.05396	0.00114	0.19954	0.12880	0.00891				

Plots of the estimated PDFs of OKIW and EKIW and histogram of the data are given in Figure 3. The plots further indicate that OKIW is superior to EKIW in terms of empirical model fitting.

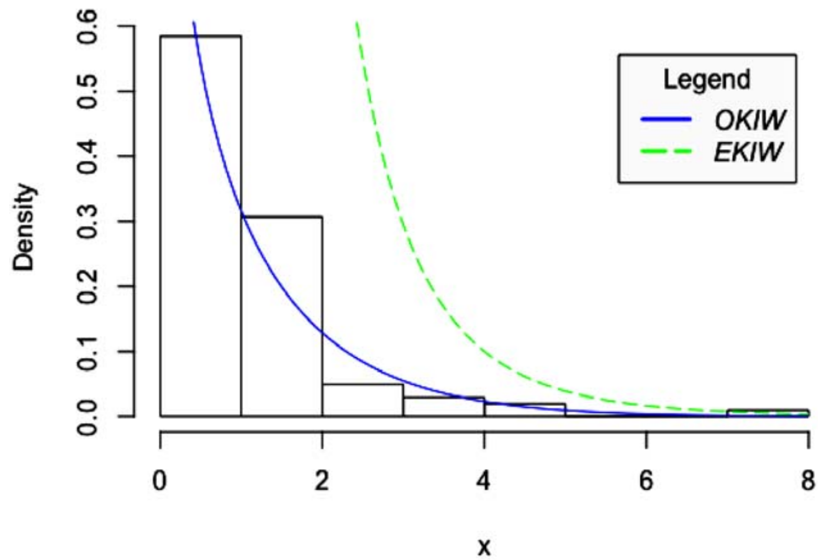

Figure 3. Histogram and estimated densities for Kevlar data.

6.2. Strength of the glass fibres data

This data set represents the strength of 1.5cm glass fibres, recorded in a laboratory [13] and is analysed in [1]. The maximum likelihood estimates of the parameters of OKIW and EKIW distributions are given in Table 3 along with standard errors, $-2\log$-likelihood, AIC, AICC and BIC. The results show that OKIW distribution provides a better fit than EKIW model but not as good as EPLG model for these data.

Table 3. MLEs estimates of OKIW and EKIW for glass fibres

	MLEs estimates of the parameters						**Statistics**		
Distribution	β	η	λ	φ	α	**–2 log *L***	AIC	AICC	BIC
OKIW	0.73504	74.65615	1.04537	1.17434	12.14228	33.04560	43.00456	44.05719	53.76127
Std. errors	0.17243	0.31757	0.73751	0.50496	10.75254				
	β	η	λ	θ	α				
EKIW	1.47210	19.13500	0.01645	0.71991	57.89700	61.97205	63.02468	72.68772	61.97205
Std. errors	0.30800	0.00139	0.01603	0.40493	0.00006				
	β	-	λ	θ	α				
EPLG*	0.9173	-	3.08735	0.94201	0.69931	23.88	31.88	-	40.45
Note:* MLEs estimates as in Alizadeh et al. [1]									

Plots of the estimated PDFs of OKIW and EKIW and histogram of the data are given in Figure 4. The plots further indicate that the OKIW is superior to EKIW in terms of empirical model fitting.

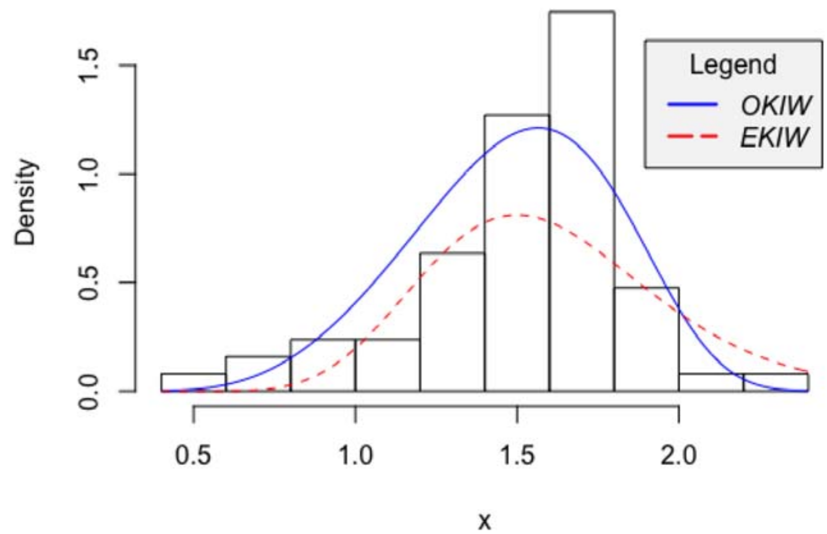

Figure 4. Histogram and estimated densities for strength of glass fibres data.

6.3. Guinea pigs data

This data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, reported by [4] and analysed by [1]. The maximum likelihood estimates of the parameters of OKIW and EKIW distributions are given in Table 4 along with standard errors, $-2\log$-likelihood, AIC, AICC and BIC. The results show that OKIW provides a better fit than EKIW and EPLG models.

Table 4. MLEs estimates of OKIW and EKIW for guinea pigs

	MLEs estimates of the parameters						Statistics		
Distribution	β	η	λ	φ	α	**–2 log *L***	AIC	AICC	BIC
OKIW	0.18836	22.74434	1.69122	3.88360	40.12850	189.054	198.9712	199.88029	210.43733
Std. errors	0.06235	0.18773	0.20959	2.93846	0.02005				
	β	η	λ	θ	α				
EKIW	0.40794	84.10154	1.19373	0.90356	49.01261	190.8867	200.66930	201.57839	212.27003
Sid. errors	0.10228	0.00248	0.26919	0.56982	0.00274				
	β	-	λ	θ	α				
EPLG*	4.34313	-	0.23122	0.99998	6.7385	849.25	857.2500	-	866.3500
Note:* MLEs estimates as in Alizadeh et al. [1]									

Plots of the estimated PDFs of OKIW and EKIW and histogram for the data are given in Figure 5. The plots further indicate that the OKIW is superior to EKIW.

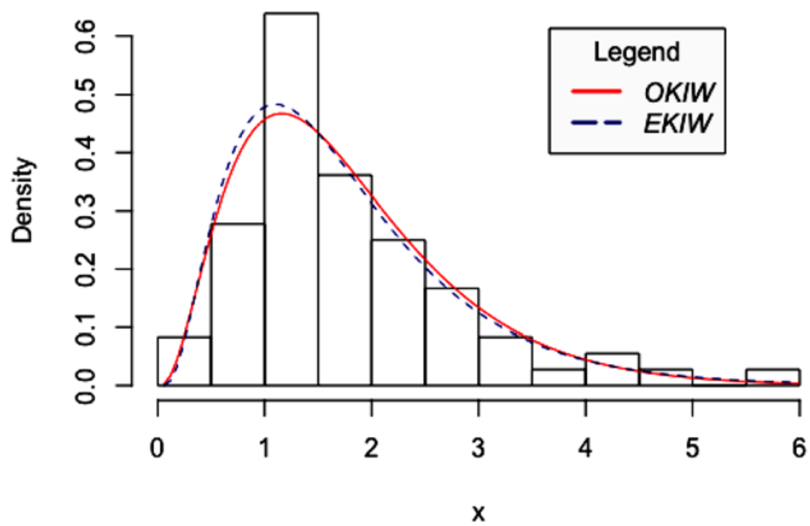

Figure 5. Histogram and estimated densities for guinea pigs data.

7. Conclusion

In this study, we propose a new five-parameter lifetime model, called the OKIW distribution, and study its mathematical and statistical properties. The model hazard function exhibits versatile behaviours: decreasing, increasing, J-shaped, reversed-J shaped, unimodal and upside-down bathtub. The PDF also has varied shapes suitable for modelling right-skewed, left-skewed, and approximately symmetric survival data and also survival data with highly varied kurtosis. We obtain point estimates of the parameters using maximum likelihood estimation. A simulation study is carried out to examine the performance of the MLEs in terms of the average biases and root mean square errors. It is established that MLEs and their asymptotic results can be utilized in constructing confidence intervals even for fairly small sample sizes. Applications of the model to real survival data prove empirically its flexibility and usefulness in modeling various types of biomedical and reliability data and that the model offers a more superior fit than EKIW distribution. Thus, we anticipate that OKIW distribution may attract wider applications in survival analysis, reliability analysis and insurance.

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