ON THE SPECTRUM OF A CLASS OF *NÖRLUND* OPERATORS

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DECLARATION

This thesis is my original work and has not been presented elsewhere for a degree in any other University.

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This thesis has been submitted for examination with our approval as University Supervisors.

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DEDICATION

To my beloved family for their support, encouragement and patience throughout my period of study.

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TABLE OF CONTENTS

SYMBOLS AND ABBREVIATIONS

SYMBOLS

1. $\mathbb{R}^+$ - the set of positive real numbers
2. $\mathbb{R}$ - the set of real numbers
3. $\mathbb{C}$ - the set of complex numbers
4. $\mathbb{k}$ - set of scalars from the set of complex numbers
5. θ - the zero vector
6. I - the identity matrix
7. $\|.\|$- norm of
8. $\rightarrow$- tend to
9. $O(1)$ - capital order i.e. $x_n = O(1)$ if there exist $M \in \mathbb{R}^+$ such that $|x_n| \leq M, \forall n$
10. $o(1)$ - small order i.e. $x_n \in o(n)$ as $n \rightarrow \infty$ if $x_n \rightarrow 0$ as $n \rightarrow \infty$
11. $\emptyset$- empty set

ABBREVIATIONS

1. s - the set of all sequences
2. c_0- the set of all sequences which converge to zero - null sequences
3. $\ell_p(0 < p < \infty)$ - sequences such that $\sum\limits_{k=0}^{\infty} |x|^p < \infty$
4. c - convergent sequences
5. ℓ_∞ - bounded sequences i.e. sequences x such that $\sup_k |x_k| < \infty$
6. bv - sequences of bounded variation i.e. sequences x such that $\sum\limits_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$
7. bv_0 - sequences of bounded variation with $x_k \rightarrow 0$ as $k \rightarrow \infty$
8. $\overline{bv_0}$ - statistically null bounded variation sequences i.e. if $\triangle x_k \in \ell_1$ such that $\delta(\{k_i \in \mathbb{N} : i \in \mathbb{N}\}) = 1$ where $\triangle x_{k_i} = x_{k_i} - x_{k_{i+1}}$for all $i \in \mathbb{N}$

9. bs - bounded series i.e. sequences x such that $\sup_{n\geq 0}\{\sum_{k=0}^{n}|x_k|\} < \infty$

10. cs - convergent series i.e sequences x such that $\sum_{k=0}^{\infty} x_k$ is convergent

11. $w_p(0 < p < \infty)$ - the space of strongly Cesaro summable complex sequences of order 1 index p i.e. the set of all sequences $x = (x_k)_{k=1}^{\infty}$such that there exist a number ℓ depending on x for which $\sum_{k=0}^{\infty} |x_k - \ell|^p = o(n)$

12. $w_p(0)$ - the space of strongly Cesaro summable complex sequences of order 1 index p such that $\ell = 0$

13. w.l.o.g - without loss of generality

ABSTRACT

Spectral theory is an important branch of Mathematics due to its application in other branches of science. In summability theory, different classes of matrices have been investigated and characterized. There are various types of summability methods e.g. *Nörlund* operators, Cesaro, Riesz, Euler, Abel and many others. This research investigates and determines the spectrum of a class of *Nörlund* operators on the sequence spaces c_0, c and bv_0. This is achieved by constructing the resolvent operator $T_\lambda = (T - I\lambda)^{-1}$, the spectrum is then given by all the values of λ for which T_λ does not exist as a bounded operator on the sequence space c_0, c and bv_0. It is shown that the spectrum consists of the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$. This will find application in the development of Tauberian and Mercerian theorems for the *Nörlund* operator which are used to determine the limit or sum of a convergent sequence or series. In addition the eigenvalues and the eigenvectors are used to solve infinite linear system of equations. Infinite dimensional linear systems appears naturally when studying control problems for systems modelled by linear partial differential equations. Many problems in dynamical systems can be written in form of infinite differential systems e.g Mathieu equation, Hill's equation.

CHAPTER ONE
INTRODUCTION AND LITERATURE REVIEW

1.1 Background of the Study

Basic concepts on the spectral theory, summability theory and some results of previous studies with a bearing on the topic of study are explored in this section.

1.1.1 Introduction

Spectral theory is an important branch of Mathematics due to its application in other branches of science. It is proved to be a standard tool of mathematical sciences because of its usefulness and application oriented scope in different fields. In numerical analysis, the spectral values may determine whether a descretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system. In quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star. In structural mechanics, it may determine whether an automobile is too noisy or whether a building will collapse during an earthquake. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In probability theory, they may be used to determine the rate of convergence of a Markov process.

Mathematics, especially mathematical analysis, develops and is maintained via the concept of convergence of sequences and series. Even in applied science and engineering, one is interested in the convergence of sequences and series of results generated during experiments. Established theorems such as ratio theorem and integral theorem are not applicable in a variety of sequences and series. Even where they apply, they just determine convergence but not the limit or sum of a convergent sequence or series. Tauberian and Mercerian theorems handles this problem well. The convergence and even limit of a convergent sequence or series is determined from the convergence of some transform of it together with a side condition.

In summability theory, different classes of matrices have been investigated and characterized into classes. There are various types of summability methods like *Nörlund* operators, Cesaro, Riesz, Euler, Abel and many others. Throughout this research $\mathbb{N}$

denotes the set of non-negative integers. A sequence of real numbers is a real valued function whose domain is the set $\mathbb{N}$. The symbol $(x_n)_1^\infty$is used to denote an infinite sequence e.g. $(x_n)_1^\infty = (\frac{1}{n})_1^\infty = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. A sequence is said to converge if $\lim_{n\to\infty} x_n = l$ where l is a finite number. A sequence diverges if it does not converge.

A series is the sum of the sequence i.e $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots$. A series is said to converge if the n^{th} partial sum $s_n = x_1 + x_2 + \cdots + x_n$ tends to a finite limit s as $n \to \infty$. If s_n does not tend to a finite limit as $n \to \infty$, then the series is said to diverge. The value of this limit is called the sum of the series denoted by $s = \sum_{n=1}^{\infty} x_n$.

1.1.2 Classical Summability

The central point of summability theory is to find means of assigning a limit to a divergent sequence or sum to a divergent series in such a way that the sequence or series can be manipulated as though it converges, (Ruckle, 1981, pg 159-161). The most common means of summing a divergent series or sequence is that of using an infinite matrix of complex numbers.

Definition 1.1.1. (Sequence to sequence Transformation)

Let $A = (a_{nk}), n, k = 0, 1, 2, \ldots$ be an infinite matrix of complex numbers. Given a sequence $x = (x_k)_{k=0}^\infty$ define

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k,\ n = 0, 1, 2, \ldots \tag{1.1.1}$$

If the series 1.1.1, converges for all n then the sequence $y = (y_n)_{n=0}^\infty$, is called the $A-$transform of the sequence $x = (x_k)_{k=0}^\infty$. If further $y_n \to a$ as $n \to \infty$, then $(x_k)_{k=0}^\infty$ is summable A to a. There are numerous sequence to sequence transformations. Below are few well known examples.

Example 1.1.2. (Cesaro opearator)

Consider the matrix $A = (a_{nk})$, where

$$a_{nk} = \begin{cases} \frac{1}{1+n}, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{1.1.2}$$

and a sequence $(x_k)_{k=0}^\infty = (1, 0, 1, 0, \ldots)$, then the sequence $(x_k)_0^\infty$is summable by A to $\frac{1}{2}$. The matrix A is called the Cesaro operator of order 1 and is usually denoted by

$(C,1)$ or C_1. Cesaro operator of other orders are also well known, the most general is (C,α) operator which are given by

$$a_{nk} = \begin{cases} \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}}, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{1.1.3}$$

where $A_n^{\alpha} = \begin{pmatrix} \alpha+n \\ n \end{pmatrix} = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}$, $\alpha > -1$(Holder operator)

Closely related to the Cesaro operator (C,α) is the Holder operator (H,k). This is simply the product of $(C,1)$ operator k times. Its matrix is given by

$$h_{nk}^{(m)} = \left(h_{nk}^{(1)}\right)\left(h_{nk}^{(m-1)}\right) \tag{1.1.4}$$

where $h_{nk}^{(1)} = (C,1)$, (Powel and Shah, 1972, pg. 46-49).

Example 1.1.3. (Holder operator)

Closely related to the Cesaro operator (C,α) is the Holder operator (H,k). This is simply the product of $(C,1)$ operator k times. Its matrix is given by

$$h_{nk}^{(m)} = \left(h_{nk}^{(1)}\right)\left(h_{nk}^{(m-1)}\right) \tag{1.1.5}$$

where $h_{nk}^{(1)} = (C,1)$, (Powel and Shah, 1972, pg. 46-49).

Example 1.1.4. (*Nörlund operator*)

Let the sequence $\{p_n\}_0^{\infty}$ be the sequence of real numbers with $p_0 > 0$, the transformation given by

$$y_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}x_k,\ n = 0,1,2,\ldots \tag{1.1.6}$$

where $P_n = p_0 + p_1 + \cdots + p_n \neq 0$, is called a *Nörlund operator* and is denoted by (N,p). Its matrix is given by

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n}, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{1.1.7}$$

In Matrix 1.1.7, if $p_0 = 1$, $p_1 = -2$, $p_2 = p_3 = p_4 = \cdots = 0$, then a_{nk} transforms the unbounded sequence $(x_k)_{k=0}^{\infty} = (1,2,4,8,\ldots)$ to zero. If $p_n = 1$ for each $n = 0,1,2,\cdots$, then $(a_{nk}) = (C,1)$, (Powel and Shah, 1972, pg. 45-46).

Similarly in matrix 1.1.7 if $p_0 = m$, $p_1 = p_2 = p_3 = p_4 = \cdots = 0$, $m \in \mathbb{R}$ this gives the identity matrix

$$a_{nk} = \begin{cases} 1, & n = k \\ 0, & Otherwise \end{cases} \tag{1.1.8}$$

That is,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & \cdots & & \end{pmatrix} \tag{1.1.9}$$

Similarly in matrix 1.1.7 if $p_0 = p_1 = m$, $p_2 = p_3 = p_4 = \cdots = 0$, $m \in \mathbb{R}$ then the matrix is given by

$$a_{nk} = \begin{cases} 1, & n = k = 0 \\ \frac{1}{2}, & n - 1 \leq k \leq n \\ 0, & Otherwise \end{cases} \tag{1.1.10}$$

That is,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots \\ & & \cdots & & \end{pmatrix} \tag{1.1.11}$$

If $p_n = \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix} = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}$, $k > 0$ then (N, p_n) is the (C, k) operator, (Hardy, 1948 pg. 64-65).

In matrix 1.1.7, if $p_0 = p_1 = p_2 = m$, $p_3 = p_4 = p_5 \cdots = 0$**,** $m \in \mathbb{R}$ then the matrix is given by

$$b_{nk} = \begin{cases} 1, & n = k = 0 \\ \frac{1}{2}, & n = 1, 0 \leq k \leq n \\ \frac{1}{3}, & n - 2 \leq k \leq n \\ 0, & Otherwise \end{cases} \tag{1.1.12}$$

That is,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & \end{pmatrix} \tag{1.1.13}$$

This matrix represents all the *Nörlund* operators in which the first three terms of the sequence $\{p_n\}_0^\infty$ are equal and the rest are zeros. It converts any sequence to a sequence of the arithmetic mean of three consecutive terms of the original sequence apart from the first term which it maintains and the second term which is the arithmetic mean of the first two terms. This research will focus on this class of matrices.

Definition 1.1.5. (Series to series transformation)

The transformation of the series $\sum_{k=0}^{\infty} x_k$ into a convergent series $\sum_{n=0}^{\infty} y_n$ by an infinite matrix $A = (a_{nk})$ so that

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k \tag{1.1.14}$$

is called series to series transformation.

1.1.3 General Results in Classical Summability

(X,Y) is the set of all matrices A which maps X into Y. (X,Y,p) is the subset of (X,Y) for which limits or sums are preserved, i.e $A \in (c,c,p)$ means that $A_n(x) \in c$ for each n whenever $x \in c$ and $A_n(x) \to l$ as $n \to \infty$ whenever $x \to l$ as $n \to \infty$.

Definition 1.1.6. (Regular Method, Conservative method)

Let $A = (a_{nk})$, $n = 0,1,2,\ldots$ be an infinite matrix of complex numbers;

i. If the $A-$transform of any convergent sequence of complex numbers exists and converges, then A is called a conservative method. It is then written as $A \in (c,c)$.

ii If A is conservative and preserves the limits i.e

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = a,\ a \in \mathbb{C} \tag{1.1.15}$$

where $(y_n)_{n=0}^{\infty}$ is the $A-$transform of the convergent sequence $(x_n)_{n=0}^{\infty}$, then A is called regular. It is then written as $A \in (c,c,p)$.

Theorem 1.1.7. *(Kojima-Shur)*

$A \in (c,c)$ if and only if

i. $\lim\limits_{n\to\infty} a_{nk} = a_k$ for each fixed $k,\ k = 0,1,2,\dots$

ii. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = a$ as $n \to \infty$.

iii. $\sup_{n\geq 0}\{\sum\limits_{k=0}^{\infty} |a_{nk}|\} < \infty$.

(Maddox, 1970, pg. 166-167).

Theorem 1.1.8. *(Silverman-Toeplitz)*

$A \in (c,c,p)$ if and only if

i. $\lim\limits_{n\to\infty} a_{nk} = 0$ for each fixed $k,\ k = 0,1,2,\dots$.

ii. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} a_{nk} = 1$ as $n \to \infty$.

iii. $\sup_{n\geq 0}\{\sum\limits_{k=0}^{\infty} |a_{nk}|\} \leq M < \infty, M \in \mathbb{R}^{+}$.

(Maddox, 1970, pg. 165-166)

Theorem 1.1.9. $A \in (c_0, c_0)$ *if and only if*

i. $\lim\limits_{n\to\infty} a_{nk} = 0$ for each fixed $k,\ k = 0,1,2,\dots$.

ii. $\sup_{n\geq 0}\{\sum\limits_{k=0}^{\infty} |a_{nk}|\} < \infty$.

(Maddox, 1970, pg. 165-167).

Theorem 1.1.10. $A \in (l_1, l_1)$ *if and only if*

i. $\sum\limits_{n=0}^{\infty} |a_{nk}| < \infty$ for each fixed k.

ii. $\sup_k\{\sum\limits_{n=0}^{\infty} |a_{nk}|\} < \infty$.

(Limaye, 1996, pg. 88-90, 154-156)

Theorem 1.1.11. $A \in (l_1, l_p)$ *if and only if*

i. $\sup_k\{\sum\limits_{k=0}^{\infty} |a_{nk}|^p\} = M < \infty, 1 \leq p < \infty$.

ii. $\sup_{n,k} |a_{nk}| < \infty,$ for the case, $p = \infty$.

(Maddox, 1970, pg 167)

Recall: If $\sum\limits_{k=0}^{\infty} |b_{nk}| < \infty$ for each n and $\sum\limits_{k=0}^{\infty} |b_{nk}| \to 0$ as $n \to \infty$, then $\sum\limits_{k=0}^{\infty} |b_{nk}|$ is uniformly convergent in n,

Theorem 1.1.12. *(Schur)*

$A \in (l_\infty, c)$ if and only if

i. $\sum_{k=0}^{\infty} |a_{nk}|$ converges uniformly in n.

ii. There exists $\lim_{n\to\infty} a_{nk}$ for each fixed k.

Definition 1.1.13. (Spaces bv and bv_0)

The sequence space bv is such that $x \in bv$ if

$$\sum_{k=0}^{\infty} |x_{k+1} - x_k| < \infty \tag{1.1.16}$$

and $x \in bv_0$ if $x \in bv$ with $x_k \to 0$ as $k \to \infty$. That is bv_0 is the space of sequences of bounded variation with limit zero.

Theorem 1.1.14. *$A \in (bv_0, bv_0)$ if and only if*

i. $\lim_{n\to\infty} a_{nk} = 0$ for each fixed $k \geq 0$.

ii. $\sup_{m\geq 0} \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} \left(a_{nk} - a_{n-1,k}\right) \right| < \infty$.

Theorem 1.1.15. *$A \in (bv, bv)$ if and only if*

i. $\sup_{m\geq 0} \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} \left(a_{nk} - a_{n-1,k}\right) \right| < \infty$.

ii. $\sum_{k=0}^{\infty} a_{nk}$ converges for all $n \geq 0$.

Moreover $\|A\|_{(bv,bv)} = \|A\|_{(bv_0,bv_0)} = \sup_{m\geq 0} \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} \left(a_{nk} - a_{n-1,k}\right) \right|$. (Jakimovski and Russel, 1972 , pg. 345-353).

Below are some of the characteristics of the spaces discussed above

i. $(c, c, p) \subset (l_\infty, l_\infty)$.

ii. $(l_\infty, c,) \subset (c, c)$.

iii. $(c, c, p) \subset (c_0, c_0)$.

iv. $(c, c, p) \cap (l_\infty, c) = \emptyset$.

v. If $A, B \in (c, c) \Rightarrow A + B, AB \in (c, c)$ where $(AB)_{nk} = \sum_{i=0}^{\infty} a_{ni} b_{ik}$.

vi. $A \in (l_\infty, c_0)$ if and only if $\sum_{k=0}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$.

1.1.4 Banach Spaces

Definition 1.1.16. (Paranorm)

A paranorm p on a linear space X, is a function $p: X \to \mathbb{R}$ such that

i. $p(\theta) = 0$, where θ denotes the zero vector

ii. $p(x) \geq 0$

iii. $p(x) = p(-x)$

iv. $p(x+y) \leq p(x) + p(y)$

v. If $(\lambda_n)_0^\infty$ is a sequence of scalars with $\lambda_n \to \lambda$ and $(x_n)_0^\infty$ is a sequence of points in X with $x_n \to x$, then $p(\lambda_n x_n - \lambda x) \to 0$ (Continuity of multiplication)

Definition 1.1.17. (Seminorm and Norm)

A seminorm p on a linear space X, is a function $p: X \to \mathbb{R}$ such that

i. $p(x) \geq 0$

ii. $p(x+y) \leq p(x) + p(y)$

iii. $p(\lambda x) = |\lambda|\, p(x),\ \lambda \in \mathbb{k}$

If in addition to these conditions, if a seminorm satisfies the condition that $p(x) = \theta$ if and only if $x = \theta$, then it is called a norm.

Definition 1.1.18. (Linear topological space)

A linear topological space is a linear space X which has a topology T, such that addition and scalar multiplication in X are continuous. If T is given a metric, it is a linear metric space.

Definition 1.1.19. (Schauder basis)

Let X be a paranormed or normed space with a paranorm p or norm $\|.\|$. A sequence $(b_k)_0^\infty$of elements of X is called a schauder basis if and only if for every $x \in X$, $\exists$ a unique sequence of scalars $(\lambda_k)_0^\infty$such that

$$x = \sum_{k=0}^{\infty} \lambda_k b_k. \tag{1.1.17}$$

That is, $p(x - \sum_{k=0}^{\infty} \lambda_k b_k) \to 0$ as $n \to \infty$ or in norm notation $\left\| x - \sum_{k=0}^{\infty} \lambda_k b_k \right\| \to 0$ as $n \to \infty$

Example 1.1.20. $\triangle = (\delta^k)_0^\infty = (\delta^0, \delta^1, \delta^2, \cdots)$ is a schauder basis for the spaces $c_0, bv_0, l_p (0 < p < 1), cs, s$.

$\triangle^+ = (\delta, \delta^0, \delta^1, \delta^2, \cdots)$ is a schauder basis for the spaces c, bv. The spaces l_∞ and bs have no schauder basis

Example 1.1.21. c_0, c, $l_p(p \geq 1)$, l_∞, bv, bv_0, cs, bs, $w_p(p \geq 1)$ are all normed linear spaces.

Their norms are as follows; c_0, c, l_∞ have the same natural norm namely $\|x\| = \sup_{n \geq 0} \{|x_n|\}$;

$l_p(1 \leq P < \infty)$ has a natural norm

$$\|x\| = \left(\sum_{k=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}}. \tag{1.1.18}$$

bv has a natural norm

$$\|x\| = \lim_{n \to \infty} |x| + \sum_{k=0}^{\infty} |x_{k+1} - x_k|. \tag{1.1.19}$$

bv_0 has a natural norm

$$\|x\| = \sum_{k=0}^{\infty} |x_{k+1} - x_k|. \tag{1.1.20}$$

cs and bs have the same natural norm given by

$$\|x\| = \sup_{n \geq 0} \left\{ \left| \sum_{k=0}^{\infty} x_k \right| \right\}. \tag{1.1.21}$$

Definition 1.1.22. (Banach Spaces)

A Banach space is a complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, where $x_n \in X$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Example 1.1.23. The spaces c_0, c, $l_p(p \geq 1)$, l_∞, bv, bv_0, cs, bs, $w_p(p \geq 1)$ are all Banach spaces under their natural norms.

Definition 1.1.24. (Frechet space), FK - space

A Frechet space is a complete linear metric space. An FK - space is a Frechet space with continuous coordinates. A normed FK - space is called a BK - Space.

NOTE: Every Frechet space with a schauder basis is an FK - space. The examples of FK - spaces are c_0, c, $l_p(p \geq 1)$, bv, bv_0, cs, $w_p(p \geq 1)$, (Bennet, 1971), (Bennet, 1972b), (Bennet, 1972a), (Bennet and Kalton, 1972), (Brown, et al, 1969) and (Maddox, 1970).

1.1.5 Linear Operators and Functionals

Definition 1.1.25. (Linear Operator)

Let X and Y be linear spaces. Then a function $f : X \to Y$ is called a linear operator or a map or a transformation if and only if for all $x, y \in X$ and all $\lambda, \mu \in \Bbbk$

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \tag{1.1.22}$$

Definition 1.1.26. (Linear Functional)

A function f is a linear functional on X if $f : X \to \Bbbk$ is a linear operator, i.e. a linear functional is a real or complex valued linear operator.

Definition 1.1.27. (Bounded linear operator)

A linear Operator $A : X \to Y$ is called bounded if there exist a constant M such that

$$\|A(x)\| \leq M \|x\|, \forall x \in X. \tag{1.1.23}$$

NOTE: A bounded functional on X satisfies,

$$|f(x)| \leq M \|x\|, \forall x \in X. \tag{1.1.24}$$

NOTATION: Let X and Y be linear spaces. Then $L(X,Y)$ denotes the set of all linear operators on X into Y. $L(X,\Bbbk)$ the set of all linear functionals on X. It is usual to denote this by X^+ and call it the algebraic dual of X.

Definition 1.1.28. (Continuous dual of X)

Let X and Y be normed spaces. Then $B(X,Y)$ denotes the set of all bounded or continuous linear operators on X into Y. $B(X,\Bbbk)$, the set of all bounded or continuous linear functionals on X.

Remark 1.1.29. Let X be a Banach space, then $B(X,X) = B(X)$, the linear space of all bounded linear operators T on X into itself is a Banach space with norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|. \tag{1.1.25}$$

(Maddox,1970, pg. 107)

This norm induces a metric topology, the uniform operator topology on $B(X)$, (Dunford and Schwatz, 1957, pg. 475)

Definition 1.1.30. (Adjoint Operator T^*)

The adjoint T^* of linear operator $T \in B(X,Y)$ is the mapping from $Y^*$to X^* defined by

$$T^* \circ f = f \circ T,\ f \in Y^*. \tag{1.1.26}$$

Theorem 1.1.31. *T^* is linear and bounded. Moreover, $\|T^*\| = \|T\|$,*

(Kreyszig, 1980, pg. 232).

Theorem 1.1.32. *A linear Operator $T \in B(X,Y)$ has a bounded inverse T^{-1} defined on all Y if and only if its adjoint T^*has a bounded inverse $(T^*)^{-1}$ defined on all of X^*. When these inverses exist, $(T^{-1})^* = (T^*)^{-1}$,*

(Goldberg, 1966, pg. 60).

Definition 1.1.33. (Resolvent Operator, $R_\lambda = (T - \lambda I)^{-1}$)

Let X be a non - empty Banach space and suppose that $T : X \to X$. With T, associated is the operator $T_\lambda = T - \lambda I,\ \lambda \in \mathbb{C},$where I is the identity operator on X. If $T_\lambda = T - \lambda I$ has an inverse, then it is denoted by $R_\lambda(T)$ or simply R_λ and call it the resolvent operator of T.

Definition 1.1.34. (Resolvent set $\rho(T)$, spectrum $\sigma(T)$)

Let X be a non - empty Banach space and suppose that $T : X \to X$. The resolvent set $\rho(T)$ of T is the set of complex numbers $\lambda$for which $(T - \lambda I)^{-1}$ exist as a bounded operator with the domain X. The spectrum $\sigma(T)$ of T is the compliment of $\rho(T)$ in $\mathbb{C}$.

Theorem 1.1.35. *The resolvent set $\rho(T)$ of a bounded linear operator T on a Banach space X is open; hence the spectrum $\sigma(T)$ of T is closed,*

(Kreyszig, 1980, pg. 376).

Theorem 1.1.36. *If X is any Banach space and $T \in B(X)$, then $\sigma(T) \neq \emptyset$,*

(Taylor and Lay, 1980, pg. 278)
The spectrum $\sigma(T)$ of a bounded linear operator $T : X \to X$ on a Banach space X is compact and lies in the disk given by:

$$|\lambda| = \|T\|. \tag{1.1.27}$$

(Kreyszig, 1980, pg. 377).

Theorem 1.1.37. *Let $T \in B(X)$, where X is any Banach space, then the spectrum of $T^*$is identical to the spectrum of T. Furthermore, $R_\lambda(T^*) = (R_\lambda(T))^*$for $\lambda \in \rho(T) = \rho(T^*)$,*

(Goldberg, 1966, pg. 71).

1.2 Literature Review

The spectra of conservative matrices and in particular the spectrum of any Hausdorff method is either uncountable or finite if it's finite then it consists of one point or two points. Let E be Hausdorff method correspoding to the seqeunce $\{1, q, q^2, q^3, \cdots\}$, $0 < q < 1$, then $\alpha I + (1-\alpha)E$ is Mercerian if and only if $|\alpha| > |1-\alpha|$. Thus for $\alpha > 0$, $\alpha I + (1-\alpha)E$ is Mercerian if and only if $\alpha > \frac{1}{2}$, (Sharma, 1972) .
Some of the well known Hausdorff matrices are Cesaro, Holder, Euler, Gamma and Generalized Cesaro. If H is a Hausdorff operator in ℓ^2_+, then there exist a bounded analytic function ψ on $E = \{z \in \mathbb{C} : |z-1| < 1\}$, with $\psi(1) = 1$ such that $H = \psi(C_1)$. Then $\psi(E)$ is an open set, the spectrum of H, $\sigma(H) = closure\, \psi(E)$, and the point spectrum $\sigma_p(H^*)$ contains the set $\psi(E)^-$ which is the complex conjugate of $\psi(E)$. Finally $\|H\| = \sup\{|\lambda| : \lambda \in \psi(E)\}$ (Deddens, 1978)
The isolated points of the spectra of conservative matrices are given by the diagonal elements of these matrices, (Sharma, 1975).
The fine spectra of C_1, the Cesaro Operator and C_1^p, (p a positive integer), the Holder summability operator of order p on c- the space of convergent sequences have been investigated. The fine spectra of C_1consists of all points exterior to the circle centered at $\left(\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$. The Holder summability operator has a fine spectra given by the closed region bounded by the closed curve given in polar cordinates by $r = \cos^p(\frac{\theta}{p})$ (Wenger, 1975). Wenger's work was extended by determining the fine spectra of weighted mean operators on c. Let A be a regular weighted mean method such that $\delta = \lim p_n/P_n$ exists. If λ satisfies $\left|\lambda - (2-\delta)^{-1}\right| < |1-\delta|/|2-\delta|$, then λ is a point of $\sigma(A)$ for which $\overline{R(T)} \neq X$ and T^{-1} exists and is continuous, (Rhoedes, 1983).
Brown and others determined the spectrum of the Cesaro operator (C_1 operator) on the space ℓ^2 of square summable sequences, they showed that it consists of all $\{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1$, (Brown et al, 1965). This was extended by determining the spectrum of the same C_1 operator on $\ell^p(\mathbb{R}^+)$ for $p \neq 2$. The spectrum is given by the set $\{\lambda : Re(\frac{1}{\lambda}) = (p-1)/p\}$ which for $p = -1$, is a circle centered at $2(p-1)/p$ and the same radius, and for $p = 1$, is the imaginary axis, (Boyd, 1968). The spectrum of the cesaro operator on c_0- the space of null sequences, consists of all λ satisfying $\left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$, (Reade, 1985). Okutoyi in the same year determined the spectrum of C_1operator on $w_p(0)(1 \leq p < \infty)$, and concluded that the spectrum is the set $\sigma(C_1) = \{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$ with no eigenvalues, (Okutoyi, 1985). The spectrum of $(I-C)^*$ as an operator in $\ell_p(1 < p < \infty)$ is $\sigma\left((I-C)^*\right) = \{z - (p/2 - 1) : |z| \leq p/2\}$, (Gonzalez, 1985). The spectrum of C_1 as an operator on the bv_0 space is $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$, (Okutoyi, 1990). The only eigenvalue of $C_1 \in B(bv)$, the space of bounded variations, is $\lambda = 1$,

and its spectrum is given by $\sigma(C_1) = \{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$, (Okutoyi, 1992). In 1996, Shafiquel Islam obtained the spectrum of the C_1 operator on ℓ_∞- the space of bounded sequences, (Shafiquel, 1996). The spectrum of the C_1 operator on $w_p(1 \leq p < \infty)$ - the space of strongly Cesaro summable complex sequences of order 1, index p, is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$ and its eigenvalue is $\lambda = 1$, (Okutoyi and Akanga, 2005). Let $C_1 : \overline{bv_0} \cap \ell_\infty \to \overline{bv_0} \cap \ell_\infty$, then the spectrum of C_1 is $\sigma(C_1) = \{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$, (Binod et al, 2013).

Let $A = \{\lambda \in \mathbb{C} : \lambda = \lambda_1.\lambda_2, \textit{ where } \lambda_1, \lambda_2 \in D\}$ and $B = \left\{\lambda \in \mathbb{C} : \lambda = \lambda_3^2, \lambda_3 \in D\right\}$ where $D = \{z \in \mathbb{C} : \left|z - \frac{1}{2}\right| \leq \frac{1}{2}\}$, then $A = B$. The spectrum of the cesaro operator of order two (C_{11} operator) on $c_0(c_0)$- the space of double null sequences, is $\sigma(C_{11}) = A$ (Okutoyi and Thorpe, 1989).

It has been shown that the spectrum of a certain mercerian *Nörlund* operator with $a_{nn} = 1$, contains negative numbers, (Dorff and Wilansky, 1960). The set of eigenvalues of a special *Nörlund* operator in which $p_k = r^k$, $0 < r < 1$ and $k \geq 1$ as a bounded operator over the sequence spaces ℓ_∞, c and bv, is the singleton set $\{1\}$, (Coskun, 2003). In the abstract of his paper, Coskun remarked that as far as he was concerned there was no investigations on the spectrum of *Nörlund* operator. The spectrum of a special *Nörlund Q* operator on the space c_0, is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$ and it has no eigenvalues, (Akanga et at, 2010). The spectrum of a special *Nörlund* operator in which $p_0 = p_1 = 1$, $p_2 = p_3 = \cdots = 0$ as a bounded operator on the sequence space c, is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$ and it has one eigenvalue $\lambda = 1$ corresponding to the eigenvector $x = (1, 1, 1 \cdots)^T$, (Akanga, 2014).

In this research, we investigate and determine the spectrum of a class B of *Nörlund* operators on three sequence spaces c_0, c and bv_0.

1.3 Statement of the Problem

Mathematics, especially mathematical analysis, develops and is maintained via the concept of convergence of sequences and series. Even in applied science and engineering, one is interested in the convergence of sequences and series of results generated during experiments. A divergent series or sequence is therefore not as useful as the convergent ones.

Spectral theory is applied in the determination of convergence or non - convergence of sequences and series.

The theorems in which ordinary convergence is deduced from the fact that one has some condition plus additional condition are called Tauberian theorems. The spectrum of an operator plays a crucial role in the development of Tauberian theory for the operator which is used to determine the limit or sum of a convergent sequence or series. Mercerian theorems are also developed using the spectrum of an operator and are used to determine the limit or sum of a convergent sequence or series. For example, let A be a coregular triangle with inverse satisfying $a_{nk}^{-1} \leq 0$, $(n < k)$, $a_{nn}^{-1} > 0$, $(n = 0, 1, 2, \cdots)$, then $I + \alpha A$ is equivalent for convergence for $Re(\alpha) > \frac{-1}{t}$ where $t = \lim Ae$, $e = \{1, 1, 1, 1, \cdots\}$ and $e_k = \{0, 0, \cdots, 1, \cdots\}$, 1 in the k^{th} position.

The spectrum and the eigenvalues of the Cesaro Operator of order 1, C_1, on the sequence spaces c_0, c, $\ell_p (0 < p < \infty)$, c, ℓ_∞, bv, bv_0, $\overline{bv}_0$, bs, cs, $w_p (0 < p < \infty)$, $w_p(0)$, have been investigated by many researchers, the recent ones being Okutoyi and Akanga (2005), Binod and Pallavi (2013). Hence various Tauberian and Mercerian theorems have been developed. On the other hand, the spectrum and eigenvalues of special *Nörlund* operators have been investigated by researchers such as Dorff and Wilansky (1960), Coskun (2003), Akanga, Mwathi and Wali (2010) and Akanga (2014). The spectrum of a general *Nörlund* operator has not been determined and therefore the Tauberian and Mercerian theorems has not been determined for that operator.

In this research, we investigate and determine the spectrum of a class B of *Nörlund* operators on the sequence spaces c_0, c and bv_0. These are the most common spaces in analysis. This will find application in the development of Tauberian and Mercerian theorems for the *Nörlund* operator which is used to determine the limit or sum of a convergent sequence or series

1.4 Justification

A part from the various applications of spectral theory mentioned in section 1.1, the spectrum of operators is a very important tool in the solution of systems of linear equations. Established theorems such as the comparison test, the ratio test and the integral test, are not applicable in a variety of sequences and series. Even where they apply, they just determine convergence but not limit or sum of a convergent sequence or series respectively. Tauberian and Mecerian theorems in summability theory handles this problem well. The convergence and even limit of a convergent sequence or series is determined from the convergence of some transform of it together with a side condition, (Boos, 2000, pg. 167-204), (Hardy, 1948, pg. 148-177), (Powell and Shah, 1972, pg. 75-92), (Maddox, 1980, pg. 65-80). Therefore the results obtained from this research will find application in the development of Tauberian and Mercerian theorems for the *Nörlund* operator. In turn, this will find applications in diverse fields such as integral transforms of Fourier analysis; in probability and statistics through such areas involving central limit theorem, almost sure convergence, summation of random series, Markov series, (Boos, 2000, pg. 256-257). The eigenvalues and the spectrum of a matrix also has numerous applications, for exapmle in solving a system of first order differential equations. The system can be written in matrix form as $y' = Ay$ where y is a function of t. The solution is given by $y = e^{\lambda_i t}$ where λ_i are the eigenvalues of A if it is a diagonal matrix. If it is not diagonal then it is diagonalized and transformed i.e

$$\begin{aligned} y &= Pw \\ y' &= Pw' \\ w' &= P^{-1}APw \\ w &= e^{\lambda_i t} \end{aligned}$$

where P is the diagonalization matrix and λ_i are the eigenvalues of the resulting diagonal matrix. Also, in Quantum mechanics, the Hamiltonian H of some system is given by an infinite matrix $H = (h_{ij}), i, j = 1, 2, \cdots$ considered as an operator on some infinite set of numbers . The possible energy values of the system are the eigenvalues of H (usually relative to ℓ^2) and the main problem of pertubation theory is to estimate these eigenvalues.

1.5 Objectives

1.5.1 General Objective

To determine the spectrum of a class of *Nörlund* operators on sequence spaces c_0 , c and bv_0

1.5.2 Specific objectives

1. To compute the spectrum of a class of *Nörlund* operators when $p_0 = m, p_1 = p_2 = p_3 = p_4 = \cdots = 0$, $m \in \mathbb{R}$ on sequence spaces c_0 and c.
2. To compute the spectrum of a class of *Nörlund* operators when $p_0 = p_1 = p_2 = m, p_3 = p_4 = p_5 \cdots = 0$, $m \in \mathbb{R}$ on sequence spaces c_0.
3. To compute the spectrum of a class of *Nörlund* operators when $p_0 = p_1 = p_2 = m, p_3 = p_4 = p_5 \cdots = 0$, $m \in \mathbb{R}$ on sequence spaces c.
4. To compute the spectrum of a class of *Nörlund* operators when $p_0 = p_1 = p_2 = m, p_3 = p_4 = p_5 \cdots = 0$, $m \in \mathbb{R}$ on sequence spaces bv_0.

CHAPTER TWO
THE SPECTRUM OF A NÖRLUND OPERATOR B ON c_0

2.1 Introduction

This chapter investigates the spectrum of the Nörlund operators I and B on the sequence space c_0 by applying theorem 1.1.9.

2.2 The spectrum of I operator on c_0

Theorem 2.2.1. *The spectrum of $I \in B(c_0)$ is the singleton set $\{1\}$.*

Proof. $I \in B(c_0)$ since $\lim_{n\to\infty} a_{nk} = 0$ for each fixed k, $k = 0,1,2,\dots$ and $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |a_{nk}|\} = 1 < \infty$,satisfying all the conditions in theorem 1.1.9.
Also $\|I\|_{c_0} = \|I^*\|_{l_1} = 1$.
Suppose $Ix = \lambda x$, $x \neq \theta$ in c_0 and $\lambda \in \mathbb{C}$,

then
$$\begin{aligned} x_0 &= \lambda x_0 \\ x_1 &= \lambda x_1 \\ x_2 &= \lambda x_2 \\ &\vdots \\ x_n &= \lambda x_n \\ &\vdots \end{aligned},$$

solving the system gives $\lambda = 1$, hence $\lambda = 1$ is an eigen value of I in c_0.
$I^* = I^T = I$, hence $\lambda = 1$ is an eigen value of I^* in c_0.
Solving the system $(I - I\lambda)x = y$ for x in terms of y to get the matrix of $(I - I\lambda)^{-1}$ we have

$$\begin{pmatrix} 1-\lambda & 0 & 0 & 0 & \cdots \\ 0 & 1-\lambda & 0 & 0 & \cdots \\ 0 & 0 & 1-\lambda & 0 & \cdots \\ 0 & 0 & 0 & 1-\lambda & \cdots \\ & & \vdots & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{pmatrix} \tag{2.2.1}$$

or

$$\begin{aligned} (1-\lambda)x_0 &= y_0 \\ (1-\lambda)x_1 &= y_1 \\ (1-\lambda)x_2 &= y_2 \\ (1-\lambda)x_3 &= y_3 \\ &\vdots \end{aligned} \tag{2.2.2}$$

which gives

$$\begin{aligned}
x_0 &= \tfrac{1}{1-\lambda}y_0 \\
x_1 &= \tfrac{1}{1-\lambda}y_1 \\
x_2 &= \tfrac{1}{1-\lambda}y_2 \\
x_3 &= \tfrac{1}{1-\lambda}y_3 \\
&\vdots
\end{aligned} \tag{2.2.3}$$

this gives the matrix

$$(I - I\lambda)^{-1} = \begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{1-\lambda} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{1-\lambda} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{1-\lambda} & \cdots \\ & & \vdots & & \end{pmatrix} \tag{2.2.4}$$

The columns of this matrix are defined for all values of $\lambda \neq 1$ and converges to zero satisfying part (i) of Theorem 1.1.9. For the second part, $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |a_{nk}|\} = \frac{1}{1-\lambda} < \infty$ provided $\lambda \neq 1$, hence $(I - I\lambda)^{-1} \in B(c_0)$ if $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$. Which implies $(I - I\lambda)^{-1} \notin B(c_0)$ when $\lambda = 1$. □

2.3 The spectrum of B operator on c_0

Refer to matrix 1.1.13

Corollary 2.3.1. $B \in B(c_0)$

Proof. $\lim_{n\to\infty} b_{nk} = 0$ for each fixed k, $k = 0, 1, 2, \dots$ and
$\|B\| = \sup_{n\geq 0}\{\sum_{k=0}^{\infty} |b_{nk}|\} = 1 < \infty$. Satisfying the conditions in theorem 1.1.9. □

Lemma 2.3.2. *Each bounded linear operator $T : X \to Y$, where $X = c_0, \ell_1, c$ and $Y = c_0, \ell_p (1 \leq p \leq \infty), \ell_\infty$* determines and is determined by an infinite matrix of complex numbers.

(Taylor, 1958, pg. 217 - 219)

2.3.1 The eigenvalues of $B \in B(c_0)$

Theorem 2.3.3. $B \in B(c_0)$ *has no eigenvalues.*

Proof. Solving the system $Bx = \lambda x$, $x \neq \theta$ in c_0 and $\lambda \in \mathbb{C}$, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} \tag{2.3.1}$$

which gives

$$\begin{gathered} x_0 = \lambda x_0 \\ \frac{1}{2}x_0 + \frac{1}{2}x_1 = \lambda x_1 \\ \frac{1}{3}x_0 + \frac{1}{3}x_1 + \frac{1}{3}x_2 = \lambda x_2 \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = \lambda x_3 \\ \vdots \\ \frac{1}{3}x_{n-2} + \frac{1}{3}x_{n-1} + \frac{1}{3}x_n = \lambda x_n \\ \vdots \end{gathered} \tag{2.3.2}$$

Solving the equation, if x_0 is the first non zero entry of x, then $\lambda = 1$. But $\lambda = 1$ implies $x_0 = x_1 = x_2 = \cdots = x_n = \cdots$, which shows that x is in the span of $\delta = (1,1,1,1,\cdots)$ hence does not tend to zero as n tends to infinity. Therefore $\lambda = 1$ is not an eigenvalue of $B \in B(c_0)$. When x_1 is the first non zero entry of x, $\lambda = \frac{1}{2}$. But $\lambda = \frac{1}{2}$ implies $x_0 = 0$, $x_2 = 2x_1$, $x_3 = 6x_1$, $x_4 = 16x_1$, $x_5 = 44x_1$, $\cdots$ which shows that x is spanned by $\{0,1,2,6,16,44,\cdots\}$ an increasing sequence hence does not tend to zero as n tends to infinity. If x_{n+2} is the first non zero entry for $n = 0,1,2,3,\cdots$, then $\lambda = \frac{1}{3}$, solving the system gives $x_n = 0$ for $n = 0,1,2,3,\cdots$ which is a cotradiction hence $\lambda = \frac{1}{3}$ cannot be an eigenvalue ☐

2.3.2 The eigenvalues of $B^* \in B(\ell_1)$

Lemma 2.3.4. *Let $T : c_0 \to c_0$ be a linear map and define $T^* : \ell_1 \to \ell_1$ by $T^* \circ g = g \circ T, g \in c_0^* = \ell_1$. Then T must be given by an infinite matrix of complex numbers and moreover $T^* : \ell_1 \to \ell_1$ is the transposed matrix of T.*

(Wilansky, 1984, pg. 266)

Corollary 2.3.5. *Let $B : c_0 \to c_0$, then $B^* = B^T \in B(\ell_1)$*

$$B^* = B^T = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix} \tag{2.3.3}$$

$\|B\| = \|B^*\| = 1$

Theorem 2.3.6. *The eigenvalues of $B^* \in B(\ell_1)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\} \cup \{1\}$*

Proof. Consider the system $B^*x = \lambda x$, $x \neq \theta$ in ℓ_1 and $\lambda \in \mathbb{C}$,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} \tag{2.3.4}$$

or

$$\begin{gathered} x_0 + \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 = \lambda x_0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 = \lambda x_1 \\ \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 = \lambda x_2 \\ \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 + \tfrac{1}{3}x_5 = \lambda x_3 \\ \cdots \\ \tfrac{1}{3}x_{n-2} + \tfrac{1}{3}x_{n-1} + \tfrac{1}{3}x_n = \lambda x_{n-2}, \ for\, n \geq 4 \end{gathered} \tag{2.3.5}$$

solving the system gives

$$\begin{gathered} x_2 = 3(\lambda - 1)x_0 - \tfrac{3}{2}x_1 \\ x_3 = 3(\lambda - \tfrac{1}{2})x_1 - x_2 \\ x_4 = 3(\lambda - \tfrac{1}{3})x_2 - x_3 \\ x_5 = 3(\lambda - \tfrac{1}{3})x_3 - x_4 \\ \cdots \\ x_n = 3(\lambda - \tfrac{1}{3})x_{n-2} - x_{n-1}, n \geq 4 \end{gathered} \tag{2.3.6}$$

i.e.

$$\begin{aligned}
x_4 &= 3(\lambda - \tfrac{1}{3})x_2 - x_3 \\
x_5 &= 3(\lambda - \tfrac{1}{3})x_3 - x_4 \\
x_6 &= 3^2(\lambda - \tfrac{1}{3})^2 x_2 - 3(\lambda - \tfrac{1}{3})x_3 - x_5 \\
x_7 &= 3^2(\lambda - \tfrac{1}{3})^2 x_3 - 3(\lambda - \tfrac{1}{3})x_4 - x_6 \\
x_8 &= 3^3(\lambda - \tfrac{1}{3})^3 x_2 - 3^2(\lambda - \tfrac{1}{3})^2 x_3 - 3(\lambda - \tfrac{1}{3})x_5 - x_7 \\
x_9 &= 3^3(\lambda - \tfrac{1}{3})^3 x_3 - 3^2(\lambda - \tfrac{1}{3})^2 x_4 - 3(\lambda - \tfrac{1}{3})x_6 - x_8 \\
x_{10} &= 3^4(\lambda - \tfrac{1}{3})^4 x_2 - 3^3(\lambda - \tfrac{1}{3})^3 x_3 - 3^2(\lambda - \tfrac{1}{3})^2 x_5 - 3(\lambda - \tfrac{1}{3})x_7 - x_9 \\
&\vdots \\
for\, n\, even,\ x_n &= 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1}x_2 - \sum_{k=0}^{\frac{n}{2}-2} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \\
for\, n\, odd,\ x_n &= 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1}x_3 - \sum_{k=0}^{\frac{n-1}{2}-2} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1),\, n\geq 4}
\end{aligned} \tag{2.3.7}$$

Each term is a geometric progression with common ratio, $r = 3(\lambda - \frac{1}{3})$

$$\sum_{n=0}^{\infty} |x_n| = |x_0| + |x_1| + |x_2| + |x_3| + \sum_{\substack{n=4 \\ n\,even}}^{\infty} \left| 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1}x_2 - \sum_{k=0}^{\frac{n}{2}-2} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right|$$

$$+ \sum_{\substack{n=5 \\ n\,odd}}^{\infty} \left| 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1}x_3 - \sum_{k=0}^{\frac{n-1}{2}-2} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right|$$

$$\leq \sum_{n=0}^{3} |x_n| + \sum_{\substack{n=4 \\ n\,even}}^{\infty} \left| 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1}x_2 \right| + \sum_{\substack{n=5 \\ n\,odd}}^{\infty} \left| 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1}x_3 \right|$$

$$+ \sum_{\substack{n=4 \\ n\,even}}^{\infty} \sum_{k=0}^{\frac{n}{2}-2} \left| 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right| + \sum_{\substack{n=5 \\ n\,odd}}^{\infty} \sum_{k=0}^{\frac{n-1}{2}-2} \left| 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right|$$

as $n \to \infty$, this is a geometric series with the common ratio, $r = 3(\lambda - \frac{1}{3})$. This series converges only if $|r| < 1$, that is $\left|3(\lambda - \frac{1}{3})\right| = 3\left|\lambda - \frac{1}{3}\right| < 1$ or $\left|\lambda - \frac{1}{3}\right| < \frac{1}{3}$. $\lambda = 1$ is an eigenvalue corresponding to the eigenvector, $(x_0, 0, 0, 0, \ldots)^T$. Hence the eigenvalues is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\} \cup \{1\}$. $\square$

2.3.3 The Spectrum of $B \in B(c_0)$

Theorem 2.3.7. *The inverse of an infinite lower triangular matrix*

$$L = \begin{pmatrix} a_0(0) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ a_0(1) & a_1(0) & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ a_0(2) & a_1(1) & a_2(0) & 0 & 0 & \cdots & 0 & 0 & \cdots \\ a_0(3) & a_1(2) & a_2(1) & a_3(0) & 0 & \cdots & 0 & 0 & \cdots \\ a_0(4) & a_1(3) & a_2(2) & a_3(1) & a_4(0) & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_0(m) & a_1(m-1) & a_2(m-2) & a_3(m-3) & a_4(m-4) & \cdots & a_m(0) & 0 & \cdots \\ 0 & a_1(m) & a_2(m-1) & a_3(m-2) & a_4(m-3) & \cdots & a_m(1) & a_{m+1}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is given by

$$L^{-1}_{nk} = \begin{cases} \frac{1}{a_n(0)}, & n = k \\ \frac{(-1)^{n-k}}{\prod_{j=k}^{n} a_j(0)} D^{(k)}_{n-k}(a\,[m]), & (0 \le k \le n-1),\ (n,k \in \mathbb{N}_0) \\ 0, & (k > n) \end{cases}$$

where

$$D^{(k)}_{n-k}(a\,[m]) = \begin{vmatrix} a_k(1) & a_{k+1}(0) & 0 & \cdots & 0 & \cdots & 0 \\ a_k(2) & a_{k+1}(1) & a_{k+2}(0) & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_k(m) & a_{k+1}(m-1) & a_{k+2}(m-2) & \cdots & a_m(0) & \cdots & 0 \\ 0 & a_{k+1}(m) & a_{k+2}(m-1) & \vdots & a_m(1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{n+k-1}(1) \end{vmatrix}$$

$n \geq 1$.

(Pinakadhar and Dutta, 2014).

Corollary 2.3.8. *For matrix B, we have* $B - I\lambda = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots \\ & & \cdots & & & \end{pmatrix}$,

$(B - I\lambda)^{-1}$ *is given by* $M_{nk} = \begin{cases} \frac{1}{a_{nn}}, & n = k \\ \frac{(-1)^{n-k}}{\prod_{j=k}^{n} a_{nn}} D^{(k)}_{n-k}, & (0 \le k \le n-1),\ (n,k \in \mathbb{N}_0) \\ 0, & (k > n) \end{cases}$ *where*

$$D_{n-k}^{(k)} = \begin{vmatrix} a_{1k} & a_{1k+1} & 0 & 0 & \cdots 0 \\ a_{2k} & a_{2k+1} & a_{2k+2} & 0 & \cdots 0 \\ 0 & a_{3k+1} & a_{3k+2} & a_{3k+3} & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots a_{n,n+k-1} \end{vmatrix}$$

for k=0, $D_1^{(0)} = |a_{10}| = \frac{1}{2}$

$$D_2^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2}-\lambda \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$

$$D_3^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2}-\lambda & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda \\ 0 & \frac{1}{3} & \frac{1}{3} \end{vmatrix} ....$$

$$D_n^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2}-\lambda & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$ *, which is an* $n \times n$ *tridiagonal matrix.*

for $k \geq 1$, $D_{n-k}^{(k)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$ *, this is an* $n-k \times n-k$ *tridiagonal matrix.*

The determinant of this tridiagonal matrix is given by,
$\det B(n) = \frac{1}{3} \det B(n-1) - \frac{1}{3} \left(\frac{1}{3} - \lambda \right) \det B(n-2)$
Substituting gives matrix M as,

$$M = \begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & \cdots \\ \frac{-1}{2(1-\lambda)(\frac{1}{2}-\lambda)} & \frac{1}{\frac{1}{2}-\lambda} & 0 & 0 & 0 & \cdots \\ \frac{\{\frac{1}{2}-(\frac{1}{2}-\lambda)\}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{-1}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{1}{\frac{1}{3}-\lambda} & 0 & 0 & \cdots \\ \frac{-\{\frac{1}{2}(1-(1-3\lambda))-(\frac{1}{2}-\lambda)}{3^2(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \frac{(1-(1-3\lambda)}{3^2(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \frac{-1}{3(\frac{1}{3}-\lambda)^2} & \frac{1}{\frac{1}{3}-\lambda} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \qquad (2.3.8)$$

for $k=0$, $m_{00}=\frac{1}{1-\lambda}$, $m_{n0}=\frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}D_n^{(0)}=\frac{(-1)^n(\frac{1}{3}D_{n-1}^{(0)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{n-2}^{0})}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$,

for $k=1$, $m_{11}=\frac{1}{\frac{1}{2}-\lambda}$, $m_{n1}=\frac{(-1)^{n-1}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}D_{n-1}^{(1)}=\frac{(-1)^{n-1}(\frac{1}{3}D_{n-2}^{(1)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{n-3}^{(1)})}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$

for $k\geq 2$, $m_{nn}=\frac{1}{\frac{1}{3}-\lambda}$, $m_{nk}=\frac{(-1)^{n-k}}{(\frac{1}{3}-\lambda)^{n-k+1}}D_{n-k}^{(k)}=\frac{(-1)^{n-k}(\frac{1}{3}D_{n-k-1}^{(k)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{n-k-2}^{(k)})}{(\frac{1}{3}-\lambda)^{n-k+1}}$

Theorem 2.3.9. *The spectrum* $\sigma(B)$ *of* $B\in B(c_0)$ is the set $\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1}{3}\right|\leq\frac{1}{3}\}\cup\{1\}$

Proof. We show that $(B-I\lambda)^{-1}\in B(c_0)$ for all $\lambda\in\mathbb{C}$ such that $\left|\lambda-\frac{1}{3}\right|>\frac{1}{3}$
for $k=0$, $D_1^{(0)}=\frac{1}{2}$
$D_2^{(0)}=\frac{1}{2}(\frac{1}{3})-(\frac{1}{2}-\lambda)\frac{1}{3}=\frac{1}{3}\{\frac{1}{2}-(\frac{1}{2}-\lambda)\}$
$D_3^{(0)}=\frac{1}{3^2}\{\frac{1}{2}-\frac{1}{2}(1-3\lambda)-(\frac{1}{2}-\lambda)\}=\frac{1}{3^2}\{\frac{1}{2}(1-(1-3\lambda))-(\frac{1}{2}-\lambda)\}$
$D_4^{(0)}=\frac{1}{3^3}\{\frac{1}{2}-\frac{2}{2}(1-3\lambda)-(\frac{1}{2}-\lambda)-(\frac{1}{2}-\lambda)(1-3\lambda)\}$
$=\frac{1}{3^3}\{\frac{1}{2}(1-2(1-3\lambda))-(\frac{1}{2}-\lambda)(1-(1-3\lambda))\}$
$D_5^{(0)}=\frac{1}{3^4}\{\frac{1}{2}-\frac{3}{2}(1-3\lambda)+\frac{1}{2}(1-3\lambda)^2-(\frac{1}{2}-\lambda)+2(\frac{1}{2}-\lambda)(1-3\lambda)\}$
$=\frac{1}{3^4}\{\frac{1}{2}(1-3(1-3\lambda)+(1-3\lambda)^2)-(\frac{1}{2}-\lambda)(1+2(1-3\lambda)\}$

$$D_n^{(0)}=\frac{1}{3^{n-1}}\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n}{2}-1}a_k(1-3\lambda)^k\right)-(\tfrac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n}{2}-1}b_k(1-3\lambda)^k\right)\right\}\text{when n is even,}$$

and

$$D_n^{(0)}=\frac{1}{3^{n-1}}\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n-1}{2}}a_k(1-3\lambda)^k\right)-(\tfrac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n-1}{2}}b_{k-1}(1-3\lambda)^{k-1}\right)\right\}\text{when n is odd,}$$

where $a_{k's}$ and $b_{k's}$ are integers.
Hence the n^{th} row is given by

$$m_{n0}=\frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}D_n^{(0)}=\frac{(-1)^n\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n}{2}-1}a_k(1-3\lambda)^k\right)-(\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n}{2}-1}b_k(1-3\lambda)^k\right)\right\}}{3^{n-1}(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}},$$

when n is even, and

$$m_{n0}=\frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}D_n^{(0)}=\frac{(-1)^n\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n-1}{2}}a_k(1-3\lambda)^k\right)-(\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n-1}{2}}b_{k-1}(1-3\lambda)^{k-1}\right)\right\}}{3^{n-1}(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}},$$

when n is odd.
As $n\to\infty$, the colums $m_{no}\to 0$ only if the denominator tends to infinity, and the denominator tends to infinity provided $\left|3(\frac{1}{3}-\lambda)\right|>1$.
Similarly for $k=1$, the denominator is given by $3^{n-1}(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}$which tends to infinity provided $\left|3(\frac{1}{3}-\lambda)\right|>1$ or $\left|\frac{1}{3}-\lambda\right|>\frac{1}{3}$
Similarly for $k\geq 2$,the denominator is given by $3^{n-k}(\frac{1}{3}-\lambda)^{n-k+1}$which tends to infinity provided $\left|3(\frac{1}{3}-\lambda)\right|>1$ or $\left|\frac{1}{3}-\lambda\right|>\frac{1}{3}$, which satisfies condition (i) of theorem 1.1.9 □

Summing the entries of the matrix 2.3.8 along the n^{th} row, we have

$$\sum_{k=0}^{\infty}|m_{nk}| = \left|\frac{(-1)^n D_n^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \left|\frac{(-1)^{n-1} D_{n-1}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right|$$

$$+\sum_{k=2}^{\infty}\left|\frac{(-1)^{n-k}(\frac{1}{3}D_{n-k-1}^{(k)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{n-k-2}^{(k)})}{(\frac{1}{3}-\lambda)^{n-k+1}}\right| = s_n \qquad (2.3.9)$$

say for $n \geq 0$. $\sup_n\{s_n\} \leq K < \infty$, provided $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$, since the columns are tending to zero as n tends to infinity. Which deals with condition (ii) of theorem 1.1.9.

Therefore $(B-I\lambda)^{-1} \in B(c_0)$ if $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. Which implies $(B-I\lambda)^{-1} \notin B(c_0)$ if $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| \leq \frac{1}{3}$. Clearly, when $\lambda = 1$, column 1 is infinite therefore the inverse does not exist. Hence $\sigma(B) = \{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

2.3.4 The spectrum of $B^* \in B(\ell_1)$

Theorem 2.3.10. *The inverse of an infinite upper triangular matrix*

$$U = \begin{pmatrix} a_0(0) & a_0(1) & a_0(2) & a_0(3) & a_0(4) & \cdots & a_0(m) & 0 & \cdots \\ 0 & a_1(0) & a_1(1) & a_1(2) & a_1(3) & \cdots & a_1(m-1) & a_1(m) & \cdots \\ 0 & 0 & a_2(0) & a_2(1) & a_2(2) & \cdots & a_2(m-2) & a_2(m-1) & \cdots \\ 0 & 0 & 0 & a_3(0) & a_3(1) & \cdots & a_3(m-3) & a_3(m-2) & \cdots \\ 0 & 0 & 0 & 0 & a_4(0) & \cdots & a_4(m-4) & a_4(m-3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_m(0) & a_m(1) & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{m+1}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is given by

$$U_{nk}^{-1} = \begin{cases} \frac{1}{a_n(0)}, & (n=k) \\ \frac{(-1)^{k-n}}{\prod_{j=n}^{k} a_j(0)} D_{k-n}^{(n)}(a[m]), & (0 \leq n \leq k-1),\ (n,k \in \mathbb{N}_0) \\ 0, & (n > k) \end{cases}$$

where

$$D_n^{(n)}(a[m]) = \begin{vmatrix} a_0(1) & a_0(2) & a_0(3) & \cdots & a_0(m-1) & \cdots & a_0(n) \\ a_1(0) & a_1(1) & a_1(2) & \cdots & a_1(m-2) & \cdots & a_1(n-1) \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_m(0) & a_m(1) & a_m(2) & \cdots & a_m(n-m) \\ 0 & \cdots & 0 & a_{m+1}(0) & a_{m+1}(1) & \cdots & a_{m+1}(n-m-1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1}(0) & a_{n-1}(1) \end{vmatrix},$$

$n \geq 1$.

(Pinakadhar and Dutta, 2014).

Corollary 2.3.11. *For matrix B^*, we have*

$$B^* - I\lambda = \begin{pmatrix} 1-\lambda & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{2}-\lambda & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{3}-\lambda & \cdots \\ & & \cdots & & & \end{pmatrix} \tag{2.3.10}$$

$(B^* - I\lambda)^{-1}$ is given by

$$U_{nk} = \begin{cases} \frac{1}{a_{nn}}, & n = k \\ \frac{(-1)^{k-n}}{\prod_{j=n}^{k} a_{nn}} D_{k-n}^{(n)}, & (0 \leq n \leq k-1),\ (n,k \in \mathbb{N}_0) \\ 0, & (n > k) \end{cases} \tag{2.3.11}$$

where

$$D_n^{(n)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots & 0 \\ \frac{1}{2}-\lambda & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ 0 & 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3}-\lambda & \cdots & \\ & & \cdots & & \vdots & \frac{1}{3} \end{vmatrix}$$

for $n = 0$, $D_1^{(0)} = \left|\frac{1}{2}\right|$

$$D_2^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2}-\lambda & \frac{1}{3} \end{vmatrix}$$

$$D_3^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2}-\lambda & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} \end{vmatrix} \ldots.$$

$$D_n^{(0)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots & 0 \\ \frac{1}{2}-\lambda & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ 0 & 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3}-\lambda & \cdots & \\ & & \cdots & & \vdots & \frac{1}{3} \end{vmatrix}$$

, which is an $n \times n$ tridiagonal ma-

trix.

this gives, $D_k^{(0)} = \frac{1}{3}D_{k-1}^{(0)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{k-2}^{0}$ *and*

$$\text{for } n \geq 1,\ D_{k-n}^{(n)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 \\ \frac{1}{3}-\lambda & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & \cdots & 0 \\ 0 & 0 & \frac{1}{3}-\lambda & \cdots & \\ & \cdots & & \vdots & \frac{1}{3} \end{vmatrix},$$

this is an $k-n \times k-n$ tridiagonal matrix.

Or $D_{k-n}^{(n)} = \frac{1}{3}D_{k-n-1}^{(n)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{k-n-2}^{(n)}$.

so that,

$$U = \begin{pmatrix} \frac{1}{1-\lambda} & \frac{-1}{2(1-\lambda)(\frac{1}{2}-\lambda)} & \frac{\{\frac{1}{2}-(\frac{1}{2}-\lambda)\}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{-\{\frac{1}{2}(1-(1-3\lambda))-(\frac{1}{2}-\lambda)}{3^2(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \cdots \\ 0 & \frac{1}{\frac{1}{2}-\lambda} & \frac{-1}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{(1-(1-3\lambda)}{3^2(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \cdots \\ 0 & 0 & \frac{1}{\frac{1}{3}-\lambda} & \frac{-1}{3(\frac{1}{3}-\lambda)^2} & \cdots \\ 0 & 0 & & \frac{1}{\frac{1}{3}-\lambda} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \tag{2.3.12}$$

for $n=0$, $u_{00} = \frac{1}{1-\lambda}$, $u_{0k} = \frac{(-1)^k}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{k-1}}D_k^{(0)} = \frac{(-1)^k(\frac{1}{3}D_{k-1}^{(0)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{k-2}^{0})}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{k-1}}$,

for $n=1$, $u_{11} = \frac{1}{\frac{1}{2}-\lambda}$, $u_{1k} = \frac{(-1)^{k-1}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}D_{k-1}^{(1)} = \frac{(-1)^{k-1}(\frac{1}{3}D_{k-2}^{(1)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{k-3}^{(1)})}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}$

for $n \geq 2$, $u_{nn} = \frac{1}{\frac{1}{3}-\lambda}$, $u_{nk} = \frac{(-1)^{k-n}}{(\frac{1}{3}-\lambda)^{k-n+1}}D_{k-n}^{(n)} = \frac{(-1)^{k-n}(\frac{1}{3}D_{k-n-1}^{(n)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{k-n-2}^{(n)})}{(\frac{1}{3}-\lambda)^{k-n+1}}$

Remark 2.3.12. For an infinite matrix A, the inverse of A^T equals the transpose of A^{-1}, i.e $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Theorem 2.3.13. *The spectrum* $\sigma(B^*)$ *of* $B^* \in B(\ell_1)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$

Proof. Applying theorem 1.1.10,we show that $(B^* - I\lambda)^{-1} \in B(\ell_1)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$

Summing the entries of the matrix along the k^{th} column, we have

$$\sum_{n=0}^{\infty} |u_{nk}| = \left|\frac{(-1)^k D_k^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \left|\frac{(-1)^{k-1}D_{k-1}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \sum_{n=2}^{\infty}\left|\frac{(-1)^{k-n}(D_{k-n}^{(n)})}{(\frac{1}{3}-\lambda)^{k-n+1}}\right|$$

$\sum_{n=0}^{\infty}|u_{nk}| = \left|\frac{D_k^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \left|\frac{D_{k-1}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \sum_{n=2}^{\infty}\left|\frac{D_{k-n}^{(n)}}{(\frac{1}{3}-\lambda)^{k-n+1}}\right|$

The first term and the second term are finite provided $\lambda \neq 1, \frac{1}{2}\,\frac{1}{3}$, we show that $\sum_{n=2}^{\infty}\left|\frac{D_{k-n}^{(n)}}{(\frac{1}{3}-\lambda)^{k-n+1}}\right| < \infty$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$

$D_1^{(n)} = \frac{1}{3}$

$D_2^{(n)} = \frac{1}{3}(\frac{1}{3}) - (\frac{1}{3}-\lambda)\frac{1}{3} = \frac{1}{3^2} - \frac{1}{3}(\frac{1}{3}-\lambda)$

$D_3^{(n)} = \frac{1}{3^2}\{\frac{1}{3} - (\frac{1}{3}-\lambda) - (\frac{1}{3}-\lambda)\} = \frac{1}{3^3} - \frac{2}{3^2}(\frac{1}{3}-\lambda)$

$D_4^{(n)} = \frac{1}{3}\{\frac{1}{3^2}\{\frac{1}{3} - 2(\frac{1}{3}-\lambda)\}\} - \frac{1}{3}\left(\frac{1}{3}-\lambda\right)\{\frac{1}{3}\{\frac{1}{3} - (\frac{1}{3}-\lambda)\}\}$

$= \frac{1}{3^4} - \frac{3}{3^3}(\frac{1}{3}-\lambda) + \frac{1}{3^2}(\frac{1}{3}-\lambda)^2$

$D_5^{(n)} = \frac{1}{3}\left\{\frac{1}{3^4} - \frac{3}{3^3}(\frac{1}{3}-\lambda) + \frac{1}{3^2}(\frac{1}{3}-\lambda)^2\right\} - \frac{1}{3}\left(\frac{1}{3}-\lambda\right)\left\{\frac{1}{3^3} - \frac{2}{3^2}(\frac{1}{3}-\lambda)\right\}$

$= \frac{1}{3^5} - \frac{4}{3^4}\left(\frac{1}{3}-\lambda\right) + \frac{3}{3^3}(\frac{1}{3}-\lambda)^2$

$\vdots$

$$D_{k-n}^{(n)} = \frac{1}{3^{k-n}} - \frac{a_1}{3^{k-n-1}}\left(\frac{1}{3}-\lambda\right) + \frac{a_2}{3^{k-n-2}}(\frac{1}{3}-\lambda)^2 - \frac{a_3}{3^{k-n-3}}(\frac{1}{3}-\lambda)^3 + \cdots (-1)^m\frac{a_m}{3^{k-n-m}}\left(\frac{1}{3}-\lambda\right)^m$$

, $m = 1, 2\cdots\frac{1}{2}(k-n)-1$, if $k-n$ is even and $m = 1, 2\cdots\frac{1}{2}(k-n-1)-1$, otherwise.
Hence $|u_{nk}| = \left|\frac{D_{k-n}^{(n)}}{(\frac{1}{3}-\lambda)^{k-n+1}}\right| =$
$\left|\frac{1}{3^{k-n}(\frac{1}{3}-\lambda)^{k-n+1}} - \frac{a_1}{3^{k-n-1}(\frac{1}{3}-\lambda)^{k-n}} + \frac{a_2}{3^{k-n-2}(\frac{1}{3}-\lambda)^{k-n-1}} - \frac{a_3}{3^{k-n-3}(\frac{1}{3}-\lambda)^{k-n-2}} + \cdots (-1)^m\frac{a_m}{3^{k-n-m}(\frac{1}{3}-\lambda)^{k-n-m+1}}\right|$
as $n\to\infty$, $|u_{nk}| \to 0$ if the denominator is tending to infinity, i.e $3\left|\frac{1}{3}-\lambda\right| > 1$ or $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. Hence $\sum_{n=0}^{\infty}|u_{nk}| < \infty$ for each fixed k all $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. Which proves part (i) of theorem 1.1.10

Again summing the entries of the matrix along the k^{th} column, we have

$$\sum_{n=0}^{\infty}|u_{nk}| = \left|\frac{(-1)^k D_k^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \left|\frac{(-1)^{k-1}D_{k-1}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \sum_{n=2}^{\infty}\left|\frac{(-1)^{k-n}(D_{k-n}^{(n)})}{(\frac{1}{3}-\lambda)^{k-n+1}}\right| \quad (2.3.13)$$

equation 2.3.13 equals equation 2.3.9 , hence $\sup_k\{\sum_{n=0}^{\infty}|u_{nk}|\} < \infty$ provided $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. which proves condition (ii) of theorem 1.1.10. Therefore $B^* \notin B(\ell_1)$, provided $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$. □

CHAPTER THREE
THE SPECTRUM OF A NÖRLUND OPERATOR B ON c

3.1 Introduction

This chapter determines the spectrum of a Nörlund matrix B as an operator on the sequence space c applying theorem 1.1.7.

3.2 The spectrum of I operator on c

Theorem 3.2.1. *The spectrum of $I \in B(c)$ is the singleton set $\{1\}$*

Proof. $I \in B(c)$ since ☐

1. $\lim_{n\to\infty} i_{nk} = 0$ for each fixed k, $k = 0, 1, 2, \ldots$
2. $\lim_{n\to\infty} \sum_{k=0}^{\infty} i_{nk} = 1$ as $n \to \infty$
3. $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |i_{nk}|\} = 1 < \infty$.

Also $\|I\|_c = \|I^*\|_{l_1} = 1$.
suppose $Ix = \lambda x$, $x \neq \theta$ in c and $\lambda \in \mathbb{C}$, then

$$\begin{aligned} x_0 &= \lambda x_0 \\ x_1 &= \lambda x_1 \\ x_2 &= \lambda x_2 \\ &\vdots \\ x_n &= \lambda x_n \\ &\vdots \end{aligned} \tag{3.2.1}$$

solving equation 3.2.1 gives $\lambda = 1$, hence $\lambda = 1$ is an eigen value of I in c.
$I^* = I$, hence $\lambda = 1$ is an eigen value of I^* in c.
Solving the system $(I - I\lambda)x = y$ for x in terms of y to get the matrix of $(I - I\lambda)^{-1}$ we have

$$\begin{pmatrix} 1-\lambda & 0 & 0 & 0 & \cdots \\ 0 & 1-\lambda & 0 & 0 & \cdots \\ 0 & 0 & 1-\lambda & 0 & \cdots \\ 0 & 0 & 0 & 1-\lambda & \cdots \\ & & \vdots & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{pmatrix} \tag{3.2.2}$$

or

$$
\begin{aligned}
(1-\lambda)x_0 &= y_0 \\
(1-\lambda)x_1 &= y_1 \\
(1-\lambda)x_2 &= y_2 \\
(1-\lambda)x_3 &= y_3 \\
&\vdots
\end{aligned}
\tag{3.2.3}
$$

therefore

$$
\begin{aligned}
x_0 &= \tfrac{1}{1-\lambda}y_0 \\
x_1 &= \tfrac{1}{1-\lambda}y_1 \\
x_2 &= \tfrac{1}{1-\lambda}y_2 \\
x_3 &= \tfrac{1}{1-\lambda}y_3 \\
&\vdots
\end{aligned}
\tag{3.2.4}
$$

this gives the matrix

$$
(I-I\lambda)^{-1} = \begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{1-\lambda} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{1-\lambda} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{1-\lambda} & \cdots \\ & & \vdots & & \end{pmatrix} \tag{3.2.5}
$$

The columns of this matrix are defined for all values of $\lambda \neq 1$ and converges to zero satisfying part (i) of theorem 1.1.7 for the second part, $\sup_{n\geq 0}\{\sum_{k=0}^{\infty} |i_{nk}|\} = \frac{1}{1-\lambda} < \infty$ provided $\lambda \neq 1$, hence $(I-I\lambda)^{-1} \in B(c)$ if $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$. Which implies $(I-I\lambda)^{-1} \notin B(c)$ when $\lambda = 1$.

3.3 The spectrum of B operator on c

The matrix $B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix}$

Corollary 3.3.1. $B \in B(c)$

1. $\lim_{n\to\infty} b_{nk} = 0$ for each fixed k, $k = 0, 1, 2, \ldots$

2. $\lim_{n\to\infty}\sum_{k=0}^{\infty} b_{nk} = 1$ as $n \to \infty$

3. $\|A\| = \sup_{n\geq 0}\{\sum_{k=0}^{\infty}|b_{nk}|\} = 1 < \infty$

Theorem 3.3.2. *The eigenvalue of $B \in B(c)$ is the singleton set $\{1\}$.*

Proof. Solving the system $Bx = \lambda x$, $x \neq \theta$ in c and $\lambda \in \mathbb{C}$, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ & & \cdots & & & \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{pmatrix} \tag{3.3.1}$$

which gives

$$\begin{gathered} x_0 = \lambda x_0 \\ \tfrac{1}{2}x_0 + \tfrac{1}{2}x_1 = \lambda x_1 \\ \tfrac{1}{3}x_0 + \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 = \lambda x_2 \\ \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 = \lambda x_3 \\ \vdots \\ \tfrac{1}{3}x_{n-2} + \tfrac{1}{3}x_{n-1} + \tfrac{1}{3}x_n = \lambda x_n \\ \vdots \end{gathered} \tag{3.3.2}$$

Solving 3.3.2, we have if x_0 is the first non zero entry of x, then $\lambda = 1$. But $\lambda = 1$ implies $x_0 = x_1 = x_2 = \cdots = x_n = \cdots$, which shows that x is in the span of $\delta = (1,1,1,1,\cdots)$ which tends to 1 as n tends to infinity. Therefore $\lambda = 1$ is an eigenvalue of $B \in B(c)$. When x_1 is the first non zero entry of x, $\lambda = \frac{1}{2}$. But $\lambda = \frac{1}{2}$ implies $x_0 = 0$, $x_2 = 2x_1$, $x_3 = 6x_1$, $x_4 = 16x_1$, $x_5 = 44x_1$, $\cdots$ which shows that x is spanned by $\{0,1,2,6,16,44,\cdots\}$ an increasing sequence which is not bounded above, hence does not converge as n tends to infinity. If x_{n+2} is the first non zero entry for $n = 0,1,2,3,\cdots$, then $\lambda = \frac{1}{3}$, solving the system gives $x_n = 0$ for $n = 0,1,2,3,\cdots$ which is a contradiction hence $\lambda = \frac{1}{3}$ cannot be an eigenvalue. ☐

Theorem 3.3.3. *Let $T : c \to c$ be a linear map and define $T^* : c^* \to c^*$ i,e $T^* : \ell_1 \to \ell_1$by $T^*(g) = g \circ T$, $g \in c^* \equiv \ell_1$. Then both T and T^*must be given by a matrix. Moreover $T^* : \ell_1 \to \ell_1$is given by the matrix,*

$$A^* = T^* = \begin{pmatrix} \chi(\lim A) & (v_n)_0^{\infty} \\ (a_k)_0^{\infty} & A^t \end{pmatrix} \tag{3.3.3}$$

$$= \begin{pmatrix} \chi(\lim A) & v_0 & v_1 & v_2 & \cdots \\ a_0 & a_{00} & a_{10} & a_{20} & \cdots \\ a_1 & a_{01} & a_{11} & a_{21} & \cdots \\ a_2 & a_{02} & a_{12} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \tag{3.3.4}$$

, where

$$\begin{gathered} \chi(\lim A) = \lim_A(\delta) - \sum_{k=0}^{\infty} \lim_A \delta^k \\ v_n = \chi(P_n \circ T) \\ \textit{and} \\ a_k = \lim_{n\to\infty} a_{nk} \end{gathered} \tag{3.3.5}$$

(Wilansky, 1984. pg. 267).

Corollary 3.3.4. *Let $B : c \to c$. Then $B^* \in B(\ell_1)$ and*

$$B^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots \\ & & & \cdots & & \end{pmatrix} \tag{3.3.6}$$

Proof. By Theorem 3.3.3

$$B^* = T^* = \begin{pmatrix} \chi(\lim B) & (v_n)_0^\infty \\ (b_k)_0^\infty & B^t \end{pmatrix} \tag{3.3.7}$$

where $\chi(\lim B) = \lim_B(\delta) - \sum_{k=0}^{\infty} \lim_B \delta^k$ is called the characteristic of a matrix B

$\delta = \{1,1,1,1,\ldots\}$, $\lim_B(\delta) = 1$ and $\delta^k = \{0,0,0,0,\ldots,1,0,0\ldots\}$, having zeros with 1 in the k^{th} position, $\lim_B \delta^k = 0$ and $\sum \lim_B \delta^k = 0$. Hence $\chi(\lim B) = 1-0 = 1$. $v_n = \chi(P_n \circ T) = (P_n \circ T)\delta - \sum_{k=0}^{\infty} (P_n \circ T)\delta^k$ but for matrix B, $(P_n \circ T)\delta = 1$, $\forall n$ and $\sum_{k=0}^{\infty} (P_n \circ T)\delta^k = 1$ i.e

$$
\begin{aligned}
v_0 &= 1-(1+0+0+0+\cdots) = 1-1 = 0\\
v_1 &= 1-\left(\tfrac{1}{2}+\tfrac{1}{2}+0+0+\cdots\right) = 1-1 = 0\\
v_2 &= 1-\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}+0+\cdots\right) = 1-1 = 0\\
v_3 &= 1-\left(0+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}+0+0\cdots\right) = 1-1 = 0\\
&\vdots\\
v_n &= 0,\ n \geq 0
\end{aligned} \tag{3.3.8}
$$

hence the matrix becomes

$$
\begin{pmatrix} 1 & \theta \\ \theta & B^T \end{pmatrix} \tag{3.3.9}
$$

□

Theorem 3.3.5. *The eigenvalues of $B^* \in B(\ell_1)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\}$*

Proof. Consider the system $B^*x = \lambda x$, $x \neq \theta$ in ℓ_1 and $\lambda \in \mathbb{C}$,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 1 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & \cdots\\
0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots\\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots\\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \cdots\\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots\\
 & & & \cdots & & &
\end{pmatrix}
\begin{pmatrix} x_0\\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ \vdots \end{pmatrix}
= \lambda
\begin{pmatrix} x_0\\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ \vdots \end{pmatrix} \tag{3.3.10}
$$

which gives

$$
\begin{gathered}
x_0 = \lambda x_0\\
x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 = \lambda x_1\\
\tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 = \lambda x_2\\
\tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 + \tfrac{1}{3}x_5 = \lambda x_3\\
\tfrac{1}{3}x_4 + \tfrac{1}{3}x_5 + \tfrac{1}{3}x_6 = \lambda x_4\\
\cdots\\
\tfrac{1}{3}x_{n-2} + \tfrac{1}{3}x_{n-1} + \tfrac{1}{3}x_n = \lambda x_{n-2},\ for\, n \geq 5
\end{gathered} \tag{3.3.11}
$$

solving the system, we have

$$
\begin{aligned}
x_3 &= 3(\lambda - 1)x_1 - \tfrac{3}{2}x_2 \\
x_4 &= 3(\lambda - \tfrac{1}{2})x_2 - x_3 \\
x_5 &= 3(\lambda - \tfrac{1}{3})x_3 - x_4 \\
x_6 &= 3(\lambda - \tfrac{1}{3})x_4 - x_5 \\
&\cdots \\
x_n &= 3(\lambda - \tfrac{1}{3})x_{n-2} - x_{n-1}, n \geq 5
\end{aligned}
\tag{3.3.12}
$$

which gives

$$
\begin{gathered}
x_5 = 3(\lambda - \tfrac{1}{3})x_3 - x_4 \\
x_6 = 3(\lambda - \tfrac{1}{3})x_4 - x_5 \\
x_7 = 3^2(\lambda - \tfrac{1}{3})^2 x_3 - 3(\lambda - \tfrac{1}{3})x_4 - x_6 \\
x_8 = 3^2(\lambda - \tfrac{1}{3})^2 x_4 - 3(\lambda - \tfrac{1}{3})x_5 - x_7 \\
x_9 = 3^3(\lambda - \tfrac{1}{3})^3 x_3 - 3^2(\lambda - \tfrac{1}{3})^2 x_4 - 3(\lambda - \tfrac{1}{3})x_6 - x_8 \\
x_{10} = 3^3(\lambda - \tfrac{1}{3})^3 x_4 - 3^2(\lambda - \tfrac{1}{3})^2 x_5 - 3(\lambda - \tfrac{1}{3})x_7 - x_9 \\
x_{11} = 3^4(\lambda - \tfrac{1}{3})^4 x_3 - 3^3(\lambda - \tfrac{1}{3})^3 x_4 - 3^2(\lambda - \tfrac{1}{3})^2 x_6 - 3(\lambda - \tfrac{1}{3})x_8 - x_{10} \\
\vdots \\
for\, n\, even,\ x_n = 3^{\frac{n}{2}-2}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-2}x_4 - \\
\sum_{k=0}^{\frac{n}{2}-3} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1)}\ n \geq 6 \\
for\, n\, odd,\ x_n = 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1}x_3 - \sum_{k=0}^{\frac{n-1}{2}-2} 3^k(\lambda - \tfrac{1}{3})^k x_{n-(2k+1),}\, n \geq 5
\end{gathered}
\tag{3.3.13}
$$

Each term is a geometric progression with common ratio, $r = 3(\lambda - \frac{1}{3})$

$$\sum_{n=0}^{\infty}|x_n| = |x_0|+|x_1|+|x_2|+|x_3|+|x_4|$$

$$+\sum_{\substack{n=6\\ n\,even}}^{\infty}\left|3^{\frac{n}{2}-2}(\lambda-\frac{1}{3})^{\frac{n}{2}-2}x_4-\sum_{k=0}^{\frac{n}{2}-3}3^k(\lambda-\frac{1}{3})^k x_{n-(2k+1)}\right|$$

$$+\sum_{\substack{n=5\\ n\,odd}}^{\infty}\left|3^{\frac{n-1}{2}-1}(\lambda-\frac{1}{3})^{\frac{n-1}{2}-1}x_3-\sum_{k=0}^{\frac{n-1}{2}-2}3^k(\lambda-\frac{1}{3})^k x_{n-(2k+1)}\right|$$

$$\leq\sum_{n=0}^{4}|x_n|+\sum_{\substack{n=6\\ n\,even}}^{\infty}\left|3^{\frac{n}{2}-2}(\lambda-\frac{1}{3})^{\frac{n}{2}-2}x_2\right|+\sum_{\substack{n=5\\ n\,odd}}^{\infty}\left|3^{\frac{n-1}{2}-1}(\lambda-\frac{1}{3})^{\frac{n-1}{2}-1}x_3\right|$$

$$+\sum_{\substack{n=6\\ n\,even}}^{\infty}\sum_{k=0}^{\frac{n}{2}-3}\left|3^k(\lambda-\frac{1}{3})^k x_{n-(2k+1)}\right|+\sum_{\substack{n=5\\ n\,odd}}^{\infty}\sum_{k=0}^{\frac{n-1}{2}-2}\left|3^k(\lambda-\frac{1}{3})^k x_{n-(2k+1)}\right|$$

as $n\to\infty$, this is a geometric series with the common ratio, $r=3(\lambda-\frac{1}{3})$. The series converges only if $|r|<1$, that is $\left|3(\lambda-\frac{1}{3})\right|=3\left|\lambda-\frac{1}{3}\right|<1$ or $\left|\lambda-\frac{1}{3}\right|<\frac{1}{3}$. $\square$

For matrix B, we have $B-I\lambda=\begin{pmatrix} 1-\lambda & 0 & 0 & 0 & 0 & \cdots\\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & 0 & 0 & \cdots\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & 0 & \cdots\\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots\\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots\\ & & \cdots & & & \end{pmatrix}$,

$M=(B-I\lambda)^{-1}$ is given by

$$m_{nk}=\begin{cases}\frac{1}{a_{nn}}, & n=k\\ \frac{(-1)^{n-k}}{\Pi_{j=k}^{n}a_{nn}}D_{n-k}^{(k)}, & (0\leq k\leq n-1),\ (n,k\in\mathbb{N}_0)\\ 0, & (k>n)\end{cases} \tag{3.3.14}$$

This gives matrix M as in 2.3.8

Direct calculations confirms that $(B-I\lambda)M = M(B-I\lambda) = I$ i.e

$$(B-I\lambda)M = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & \cdots \\ & & \cdots & & & \end{pmatrix} *$$

$$\begin{pmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & \cdots \\ \frac{-1}{2(1-\lambda)(\frac{1}{2}-\lambda)} & \frac{1}{\frac{1}{2}-\lambda} & 0 & 0 & 0 & \cdots \\ \frac{\{\frac{1}{2}-(\frac{1}{2}-\lambda)\}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{-1}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} & \frac{1}{\frac{1}{3}-\lambda} & 0 & 0 & \cdots \\ \frac{-\{\frac{1}{2}(1-(1-3\lambda))-(\frac{1}{2}-\lambda)}{3^2(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \frac{(1-(1-3\lambda)}{3^2(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^2} & \frac{-1}{3(\frac{1}{3}-\lambda)^2} & \frac{1}{\frac{1}{3}-\lambda} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

in row 0 column 0, we have $(1-\lambda)(\frac{1}{1-\lambda}) = 1$, the other row elements are zeros

Row 1 column 0, we have $\frac{1}{2}(\frac{1}{1-\lambda}) - \left(\frac{1}{2}-\lambda\right)\left(\frac{1}{2(1-\lambda)(\frac{1}{2}-\lambda)}\right) = 0$ and

Row 1 column 1, $(\frac{1}{2}-\lambda)(\frac{1}{\frac{1}{2}-\lambda}) = 1$, the other row elements are zeros

$\vdots$

row n, $\begin{pmatrix} 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \end{pmatrix}$ with non zero entries when $k = n-2, n-1, n$

hence row n column 0 gives

$$\begin{pmatrix} 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ \frac{-1}{2(1-\lambda)(\frac{1}{2}-\lambda)} \\ \frac{\{\frac{1}{2}-(\frac{1}{2}-\lambda)\}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} \\ \vdots \\ \frac{(-1)^{n-2}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-3}} D_{n-2}^{(0)} \\ \frac{(-1)^{n-1}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} D_{n-1}^{(0)} \\ \frac{(-1)^{n}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D_{n}^{(0)} \\ \vdots \end{pmatrix} \tag{3.3.15}$$

w.l.o.g suppose n is even then,

$$\frac{D_{n-2}^{(0)}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-3}} - \frac{D_{n-1}^{(0)}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} + \frac{\left(\frac{1}{3}-\lambda\right)D_n^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} = \frac{\left(\frac{1}{3}-\lambda\right)D_{n-2}^{(0)} - D_{n-1}^{(0)} + 3D_n^{(0)}}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}$$

$$= \frac{\left(\frac{1}{3}-\lambda\right)D^{(0)}_{n-2}-D^{(0)}_{n-1}+3(\frac{1}{3}D^{(0)}_{n-1}-\frac{1}{3}(\frac{1}{3}-\lambda)D^{0}_{n-2})}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}$$

$$= \frac{\left(\frac{1}{3}-\lambda\right)D^{(0)}_{n-2}-D^{(0)}_{n-1}+D^{(0)}_{n-1}-(\frac{1}{3}-\lambda)D^{0}_{n-2})}{3(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} = 0 \text{ for all } n \geq 1.$$

Similarly, row n column 1 we have

$$\begin{pmatrix} 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\frac{1}{2}-\lambda} \\ \frac{-1}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)} \\ \vdots \\ \frac{(-1)^{n-3}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-3}}D^{(1)}_{n-3} \\ \frac{(-1)^{n-2}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}D^{(1)}_{n-2} \\ \frac{(-1)^{n-1}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}D^{(1)}_{n-1} \\ \vdots \end{pmatrix} \tag{3.3.16}$$

w.l.o.g suppose n is even then,

$$\frac{D^{(1)}_{n-3}}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-3}} - \frac{D^{(1)}_{n-2}}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} + \frac{\left(\frac{1}{3}-\lambda\right)D^{(1)}_{n-1}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} = \frac{\left(\frac{1}{3}-\lambda\right)D^{(1)}_{n-3}-D^{(1)}_{n-2}+3D^{(1)}_{n-1}}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}$$

$$= \frac{\left(\frac{1}{3}-\lambda\right)D^{(1)}_{n-3}-D^{(1)}_{n-2}+3(\frac{1}{3}D^{(1)}_{n-2}-\frac{1}{3}(\frac{1}{3}-\lambda)D^{(1)}_{n-3})}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}$$

$$= \frac{\left(\frac{1}{3}-\lambda\right)D^{(1)}_{n-3}-D^{(1)}_{n-2}+D^{(1)}_{n-2}-(\frac{1}{3}-\lambda)D^{1}_{n-3})}{3(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} = 0 \text{ for all } n \geq 2,$$

row n column k, $2 \leq k < n$,

$$\begin{pmatrix} 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\frac{1}{3}-\lambda} \\ \vdots \\ \frac{(-1)^{n-k-2}}{(\frac{1}{3}-\lambda)^{n-k-1}}D^{(k)}_{n-k-2} \\ \frac{(-1)^{n-k-1}}{(\frac{1}{3}-\lambda)^{n-k}}D^{(k)}_{n-k-1} \\ \frac{(-1)^{n-k}}{(\frac{1}{3}-\lambda)^{n-k+1}}D^{(k)}_{n-k} \\ \vdots \end{pmatrix} \tag{3.3.17}$$

w.l.o.g suppose n is even then,

$$\frac{D^{(k)}_{n-k-2}}{3(\frac{1}{3}-\lambda)^{n-k-1}} - \frac{D^{(k)}_{n-k-1}}{3(\frac{1}{3}-\lambda)^{n-k}} + \frac{\left(\frac{1}{3}-\lambda\right)D^{(k)}_{n-k}}{(\frac{1}{3}-\lambda)^{n-k+1}} = \frac{\left(\frac{1}{3}-\lambda\right)D^{(k)}_{n-k-2}-D^{(k)}_{n-k-1}+3D^{(k)}_{n-k}}{3(\frac{1}{3}-\lambda)^{n-k}}$$

$$= \frac{(\frac{1}{3}-\lambda)D^{(k)}_{n-k-2} - D^{(k)}_{n-k-1} + 3(\frac{1}{3}D^{(k)}_{n-k-1} - \frac{1}{3}(\frac{1}{3}-\lambda)D^{(k)}_{n-k-2})}{3(\frac{1}{3}-\lambda)^{n-k}}$$

$$= \frac{(\frac{1}{3}-\lambda)D^{(k)}_{n-k-2} - D^{(k)}_{n-k-1} + D^{(k)}_{n-k-1} - (\frac{1}{3}-\lambda)D^{(k)}_{n-k-2})}{3(\frac{1}{3}-\lambda)^{n-k}} = 0$$

row n column k, for $k = n$,

$$\begin{pmatrix} 0 & \cdots & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda & 0 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \frac{1}{\frac{1}{3}-\lambda} \\ \vdots \end{pmatrix} \tag{3.3.18}$$

$$= \tfrac{1}{3}(0) + \tfrac{1}{3}(0) + \left(\tfrac{1}{3} - \lambda\right)\left(\frac{1}{\frac{1}{3}-\lambda}\right) = 1.$$

This gives a matrix with $a_{nk} = 0$ for all $k \neq n$ and $a_{nn} = 1$, which is the identity matrix. Similar calculations shows that $M(B - I\lambda) = I$, hence $M = (B - I\lambda)^{-1}$

Theorem 3.3.6. *The spectrum $\sigma(B)$ of $B \in B(c)$* is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$

Proof. We show that $(B - I\lambda)^{-1} \in B(c)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$
for $k = 0$, $D^{(0)}_1 = \frac{1}{2}$
$D^{(0)}_2 = \frac{1}{2}(\frac{1}{3}) - (\frac{1}{2} - \lambda)\frac{1}{3} = \frac{1}{3}\{\frac{1}{2} - (\frac{1}{2} - \lambda)\}$
$D^{(0)}_3 = \frac{1}{3^2}\{\frac{1}{2} - \frac{1}{2}(1 - 3\lambda) - (\frac{1}{2} - \lambda)\} = \frac{1}{3^2}\{\frac{1}{2}(1 - (1 - 3\lambda)) - (\frac{1}{2} - \lambda)\}$
$D^{(0)}_4 = \frac{1}{3^3}\{\frac{1}{2} - \frac{2}{2}(1 - 3\lambda) - (\frac{1}{2} - \lambda) - (\frac{1}{2} - \lambda)(1 - 3\lambda)\} = \frac{1}{3^3}\{\frac{1}{2}(1 - 2(1 - 3\lambda)) - (\frac{1}{2} - \lambda)(1 - (1 - 3\lambda))\}$
$D^{(0)}_5 = \frac{1}{3^4}\{\frac{1}{2} - \frac{3}{2}(1 - 3\lambda) + \frac{1}{2}(1 - 3\lambda)^2 - (\frac{1}{2} - \lambda) + 2(\frac{1}{2} - \lambda)(1 - 3\lambda)\} = \frac{1}{3^4}\{\frac{1}{2}(1 - 3(1 - 3\lambda) + (1 - 3\lambda)^2) - (\frac{1}{2} - \lambda)(1 + 2(1 - 3\lambda)\}$

$$D^{(0)}_n = \frac{1}{3^{n-1}}\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n}{2}-1} a_k(1 - 3\lambda)^k\right) - (\tfrac{1}{2} - \lambda)\left(\sum_{k=0}^{\frac{n}{2}-1} b_k(1 - 3\lambda)^k\right)\right\} \text{when n is even,}$$

and

$$D^{(0)}_n = \frac{1}{3^{n-1}}\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n-1}{2}} a_k(1 - 3\lambda)^k\right) - (\tfrac{1}{2} - \lambda)\left(\sum_{k=0}^{\frac{n-1}{2}} b_{k-1}(1 - 3\lambda)^{k-1}\right)\right\} \text{when n is odd,}$$

where $a_{k's}$ and $b_{k's}$ are integers.
Hence the n^{th} row is given by

$$m_{n0} = \frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D^{(0)}_n = \frac{(-1)^n\left\{\frac{1}{2}\left(\sum_{k=0}^{\frac{n}{2}-1} a_k(1-3\lambda)^k\right) - (\frac{1}{2}-\lambda)\left(\sum_{k=0}^{\frac{n}{2}-1} b_k(1-3\lambda)^k\right)\right\}}{3^{n-1}(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}},$$

$$m_{n0} = \frac{(-1)^n}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} D_n^{(0)} = \frac{(-1)^n \left\{ \frac{1}{2} \left(\sum_{k=0}^{\frac{n-1}{2}} a_k (1-3\lambda)^k \right) - (\frac{1}{2}-\lambda) \left(\sum_{k=0}^{\frac{n-1}{2}} b_{k-1} (1-3\lambda)^{k-1} \right) \right\}}{3^{n-1}(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}},$$

as $n \to \infty$, the colums $m_{no} \to 0$ only if the denominator tends to infinity, and the denominator tends to infinity provided $\left|3(\frac{1}{3}-\lambda)\right| > 1$.
Similarly for $k = 1$, the denominator is given by $3^{n-1}(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}$ which tends to infinity provided $\left|3(\frac{1}{3}-\lambda)\right| > 1$ or $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$
And for $k \geq 2$, the denominator is given by $3^{n-k}(\frac{1}{3}-\lambda)^{n-k+1}$ which tends to infinity provided $\left|3(\frac{1}{3}-\lambda)\right| > 1$ or $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. □

Which proves theorem 1.1.7 (i). Summing the entries of the matrix 2.3.8 along the n^{th} row
$\sum_{k=0}^{\infty} |m_{nk}| =$
$\left|\frac{(-1)^n D_n^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \left|\frac{(-1)^{n-1} D_{n-1}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right| + \sum_{k=2}^{n} \left|\frac{(-1)^{n-k}(\frac{1}{3}D_{n-k-1}^{(k)} - \frac{1}{3}(\frac{1}{3}-\lambda)D_{n-k-2}^{(k)})}{(\frac{1}{3}-\lambda)^{n-k+1}}\right| = s_n$, say
for $n \geq 0$ $\sup_n\{s_n\} \leq K < \infty$, provided $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$, hence satisfies part (iii). For part (ii), we have $M = (B-I\lambda)^{-1}$ and $(B-I\lambda)(B-I\lambda)^{-1} = I$. Now $M\delta = \sum_{k=0}^{n} m_{nk}$, where $\delta = (1, 1, ,1\cdots)^T$. Also $(B-I\lambda)^{-1}(B-I\lambda) = M(B-I\lambda) = I$, multiplying by δ on both sides $M(B-I\lambda)\delta = I\delta$. Since $B\delta = \delta$, we have $M(\delta - \lambda\delta) = \delta$ or $M(1-\lambda)\delta = \delta$. Therefore

$$M\delta = \frac{1}{1-\lambda}\delta \tag{3.3.19}$$

That is

$$\sum_{k=0}^{n} m_{nk} = \frac{1}{1-\lambda} \tag{3.3.20}$$

hence $\lim_n \sum_{k=0}^{\infty} m_{nk} = \lim_n \frac{1}{1-\lambda} = \frac{1}{1-\lambda} < \infty$ provided $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$.
Therefore $(B-I\lambda)^{-1} \in B(c)$ if $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. Which implies $(B-I\lambda)^{-1} \notin B(c)$ if $\lambda \in \mathbb{C}$ such that $\left|\frac{1}{3}-\lambda\right| \leq \frac{1}{3}$. Clearly, when $\lambda = 1$, column 0 is infinite therefore the inverse does not exist. Hence $\sigma(B) =\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

CHAPTER FOUR
THE SPECTRUM OF A NÖRLUND OPERATOR B ON bv_0

4.1 Introduction

In this chapter, we determine the eigenvalues and the spectrum of the matrix B on the sequence space bv_0 by using Theorem 1.1.14:
and $\|B\|_{(bv_0,bv_0)} = \sup_{m\geq 0} \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} \left(b_{nk} - b_{n-1,k}\right) \right|$.

4.2 The spectrum of $B \in B(bv_0)$

Theorem 4.2.1. *$B : bv_0 \to bv_0$ and $B \in B(bv_0)$ with $\|B\|_{bv_0} = 1$.*

.

Proof. Using matrix B, let $y_n = \sum_{k=0}^{\infty} b_{nk}x_k$, where $x_k \in bv_0$, we have

$$\begin{aligned} y_0 &= x_0 \\ y_1 &= \tfrac{1}{2}(x_0 + x_1) \\ y_2 &= \tfrac{1}{3}(x_0 + x_1 + x_2) \\ y_3 &= \tfrac{1}{3}(x_1 + x_2 + x_3) \\ y_4 &= \tfrac{1}{3}(x_2 + x_3 + x_4) \\ &\vdots \\ y_n &= \tfrac{1}{3}(x_{n-2} + x_{n-1} + x_n), \ n \geq 2 \end{aligned} \tag{4.2.1}$$

In general,

$$\begin{aligned} |y_n - y_{n+1}| &= \tfrac{1}{3}\left|(x_{n-2} + x_{n-1} + x_n) - (x_{n-1} + x_n + x_{n+1})\right| \\ &= \tfrac{1}{3}\left|x_{n-2} - x_{n+1}\right|, \ n \geq 2 \end{aligned} \tag{4.2.2}$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} |y_n - y_{n+1}| \leq \sum_{n=2}^{\infty} |y_n - y_{n+1}| &= \tfrac{1}{3}\left|(x_0 + x_1 + x_2) - (x_1 + x_2 + x_3)\right| + \\ &\tfrac{1}{3}\left|(x_1 + x_2 + x_3) - (x_2 + x_3 + x_4)\right| + \cdots + \\ &\tfrac{1}{3}\left|(x_{n-2} + x_{n-1} + x_n) - (x_{n-1} + x_n + x_{n+1})\right| + \cdots \\ &\leq \tfrac{1}{3}|x_0 - x_1| + \tfrac{1}{3}|x_1 - x_2| + \tfrac{1}{3}|x_2 - x_3| + \tfrac{1}{3}|x_1 - x_2| + \tfrac{1}{3}|x_2 - x_3| + \\ &\tfrac{1}{3}|x_3 - x_4| + \tfrac{1}{3}|x_2 - x_3| + \tfrac{1}{3}|x_3 - x_4| + \tfrac{1}{3}|x_4 - x_5| + \cdots \tfrac{1}{3}|x_n - x_{n+1}| + \cdots \\ &\leq \sum_{n=2}^{\infty} |x_n - x_{n+1}| \end{aligned} \tag{4.2.3}$$

i.e

$$\sum_{n=0}^{\infty}|y_n-y_{n+1}|\leq\sum_{n=2}^{\infty}|x_n-x_{n+1}|<\infty \tag{4.2.4}$$

and $\left|\frac{y_{n+1}}{y_n}\right|=\left|\frac{\frac{1}{3}(x_{n-1}+x_n+x_{n+1})}{\frac{1}{3}(x_{n-2}+x_{n-1}+x_n)}\right|<\left|\frac{3x_n}{3x_n}\right|=1$, hence

$$\left|\frac{y_{n+1}}{y_n}\right|<1,\, n\geq 1 \tag{4.2.5}$$

This is the case since $x_n\to 0$ as $n\to\infty$ so that $x_n>x_{n+1}$ and $x_n<x_{n-1}$. Hence $y_n\to 0$ as $n\to\infty$. Therefore $y=Bx\in bv_0$. Direct computation shows that

$$\|B\|_{(bv_0,bv_0)}=\sup_m\sum_{n=0}^{\infty}\left|\sum_{k=0}^{m}\left(b_{nk}-b_{n-1,k}\right)\right|=\sup\left(1,1,1,1,\cdots\right)=1$$

and $\lim_{n\to\infty}b_{nk}=0$, $\forall k\geq 0$, hence all conditions of Theorem 1.1.14 are satisfied. Therefore $B\in B\left(bv_0,bv_0\right)$ □

Corollary 4.2.2. *$B\in B\left(bv_0\right)$ has no eigenvalues.*

Proof. $bv_0\subset c_0$, $B\in B\left(c_0\right)$has no eigenvalues, see theorem 2.3.3 □

Theorem 4.2.3. *Let $T: bv_0\to bv_0$ be given by a matrix $A=(a_{nk})$. Then $T^*: bv_0^*\to bv_0^*$ is also given by a matrix. Moreover $T^*$is the transpose of the matrix A acting on bs i.e*

$$T^*=A^T=\begin{pmatrix} a_{00} & a_{10} & a_{20} & \cdots \\ a_{01} & a_{11} & a_{21} & \cdots \\ a_{02} & a_{12} & a_{22} & \cdots \\ & \cdots & & \end{pmatrix} \tag{4.2.6}$$

, (Akanga, 2014)

Corollary 4.2.4. *Let $B: bv_0\to bv_0$, then $B^*: bv_0^*\to bv_0^*$ moreover, $B^*=B^T$ acting on bs.*

Proof. Replace the matrix A by matrix B in theorem 4.2.3 □

Theorem 4.2.5. *The eigenvalues of $B^T\in B\left(bs\right)$ are all $\lambda\in\mathbb{C}$ satisfying the inequality $\left|\lambda-\frac{1}{3}\right|<\frac{1}{3}$*

Proof. Suppose $B^Tx=\lambda x$, solving for x, we have

$$\begin{aligned} &for\, n\, even,\ x_n=3^{\frac{n}{2}-1}(\lambda-\tfrac{1}{3})^{\frac{n}{2}-1}x_2-\sum_{k=0}^{\frac{n}{2}-2}3^k(\lambda-\tfrac{1}{3})^kx_{n-(2k+1)} \\ &for\, n\, odd,\ x_n=3^{\frac{n-1}{2}-1}(\lambda-\tfrac{1}{3})^{\frac{n-1}{2}-1}x_3-\sum_{k=0}^{\frac{n-1}{2}-2}3^k(\lambda-\tfrac{1}{3})^kx_{n-(2k+1),\, n\geq 4} \end{aligned} \tag{4.2.7}$$

which is a geometric series with a common ratio $r = 3\left(\lambda - \frac{1}{3}\right)$.

$$\sum_{n=0}^{m} |x_n| = |x_0| + |x_1| + |x_2| + |x_3| + \sum_{\substack{n=4 \\ n\,even}}^{m} \left| 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1} x_2 - \sum_{k=0}^{\frac{n}{2}-2} 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right| +$$

$$\sum_{\substack{n=5 \\ n\,odd}}^{m} \left| 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1} x_3 - \sum_{k=0}^{\frac{n-1}{2}-2} 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right|$$

$$\leq \sum_{n=0}^{3} |x_n| + \sum_{\substack{n=4 \\ n\,even}}^{m} \left| 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1} x_2 \right| + \sum_{\substack{n=5 \\ n\,odd}}^{m} \left| 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1} x_3 \right| +$$

$$\sum_{\substack{n=4 \\ n\,even}}^{m} \sum_{k=0}^{\frac{n}{2}-2} \left| 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right| + \sum_{\substack{n=5 \\ n\,odd}}^{m} \sum_{k=0}^{\frac{n-1}{2}-2} \left| 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right| < \infty,$$

provided $\left|3\left(\lambda - \frac{1}{3}\right)\right| < 1$ or $\left|\lambda - \frac{1}{3}\right| < \frac{1}{3}$, hence the supremum exist.

Alternatively $\sum_{n=0}^{\infty} |x_n| = |x_0| + |x_1| + |x_2| + |x_3| +$

$$\sum_{\substack{n=4 \\ n\,even}}^{\infty} \left| 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1} x_2 - \sum_{k=0}^{\frac{n}{2}-2} 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right| +$$

$$\sum_{\substack{n=5 \\ n\,odd}}^{\infty} \left| 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1} x_3 - \sum_{k=0}^{\frac{n-1}{2}-2} 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right|$$

$$\leq \sum_{n=0}^{3} |x_n| + \sum_{\substack{n=4 \\ n\,even}}^{\infty} \left| 3^{\frac{n}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n}{2}-1} x_2 \right| + \sum_{\substack{n=5 \\ n\,odd}}^{\infty} \left| 3^{\frac{n-1}{2}-1}(\lambda - \tfrac{1}{3})^{\frac{n-1}{2}-1} x_3 \right| +$$

$$\sum_{\substack{n=4 \\ n\,even}}^{\infty} \sum_{k=0}^{\frac{n}{2}-2} \left| 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right| + \sum_{\substack{n=5 \\ n\,odd}}^{\infty} \sum_{k=0}^{\frac{n-1}{2}-2} \left| 3^k (\lambda - \tfrac{1}{3})^k x_{n-(2k+1)} \right|$$

this is a geometric series with the common ratio, $r = 3(\lambda - \frac{1}{3})$. This series converges only if $|r| < 1$, that is $\left|3(\lambda - \frac{1}{3})\right| = 3\left|\lambda - \frac{1}{3}\right| < 1$ or $\left|\lambda - \frac{1}{3}\right| < \frac{1}{3}$. Hence the partial sums are bounded whenever $\left|\lambda - \frac{1}{3}\right| < \frac{1}{3}$. □

Theorem 4.2.6. *Let $B : bv_0 \to bv_0$, the spectrum $\sigma(B)$ of $B \in B(bv_0)$ is the set $\lambda \in \mathbb{C}$ such that $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.*

Proof. we show that $(B-\lambda I)^{-1} \in B(bv_0)$, for all $\lambda \in \mathbb{C}$ satisfying $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$ refer to matrix 2.3.8

The columns of M are null provided $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$and $\lambda \neq 1$, hence satisfying condition (i) of theorem 1.1.14. Direct computation shows that

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{N} \left(m_{nk} - m_{n-1,k}\right) \right| \tag{4.2.8}$$

We have

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{N} \left(m_{nk} - m_{n-1,k}\right) \right| = \sum_1 + \sum_2 + \sum_3 \tag{4.2.9}$$

Where

$$\sum_1 = \sum_{n=0}^{N} \left| \sum_{k=0}^{n} m_{nk} - \sum_{k=0}^{n-1} m_{n-1,k} \right|, \; 0 \leq n \leq N \tag{4.2.10}$$

Recall equation 3.3.13, $\sum\limits_{k=0}^{n} m_{nk} = \frac{1}{1-\lambda}$ so that we have,

$$\sum_1 = |m_{00}| + \sum_{n=1}^{N} \left| \frac{1}{1-\lambda} - \frac{1}{1-\lambda} \right| = |m_{00}| = \left| \frac{1}{1-\lambda} \right| \tag{4.2.11}$$

And

$$\sum_2 = \left| \sum_{k=0}^{N+1} m_{N+1,k} - m_{N+1,N+1} - \sum_{k=0}^{N} m_{N,k} \right| = \left| \frac{1}{1-\lambda} - \frac{1}{\frac{1}{3}-\lambda} - \frac{1}{1-\lambda} \right| = \left| \frac{1}{\frac{1}{3}-\lambda} \right|, \; n = N+1 \tag{4.2.12}$$

While

$$\sum_3 = \sum_{n=N+2}^{\infty} \left| \sum_{k=0}^{N} \left(m_{nk} - m_{n-1,k}\right) \right| \tag{4.2.13}$$

$$= \sum_{n=N+2}^{\infty} \left| m_{n0} + m_{n1} + m_{n2} - m_{n-1,0} - m_{n-1,1} - m_{n-1,N} \right|$$

which gives

$$\sum_{n=N+2}^{\infty} \left| \frac{(-1)^n \left(D_n^{(0)} - (1-\lambda) D_{n-1}^{(1)}\right)}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}} + \frac{(-1)^{n-2} D_{n-2}^{(2)}}{(\frac{1}{3}-\lambda)^{n-1}} - \frac{(-1)^{n-1} D_{n-1}^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} - \frac{(-1)^{n-2} D_{n-2}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}} - \frac{(-1)^{n-1-N} D_{n-1-N}^{(N)}}{(\frac{1}{3}-\lambda)^{n-N}} \right|$$

But,

$$\frac{(-1)^n\left(D_n^{(0)}-(1-\lambda)D_{n-1}^{(1)}\right)}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}+\frac{(-1)^{n-2}D_{n-2}^{(2)}}{(\frac{1}{3}-\lambda)^{n-1}}-\frac{(-1)^{n-1}D_{n-1}^{(0)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}-\frac{(-1)^{n-2}D_{n-2}^{(1)}}{(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-2}}$$

$$=(-1)^n\left[\frac{D_n^{(0)}-(1-\lambda)D_{n-1}^{(1)}+(1-\lambda)(\frac{1}{2}-\lambda)D_{n-2}^{(2)}+\left(\frac{1}{3}-\lambda\right)D_{n-1}^{(0)}-(1-\lambda)(\frac{1}{3}-\lambda)D_{n-2}^{(1)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right]$$

Also
$D_n^{(0)}=\frac{1}{2}D_{n-1}^{(1)}-\frac{1}{3}(\frac{1}{2}-\lambda)D_{n-2}^{(2)}$
consequetly
$D_{n-1}^{(0)}=\frac{1}{2}D_{n-2}^{(2)}-\frac{1}{3}(\frac{1}{2}-\lambda)D_{n-3}^{(2)}$
substituting in **??** gives,

$$(-1)^n\left[\frac{\frac{1}{2}D_{n-1}^{(1)}-\frac{1}{3}(\frac{1}{2}-\lambda)D_{n-2}^{(2)}-(1-\lambda)D_{n-1}^{(1)}+(1-\lambda)(\frac{1}{2}-\lambda)D_{n-2}^{(2)}+\left(\frac{1}{3}-\lambda\right)\frac{1}{2}D_{n-2}^{(2)}-\frac{1}{3}(\frac{1}{2}-\lambda)D_{n-3}^{(2)}-(1-\lambda)(\frac{1}{3}-\lambda)D_{n-2}^{(1)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right]$$

$$=(-1)^n\left[\frac{-(\frac{1}{2}-\lambda)D_{n-1}^{(2)}+\frac{1}{3}\left(\frac{1}{2}-\lambda\right)D_{n-2}^{(2)}-\frac{1}{3}(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)D_{n-3}^{(2)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right] \quad (4.2.14)$$

Again
$D_{n-1}^{(2)}=\frac{1}{3}D_{n-2}^{(2)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{n-3}^{(2)}$,
substituting in 4.2.14, we have

$$(-1)^n\left[\frac{-(\frac{1}{2}-\lambda)\left(\frac{1}{3}D_{n-2}^{(2)}-\frac{1}{3}(\frac{1}{3}-\lambda)D_{n-3}^{(2)}\right)+\frac{1}{3}\left(\frac{1}{2}-\lambda\right)D_{n-2}^{(2)}-\frac{1}{3}(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)D_{n-3}^{(2)}}{(1-\lambda)(\frac{1}{2}-\lambda)(\frac{1}{3}-\lambda)^{n-1}}\right]=0 \quad (4.2.15)$$

Hence

$$\sum_3=\sum_{n=N+2}^{\infty}\left|-\frac{(-1)^{n-1-N}D_{n-1-N}^{(N)}}{(\frac{1}{3}-\lambda)^{n-N}}\right| \quad (4.2.16)$$

Substituting for $n=N+2,\ N+3,\ N+4,\ N+5,\cdots$, we get

$$\sum_3=\frac{D_1^{(N)}}{(\frac{1}{3}-\lambda)^2}+\frac{D_2^{(N)}}{(\frac{1}{3}-\lambda)^3}+\frac{D_3^{(N)}}{(\frac{1}{3}-\lambda)^4}+\frac{D_4^{(N)}}{(\frac{1}{3}-\lambda)^5}+\cdots \quad (4.2.17)$$

which gives
$\frac{1}{3(\frac{1}{3}-\lambda)^2}+\frac{1}{3^2(\frac{1}{3}-\lambda)^3}-\frac{1}{3(\frac{1}{3}-\lambda)^2}+\frac{1}{3^3(\frac{1}{3}-\lambda)^4}-\frac{2}{3^2(\frac{1}{3}-\lambda)^3}+\frac{1}{3^4(\frac{1}{3}-\lambda)^5}-\frac{3}{3^3(\frac{1}{3}-\lambda)^4}+\frac{1}{3^2(\frac{1}{3}-\lambda)^3}+\cdots$

$$=\frac{-2}{3^3(\frac{1}{3}-\lambda)^4}+\frac{1}{3^4(\frac{1}{3}-\lambda)^5}+\cdots \quad (4.2.18)$$

$$= \sum_{m=3}^{\infty} \frac{a_m}{3^m(\frac{1}{3}-\lambda)^{m+1}} \tag{4.2.19}$$

where $a'_m s$ are constants.

This is a geometric series which converges if $\left|3\left(\frac{1}{3}-\lambda\right)\right| > 1$, hence the supremum exist provided $\left|3\left(\frac{1}{3}-\lambda\right)\right| > 1$ or $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$. Therefore

$$\sup_{N\geq 0}\left\{\left|\frac{1}{1-\lambda}\right| + \left|\frac{1}{\frac{1}{3}-\lambda}\right| + \sum_{n=N+2}^{\infty}\left|\frac{D^{(N)}_{n-1-N}}{(\frac{1}{3}-\lambda)^{n-N}}\right|\right\} \tag{4.2.20}$$

exists for all λ satisfying $\left|\frac{1}{3}-\lambda\right| > \frac{1}{3}$ and $\lambda \neq 1$. Therefore $M = (B-I\lambda)^{-1} \in (bv_0)$ for all $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{3}\right| > \frac{1}{3}$ and $\lambda \neq 1$. Hence $M = (B-I\lambda)^{-1} \notin (bv_0)$ for all $\lambda \in \mathbb{C}$ such that $\left\{\left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\right\} \cup \{1\}$. □

CHAPTER FIVE
CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

Summary of results, areas of applications and suggestions of areas for further research are given in this chapter

5.2 Summary of results obtained

These are the results obtained chapter by chapter.
In chapter two, we obtained the following results;

i. The eigen value of $I \in B(c_0)$ is the singleton set $\{1\}$.
ii. The spectrum of $I \in B(c_0)$ is the singleton set $\{1\}$.
iii. $B \in B(c_0)$ has no eigenvalues.
iv. The eigenvalues of $B^* \in B(\ell_1)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\} \cup \{1\}$.
v. The spectrum $\sigma(B)$ of $B \in B(c_0)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

In chapter three, the following results were obtained;

i. The eigen value of $I \in B(c)$ is the singleton set $\{1\}$.
ii. The spectrum of $I \in B(c)$ is the singleton set $\{1\}$.
iii. The eigenvalue of $B \in B(c)$ is the singleton set $\{1\}$.
iv. The eigenvalues of $B^* \in B(\ell_1)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\}$.
v. The spectrum $\sigma(B)$ of $B \in B(c)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

In chapter four, the following results were obtained;

i. $B \in B(bv_0)$ has no eigenvalues.
ii. The eigenvalues of $B^* \in B(bs)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| < \frac{1}{3}\}$.
iii. The spectrum $\sigma(B)$ of $B \in B(c)$ is the set $\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{3}\right| \leq \frac{1}{3}\} \cup \{1\}$.

In conclusion, the spectrum is the same in all the cases, however the sets of eigenvalues differs.

5.3 Recommendations

The following are recommendation for future research:

(a) Investigating the spectrum of the operator B on the other sequence spaces
(b) Constructing the fine spectrum of the operator B on sequence spaces

(c) Investigating the spectrum of a general N0̈rlund operator

5.4 Areas of Application

The eigenvalues and the spectrum of a matrix has numerous applications in various fields, a few areas are mentioned below.

5.4.1 Modelling population growth

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For instance, if the lifespan of a member is L years, then the following n intervals represent the age classes, $[0, \frac{L}{n})$, $[\frac{L}{n}, \frac{2L}{n}) \cdots [\frac{(n-1)L}{n}, L]$, the age distribution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ represent the number of population members in each group, the transition matrix is given by average number of offspring produced by a member of the ith age in the first row and the probability that a member of the ith age class will survive to become a member of the i+1 th age class are in the other rows. The $Ax_i = x_{i+1}$ produces the age distribution vector for the next period. To obtain a stable gowth pattern, then $\lambda x_i = Ax_i = x_{i+1}$ i.e $x_{i+1} = \lambda x_i$ a scalar multiple of the previous distribution.

5.4.2 Solution of system of first order linear differential equations

The system can be written in matrix form as $y' = Ay$ where y is a function of t. The solution is given by $y = e^{\lambda_i t}$ where λ_i are the eigenvalues of A if it is a diagonal matrix. If it is not diagonal then it is diagonalized and transformed i.e

$$\begin{aligned} y &= Pw \\ y' &= Pw' \\ w' &= P^{-1}APw \\ w &= e^{\lambda_i t} \end{aligned} \tag{5.4.1}$$

where P is the diagonalization matrix and λ_i are the eigenvalues of the resulting diagonal matrix.

5.4.3 Principal Axis Theorem

Principal Axis Theorem states that for a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by $X = PX'$ eliminates the $xy-$term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes A i.e $P^T AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$.

5.4.4 Application to summability of Sequences and Series

A divergent sequence has no limit in the usual sense, in summability method, one aims at associating with a divergent sequence a limit or a divergent series a sum for instance a divergent sequence $x = (x_n)$ being given, we may calculate the sequence (y_n) of the arithmetic means $y_n = Ax_n$,, if y_n converges to y then we say that x_n is summable to y by A.

5.4.5 Quantum Mechanics

Let the Hamiltonian H of some system be given by an infinite matrix $H = (h_{ij}), i, j = 1, 2, \cdots$ considered as an operator on some infinite set of numbers . The possible energy values of the system are the eigenvalues of H (usually relative to ℓ^2) and the main problem of pertubation theory is to estimate these eigenvalues.

5.4.6 Solution of Infinite Linear Systems

Infinite dimensional linear systems appear naturally when studying control problems for systems modelled by linear partial differential equations. Many problems in dynamic systems can be written in form of infinite differential systems which leads to infinite differential systems e.g Mathieu equation, Hill's equation e,t,c

REFERENCES

Akanga, J. R. (2014). On the spectrum of a special *Nörlund* means, *Journal of Mathematical Sciences*, *25*(2), 59 - 67.

Akanga, J. R., Mwathi, C. and Wali, A. N. (2010). The Spectrum of the *Nörlund Q* operator on c_0 space, *JAGST*, *12*(2), 92 - 104.

Bennett, G. (1971). Distinguished Subsest and Summability Invariants, *Studia Math.* *60*, 225 - 234.

Bennett, G. (1972a). The Gliding Hump Technique for FK-spaces, *Trans. Amer. Math. Soc.* *166*, 285 - 292

Bennett, G. (1972b). A Representation Theorem for Summability Domains, *Proc. London Math. Soc.* *3*(24), 193 - 202

Bennett, G. and Kalton N. (1972). FK - Spaces containg c_0, *Duke Math. Jour.* *39*, 561 - 582, 819 - 821

Binod Chandra Tripathy and Pallavi Saikia, (2013). On the spectrum of the Cesaro Operator C_1 on $\overline{b}v_0 \cap \ell_\infty$, *Mathematica Slovaka*, *63*(3), 563 - 572.

Boos, J. (2000). *Classical and Modern methods in summability*, New York: Oxford Univ. Press Inc.

Boyd, D. (1968). The spectrum of the Cesaro operator, *Acta Scientiarum Mathematicarum*, *29*, 31 - 34

Brown, A. Halmos, P. and Shields, A. (1965). Cesaro Operators, *Acta. Sci. Math.* *26*, 125 - 137

Brown, H. Kerr, D. and Straton, H. (1969). The structure of B(c) and Extensions of the concept of the Conull Matrix, *Proc. Amer. Math. Soc.* *22*, 7-15

Coskun, C. (2003). A special *Nörlund* mean and its Eigenvalues, Communication Faculty University Ankara, *Ser. Al*, *Math. Stat* *52*(1), 27 - 31.

Deddens, J. A. (1978). On the Spectra of Hausdorff operators on ℓ_+^2, *Proceedings of the American Mathematical Society*, *72*(1).

Dorf, E. K. and Wilansky, A. (1960). Remarks on Summability, *JLMS*, *35*, 234 - 236.

Dunford, N. and Schwartz, J. (1957). *Linear Operators, part I, General Theory,* New York: Interscience Publishers, Inc.

Goldberg, S. (1966). *Unbounded Linear Operators – Theory and Applications,* McGraw Hill Book Co.

Gonzalez, M. (1985). The Fine Spectrum of the Cesaro operator in $\ell_p(1 < p < \infty)$, *Arch. Math. 44*, 355 - 358.

Hardy, G. (1948). *Divergent series*, Oxford University Press.

Jakimovski, A and Russel, (1972). Matrix Mappings between B.K - spaces, *Bull. London Math.Soc. 4*, 345 - 353.

Kreyszig, E. (1980). *Introductory Functional Analysis with Applications,* John Wiley and Sons.

Limaye B. (1996). *Functional Analysis*, New Delhi: New Age International Ltd.

Maddox, I. (1970). *Elements of Functional Analysis,* UK: Cambridge University Press.

Maddox, I. (1980). *Infinite Matrices of Operators*, Berlin Heidelberg: Springer - Verlag.

Okutoyi, J. (1985). The Spectrum of C_1as an operator on $w_p(0), (1 \leq p < \infty)$, *Bullet of the Calcutta Mathematical Society, 17*, 154 - 263.

Okutoyi, J. (1992). On the Spectrum of C_1 as an operator on *bv*, *Commun. Fac. Sci. Univ. Ank. 41*.

Okutoyi, J. I. (1990). On the Spectrum of C_1 as an operator on bv_0, *Journal of Australian Mathematical Society (SeriesA), 48*, 79 - 86.

Okutoyi, J. I and Akanga J. R. (2005). The Spectrum of C_1 as an operator on $w_p(1 \leq p < \infty)$, *East African Journal of physical sciences* , *6*(1), 33 - 48

Okutoyi, J. I and Thorpe B. (1989). Spectrum of the Cesaro operator on $c_0(c_0)$, *Math. Proc. Camb. Philo. Soc. 105*, 123 - 129.

Pinakadhar, B. and Dutta, S. (2015). On an explicit formula for inverse of triangular matrices, *Journal of the Egyptian Mathematical society, 23,* 297-302.

Powell, R. and Shah, S. (1972). *Summability Theory and Applications*, Van Nostrand

Reinhold Company Ltd.

Reade, J. (1985). On The Spectrum of the Cesaro Operator, *Bull. Lond. Math. Soc. 17*, 263 - 267.

Rhoades, B. E. (1983). The Fine Spectrum of Weighted Mean Operators, *Pacific Journal of Maths, 104*, 219 - 230.

Ruckle, W. (1981). *Sequence Spaces*, Pitman Publishing Ltd.

Shafiquel, I. (1996). Spectrum of C_1 on $\ell_\infty -$the Space of Bounded Sequences, *Sust. Studies*, *1*(1), 48 - 54.

Sharma, N. K. (1972). The Spectra of Conservative Matrices, *Proc. of the American Mathematical Society 35*(2).

Sharma, N. K. (1975). Isolated points of the spectra of conservative matrices, *Proc. of the American Mathematical Society, 51*(1).

Taylor, A. E. (1958). *Introduction to Functional Analysis*, John Wiley and sons Inc.

Taylor, A. and Lay, D. (1980). *Introduction to Functional Analysis*, 2^{nd} edition, John Wiley and Sons.

Wenger, R. B. (1975). The fine spectra of the holder summability operators, Indian *Journal of Pure and Applied Maths, 6*, 695 - 710.

Wilansky, A. (1984). *Summability Through Functional Analysis*, North Holland Publishers.