A NEW GENERALIZATION OF TRANSFORMED-TRANSFORMER FAMILY OF DISTRIBUTIONS

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A New Generalization of Transformed-Transformer Family of Distributions

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DECLARATION

This thesis is my original work and no part of it has been presented for another degree award in any other university.

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DEDICATION

To my beloved wife Karim Zenabu, lovely daughter Nasira Nasiru, father Yakubu Suleman and mother Sekinatu Seidu.

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LIST OF ABBREVIATIONS

AB	Average Bias
AE	Average Estimate
AIC	Akaike Information Criterion
AICc	Corrected Akaike Information Criterion
BD	Burr III Distribution
BFGS	Broyden-Fletcher-Goldfarb-Shanno
BIC	Bayesian Information Criterion
CDF	Cumulative Distribution Function
DD	Dagum Distribution
EG	Exponentiated Generalized
EGB	Exponentiated Generalized Binomial
EGBD	Exponentiated Generalized Burr III Distribution
EGBIE	Exponentiated Generalized Binomial Inverse Exponential
EGDD	Exponentiated Generalized Dagum Distribution
EGE	Exponentiated Generalized Exponential
EGEBD	Exponentiated Generalized Exponential Burr III Distribution
EGED	Exponentiated Generalized Exponential Dagum
EGEFD	Exponentiated Generalized Exponential Fisk Distribution

EGEIE	Exponentiated Generalized Exponential Inverse Exponential
EGEIR	Exponentiated Generalized Exponential Inverse Rayleigh
EGFD	Exponentiated Generalized Fisk Distribution
EGG	Exponentiated Generalized Geometric
EGGIE	Exponentiated Generalized Geometric Inverse Exponential
EGHL	Exponentiated Generalized Half Logistic
EGHLBX	Exponentiated Generalized Half Logistic Burr X
EGHLF	Exponentiated Generalized Half Logistic Frétchet
EGIE	Exponentiated Generalized Inverse Exponential
EGIR	Exponentiated Generalized Inverse Rayleigh
EGL	Exponentiated Generalized Logarithmic
EGLIE	Exponentiated Generalized Logarithmic Inverse Exponential
EGMIR	Exponentiated Generalized Modified Inverse Rayleigh
EGP	Exponentiated Generalized Poisson
EGPIE	Exponentiated Generalized Poisson Inverse Exponential
EGPS	Exponentiated Generalized Power Series
EGSHLBX	Exponentiated Generalized Standardized Half Logistic Burr X
EHLBX	Exponentiated Half Logistic Burr X
EKD	Exponentiated Kumaraswamy Dagum

FD	Fisk Distribution
HLBX	Half Logistic Burr X
IE	Inverse Exponential
IR	Inverse Rayleigh
K-S	Kolmogorov-Smirnov
LRT	Likelihood Ratio Test
McD	Mc-Dagum
MGF	Moment Generating Function
MIR	Modified Inverse Rayleigh
NEGMIR	New Exponentiated Generalized Modified Inverse Rayleigh
NGIW	New Generalized Inverse Weibull
PDF	Probability Density Function
PS	Power Series
RMSE	Root Mean Square Error
SHLBX	Standardized Half Logistic Burr X
SIC	Schwarz Information Criterion
TTT	Total Time on Test
T-X	Transformed-Transformer
WBXII	Weibull Burr XII

ABSTRACT

The development of generalized classes of distributions have attracted the attention of both theoretical and applied statisticians in recent times due to their flexible statistical properties. In this study, the exponentiated generalized transformed-transformer family of distributions was proposed and studied. The statistical properties of the new family were derived and various sub-families were defined. Some of the sub-families were used to develop the exponentiated generalized exponential Dagum, new generalized modified inverse Rayleigh and exponentiated generalized half logistic Burr X distributions. The statistical properties of the proposed distributions were studied and inferences were made on them. An extension of a sub-family of the exponentiated generalized transformed-transformer was developed by compounding it with the power series class to obtain the exponentiated generalized power series family of distributions. Monte Carlo simulations were performed to investigate the properties of the maximum likelihood estimators for the parameters of the developed distributions. The results revealed that the maximum likelihood estimators for the parameters were consistent. Applications of the proposed distributions were demonstrated using real data sets and their performance were compared to other known competing models. The proposed distributions showed greater flexibility and can be used to model different kinds of real data sets.

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Parametric statistical inferences and modeling of data sets require the knowledge of appropriate distributional assumptions of the data sets. Thus, classical statistical distributions have been used in many areas of applied and social sciences to makes inferences and model data. The usefulness of statistical distributions in several areas of research includes: modeling environmental pollution in environmental science, modeling duration without claims in actuarial science, modeling machine life cycle in engineering, modeling survival times of patients after surgery in the medical science, modeling failure rate of software in computer science and average time from marriage to divorce in the social science. However, the data generating process in many of these areas are characterized with varied degrees of skewness and kurtosis. Also, the data may exhibit non-monotonic failure rates such as the bathtub, unimodal or modified unimodal failure rates. Hence, modeling the data with the existing classical distributions does not provide a reasonable parametric fit and is often an approximation rather than reality.

An alternative approach to overcome these challenges is to use nonparametric methods to model the data sets since they do not depend on distributional assumptions like the parametric methods. However, the nonparametric methods have their own drawbacks including: loss in power when the parametric method is appropriate, lack of imprecision measurement, computational difficulties, difficult to calculate residual variability and loss information (Allison, 1995; Blossfeld and Rohwer, 1995; Mallet, 1986; Schumitzky, 1991). Because of these, the statistical literature in the recent decades has been continuously flooded with barrage of methods for modifying existing classical distributions to make them more flexible or developing new statistical distributions for modeling data sets from different fields of study. Most of the techniques are geared towards producing distributions with heavy tails, monotonic and non-monotonic failure rates, tractable cumulative distribution function (CDF) to make simulation easy and to model data with different degrees of skewness and kurtosis. Some of the modified distributions in literature are: exponentiated exponential distribution (Gupta and Kundu, 1999); exponentiated Weibull distribution (Mudholkar and Srivastava, 1993); beta-normal (Eugene et al., 2002); and beta-Pareto distribution (Akinsete et al., 2008).

The techniques for modifying the classical distributions are usually referred to as generators in literature and are capable of improving the goodness-of-fit of the modified distributions. These features have been established by the results of many generators (Cordeiro and de Castro, 2011; Jones, 2009; Eugene et al., 2002). Recently, Alzaatreh et al. (2013) proposed an extension of the beta-generated family of distributions developed by Eugene et al. (2002) and called it the transformed-transformer (T-X) family of distributions. According to Alzaatreh et al. (2013), the CDF of the T-X family is defined as

$$G(x) = \int_0^{-\log(1-F(x))} r(t)\, dt = R\left\{-\log(1-F(x))\right\}, \tag{1.1}$$

where $r(t)$ is the probability density function (PDF) of the random variable T, $R(t)$ is the CDF of T and $F(x)$ is the CDF of the random variable X. Although, the T-X method have been embraced by several researchers, Alzaghal et al. (2013) proposed an extension of it in order to improve on some of the drawbacks of the T-X method. The new family defined by Alzaghal et al. (2013) is called the exponentiated T-X family of distributions. The CDF of this new family is given by

$$G(x) = \int_0^{-\log(1-F^c(x))} r(t)\,dt = R\{-\log(1-F^c(x))\}, \tag{1.2}$$

where c is a shape parameter introduced in the family to make it more flexible. However, both the T-X family and the exponentiated T-X family still have some drawbacks that need to be addressed.

1.2 Statement of the Problem

The problem associated with the T-X method of Alzaatreh et al. (2013) is that the CDF of the family has no extra shape parameters for improving the flexibility of modified distributions. For instance if the distribution of the random variable T follows standard exponential and that of X is exponential, then the resulting distribution will have no shape parameter. However, to produce distributions with heavy tails and model data with non-monotonic failure rates the resulting modified distribution should have extra shape parameters. Thus, if the distributions of T and X in the T-X family have no shape parameters, then no additional flexibility is added to the new distribution. To overcome these drawbacks, Alzaghal et al. (2013) introduced a new shape parameter c in the T-X and called the new family exponentiated T-X family.

The limitations of adding a single shape parameter is that it is unable to produce distributions with heavy tails and control both skewness and kurtosis at the same. The need for an additional shape parameter to produce heavy tail distribution and control both skewness and kurtosis is important since lifetime data often exhibit these traits. This study develops a new generalization of the T-X family of distributions by adding an extra shape parameter to provide greater flexibility and improve goodness-of-fit of modified distributions when modeling lifetime data sets.

1.3 General Objective

The main objective of this study is to develop and derive the statistical properties of a new generalization of the T-X family of distributions.

1.4 Specific Objectives

The specific objectives are:

1. To develop a new exponentiated generalized T-X family of distributions.
2. To derive the statistical properties of some distributions arising from this new family of distributions.
3. To develop maximum likelihood estimators for the parameters of the new distributions.
4. To investigate the statistical properties of the estimators for the parameters using simulation.
5. To demonstrate the applications of the new distributions using real data sets.

1.5 Significance of the Study

Statistical probability distributions are the foundation of statistical methodology in both theory and practice. They form the backbone to every parametric statistical method including inference, modeling, survival analysis, reliability analysis among others. For instance, statistical distributions have been used in the engineering sciences to model the life cycle of a machine. In the medical sciences, statistical distributions have been used to study duration to recurrence of cancer after surgical removal. Another important area of study where statistical distributions play a key role is extreme value theory. Statistical distributions have been used in modeling extreme events such as earthquakes and floods.

The knowledge of appropriate distribution of real data sets greatly improves the sensitivity, power and efficiency of the statistical test associated with the data sets. Hence, developing new generators for modifying existing distributions to improve their goodness-of-fit is imperative. Thus, in this study a new method for generalizing distributions called exponentiated generalized T-X was developed and studied.

1.6 Literature Review

Several methods for developing new distributions have been proposed in literature. This section presents a review on some of the general methods developed before 1980 and those proposed since the 1980s. Those methods developed before the 1980 are: method of differential equation, method of transformation and method of quantile function. The methods developed since the 1980s include: method of generating skewed distributions, method of adding parameters to existing distributions, beta-generated method and the

T-X method.

1.6.1 Method of Differential Equation

The methods developed prior to the 1980s can be classified as method of differential equations, method of transformation and method of quantile. The early works of Pearson (1895) can be seen as the break through in this field. He proposed the differential equation approach for developing probability distributions. With this approach every member of a PDF satisfies a differential equation. Pearson (1895, 1901) classified these distributions based on the shape of the PDF into a number of types known as Pearson types I-VI. Later in another paper, Pearson (1916) defined more special cases and subtypes known as the Pearson types VII-XII. Several well known distributions belong to the Pearson type distributions. Among them are: normal and student t distributions (Pearson type VII), beta distribution (Pearson type I) and gamma distribution (Pearson type III). In addition to the differential equation approach, Burr (1942) developed another method for developing probability distributions using different form of differential equations. Burr's system gave twelve solutions to the differential equation in the form of CDF. Some common distributions from the Burr's family are: the uniform, Burr III, Burr X and Burr XII distributions.

1.6.2 Method of Transformation

Johnson (1949) proposed the method of transformation which sometimes is referred to as translation in literature using normalization transformation. The Johnson's family include many commonly used distributions such normal, lognormal, gamma, beta, exponential among others. The Birnbaum-Saunders distribution is an important lifetime distribution that belongs to the Johnson's family and was originally developed to model material fatigue (Birnbaum and Saunders, 1969). Athayde et al. (2012) employed the Johnson's system to develop various generalizations of the Birnbaum-Saunders distribution including families of location-scale Birnbaum-Saunders, non-central Birnbaum-Saunders and four parameter generalized Birnbaum-Saunders distributions.

1.6.3 Method of Quantile Function

The development of the lambda distribution led to the quantile method of proposing probability distributions (Hastings et al., 1947; Tukey, 1960). The lambda distribution was later generalized as the so called lambda distributions which were defined in terms of percentile function (Ramberg and Schmeiser, 1972, 1974; Ramberg et al., 1979). Freimer et al. (1988) addressed the similarities between the Pearson's system and the generalized lambda distribution. They indicated that the Pearson's family does not include logistic distribution while the generalized lambda distribution does not cover all skewness and kurtosis values. In order to overcome these weaknesses an extended generalized lambda was proposed by Karian and Dudewicz (2000) which contains both generalized lambda distribution and generalized beta distribution. Some examples of other works based on quantiles from Tukey's lambda distribution can be found in Tuner and Pruitt (1978), Morgenthaler and Tukey (2000) and Jones (2002).

1.6.4 Method of Generating Skewed Distributions

Azzalini (1985) proposed a method for developing skewed distributions by combining two symmetric distributions. The initial idea of generating this family of distributions appeared in an article by O'Hagan and Leonard (1976). This class of distributions were called the skewed normal family. Azzalini (1986) stated that the earlier class of distributions can only produce tails thinner than the normal ones and proposed a broader class of densities. Various generalizations of the skewed family have been proposed and studied extensively. Chang and Genton (2007) proposed a weighted approach of generating skewed distributions that include Azzalini's framework as a special case. Mudholkar and Hutson (2000) developed the epsilon-skew normal family of distributions which contains additional parameter to control the magnitude of skewness. Salinas et al. (2007) proposed another extension by combining Azzalini's skew normal family, the epsilon-skew normal into a broad family of skewed distributions. Fernández and Steel (1998) developed a method of introducing skewness into any continuous unimodal and symmetric distribution by using inverse scaling of the PDF on both sides of the mode. Their method does not affect the unimodality and at the same time permits the creation of flexible distribution shape by a single scalar parameter. Ferreira and Steel (2006) introduced a general framework of adding skewness into symmetric distribution based on inverse probability integral transformation. The Azzalini (1985) family and the inverse scale family of Fernández and Steel (1998) are members of this new family of distributions.

1.6.5 Method of Adding Parameters

The addition of parameters to existing distributions to make them more flexible started with the work of Mudholkar and Srivastava (1993) on the exponentiated Weibull distribution. Gupta et al. (1998) gave a detail explanation of the exponentiated family of distributions. Gupta and Kundu (1999, 2001) studied the exponentiated exponential distribution. Nadarajah and Kotz (2006) studied a list of exponentiated distributions including exponentiated exponential, gamma, Weibull, Gumbel and Fréchet distributions.

Marshall and Olkin (2007) proposed another method of adding an extra parameter to a lifetime distribution. They studied in details the case of exponential and Weibull distribution.

1.6.6 Beta-Generated Method

The beta-generated family of distribution was proposed by Eugene et al. (2002) by using the beta distribution as a generator. This family of distributions can be described as a generalization of the distribution of order statistics (Jones, 2004). Since the development of the beta-generated family of distributions, many beta generated distributions have been proposed in literature. Among them are: beta-normal (Eugene et al., 2002); beta-Gumbel (Nadarajah and Kotz, 2004); beta-Fréchet (Nadarajah and Gupta, 2004); beta-exponential (Nadarajah and Kotz, 2005); beta-Weibull (Famoye et al., 2005); beta-Cauchy (Alshawarbeh et al., 2012); beta-exponentiated Pareto (Zea et al., 2012); beta-generalized logistic (Morais et al., 2013); and beta-extended Weibull (Cordeiro et al., 2012). Several generalized versions of the beta-generated families have been defined in literature by changing the beta distribution with any distribution defined on a finite support. Jones (2009) and Cordeiro and de Castro (2011) independently proposed the Kumaraswamy generated family of distributions by replacing the beta distribution with the Kumaraswamy distribution (Kumaraswamy, 1980). Alexander et al. (2012) defined another generalization of the beta-generated family using the generalized beta type-I distribution as the generator instead of the beta distribution.

1.6.7 Transformed-Transformer Method

The beta-generated and Kumaraswamy-generated families were developed using distributions defined on the support [0, 1] as the generators. Alzaatreh et al. (2013) developed a general method that permits the use of any continuous distribution as the generator. This new method is referred to as the T-X family of distributions. They defined many families of distributions including the gamma-X, Weibull-X and beta-exponential-X families. Due to the shortcomings of the T-X family, Alzaghal et al. (2013) proposed a new generalization of it and named it exponentiated T-X family. Some members of this family include: exponentiated gamma-X, exponentiated Weibull-X, exponentiated Lomax-X and exponentiated log-logistic-X families.

1.6.8 Summary of Review

Although new distributions continue to appear in literature using the methods developed before the 1980, it is difficult to generate new distributions that are more flexible and simple enough for all practical needs. The drawbacks compelled researchers to shift from using these methods to the methods developed since the 1980s. However, among those methods proposed from the 1980 onwards, the T-X method provides greater flexibility for modifying distributions and also generalizing most of the methods.

1.7 Thesis Outline

The thesis consists of eight chapters including this one. Chapter 2 presents some important concepts used throughout the thesis. Chapter 3 presents the exponentiated generalized T-X family of distributions. Chapter 4 presents the exponentiated generalized exponential Dagum distribution. Chapter 5 presents the new exponentiated generalized modified inverse Rayleigh distribution. Chapter 6 presents the exponentiated generalized half logistic Burr X distribution. Chapter 7 presents the exponentiated generalized power series family of distributions. Finally, chapter 8 presents the conclusions and recommendations of the study.

CHAPTER 2

BASIC CONCEPTS AND METHODS

2.1 Introduction

This chapter presents the concepts on the methods that were used to achieve the objectives of the study. The topics discussed include the principle behind the maximum likelihood estimation, Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm, goodness-of-fit tests, information criteria and total time on test (TTT).

2.2 Maximum Likelihood Estimation

The maximum likelihood estimation method is the most widely used classical approach for estimating the parameters of a probability distribution model and is based on a likelihood function. The likelihood function attains its maximum at a specific value of the parameters. Suppose that X_1, $X_2, \ldots,$ X_n are independent and identically distributed random variables of size n with PDF $\mathrm{g}(x;\, \boldsymbol{\vartheta})$ where $\boldsymbol{\vartheta} = (\vartheta_1,\, \vartheta_2,\, \ldots,\, \vartheta_k)'$, $k < n$, is the vector of parameters that govern the PDF. The joint PDF can be written as

$$\mathrm{g}(x|\boldsymbol{\vartheta}) = \prod_{i=1}^{n} \mathrm{g}(x_i;\, \boldsymbol{\vartheta}). \tag{2.1}$$

When the random sample is collected, the joint PDF becomes a function of $\boldsymbol{\vartheta}$ and this function is called the likelihood function. The likelihood function is then defined as

$$L(\boldsymbol{\vartheta}|x) = \prod_{i=1}^{n} \mathrm{g}(x_i;\, \boldsymbol{\vartheta}). \tag{2.2}$$

Practically, it is more convenient to deal with the logarithm of the likelihood function, the log-likelihood function, denoted as ℓ and given by

$$\ell(\boldsymbol{\vartheta}|x_1,\, x_2,\, \ldots,\, x_n) = \sum_{i=1}^{n} \log \mathrm{g}(x_i;\, \boldsymbol{\vartheta}). \tag{2.3}$$

Since logarithm is a monontone function, when the likelihood function is maximized, the log-likelihood function is also maximized and vice versa. The estimates $\hat{\boldsymbol{\vartheta}}$ are the values of $\boldsymbol{\vartheta}$ that maximize the likelihood function. The likelihood equations are obtained by setting the first partial derivatives of ℓ with respect to $\vartheta_1,\, \vartheta_2,\, \ldots,\, \vartheta_k$ to zero; that are

$$\frac{\partial \ell(\boldsymbol{\vartheta}|x_1,\, x_2,\, \ldots,\, x_n)}{\partial \vartheta_i} = 0,\; i = 1,\, 2,\, \ldots,\, k. \tag{2.4}$$

Solving the system of likelihood equations in (2.4) for $\vartheta_1,\, \vartheta_2,\, \ldots,\, \vartheta_k$, the maximum likelihood estimates for the parameters are obtained.

2.2.1 Properties of Maximum Likelihood Estimators

The maximum likelihood estimators have some desirable properties under certain general conditions. In this subsection, those properties are explained.

2.2.1.1 Consistency

Suppose X_1, $X_2, \ldots$, X_n are independent identically distributed random sample from a population X with density $g(x, \boldsymbol{\vartheta})$. If $\hat{\boldsymbol{\vartheta}}$ is an estimator based on the sample size n, then it depends on the sample size n. To show the dependency of $\hat{\boldsymbol{\vartheta}}$ on n, $\hat{\boldsymbol{\vartheta}}$ is written as $\hat{\boldsymbol{\vartheta}}_n$. A sequence of estimators $\left\{\hat{\boldsymbol{\vartheta}}_n\right\}$ of $\boldsymbol{\vartheta}$ is consistent for $\boldsymbol{\vartheta}$ if and only if the sequence $\left\{\hat{\boldsymbol{\vartheta}}_n\right\}$ converges in probability to $\boldsymbol{\vartheta}$, that is , for any $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\hat{\boldsymbol{\vartheta}}_n - \boldsymbol{\vartheta}\right| \geq \epsilon\right) = 0. \tag{2.5}$$

It is worth noting that if the mean squared error goes to zero as n goes to infinity, then $\left\{\hat{\boldsymbol{\vartheta}}_n\right\}$ converges in probability to $\boldsymbol{\vartheta}$. Thus, if the variance of $\hat{\boldsymbol{\vartheta}}_n$ exist for each n and is finite, then

$$\lim_{n\to\infty} E\left[\left(\hat{\boldsymbol{\vartheta}}_n - \boldsymbol{\vartheta}\right)^2\right] = 0, \tag{2.6}$$

and for any $\epsilon > 0$, implies

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\hat{\boldsymbol{\vartheta}}_n - \boldsymbol{\vartheta}\right| \geq \epsilon\right) = 0.$$

The maximum likelihood estimators converge to the true parameter value as the sample size increases.

2.2.1.2 Asymptotic Normality

The distribution of the maximum likelihood estimators converges to a multivariate normal variate as the sample size increases. Hence,

$$\sqrt{n}(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}) \xrightarrow{\text{Dist}} N(\mathbf{0},\, I^{-1}(\boldsymbol{\vartheta})),$$

where $\mathbf{0}$ is a k-dimensional mean zero vector, $\xrightarrow{\text{Dist}}$ represents convergence in distribution and $I(\boldsymbol{\vartheta})$ is the $k \times k$ dimensional Fisher information matrix. The Fisher Information matrix is defined as the negative expected value of the second partial derivate matrix of the log-likelihood function evaluated at the true parameter $\boldsymbol{\vartheta}$. Thus,

$$I(\boldsymbol{\vartheta}) = -E\left[\frac{\partial^2 \mathrm{g}(x|\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'}\right] = -\int_{-\infty}^{\infty} \left[\frac{\partial^2 \mathrm{g}(x|\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'}\right] \mathrm{g}(x)\, dx. \tag{2.7}$$

The inverse of the Fisher information matrix yields the variance-covariance matrix of the parameters.

2.2.1.3 Asymptotic Efficiency

In practice, it is possible to get more than one consistent estimator in a class of unbiased estimators. Hence, the need to compare those estimators and select the one with the least variance. The estimator with the least variance in this class of unbiased estimators is referred to as the most efficient estimator. The maximum likelihood estimators are asymptotically most efficient. Mathematically, if there exist an alternative unbiased estimator $\bar{\boldsymbol{\vartheta}}$, such that

$$\sqrt{n}(\bar{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}) \xrightarrow{\text{Dist}} N(\mathbf{0},\, I^{-1}(\boldsymbol{\Omega})), \tag{2.8}$$

then $I^{-1}(\boldsymbol{\Omega})$ is greater than or equal to $I^{-1}(\boldsymbol{\vartheta})$ always.

2.2.1.4 Invariance Property

Suppose that $f(\boldsymbol{\vartheta})$ is a differentiable function, then the maximum likelihood estimator of $f(\boldsymbol{\vartheta})$ is equal to the function evaluated at the maximum likelihood estimator of $\boldsymbol{\vartheta}$. This means that if $\hat{\boldsymbol{\vartheta}}$ is the maximum likelihood estimator of $\boldsymbol{\vartheta}$, then $f(\hat{\boldsymbol{\vartheta}})$ is the maximum likelihood estimator of $f(\boldsymbol{\vartheta})$, and further

$$\sqrt{n}\left(f(\hat{\boldsymbol{\vartheta}}) - f(\boldsymbol{\vartheta})\right) \xrightarrow{\text{Dist}} N\left(\mathbf{0},\ \left[\frac{\partial f(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}\right] I^{-1}(\boldsymbol{\vartheta}) \left[\frac{\partial f(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}\right]'\right). \tag{2.9}$$

2.2.2 Confidence Intervals for Parameters

Suppose $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_k$ are the parameters of the distribution and $\Sigma_{11},\ \Sigma_{22},\ \ldots,\ \Sigma_{kk}$ are their corresponding variances. Making use of the multivariate normal approximation, the approximate $100(1-\eta)\%$ confidence intervals for the parameters are estimated as: $\gamma_1 \in \hat{\gamma}_1 \mp z_{\eta/2}\sqrt{\Sigma_{11}},\ \gamma_2 \in \hat{\gamma}_2 \mp z_{\eta/2}\sqrt{\Sigma_{22}}, \ldots, \gamma_k \in \hat{\gamma}_k \mp z_{\eta/2}\sqrt{\Sigma_{kk}}$, where $z_{\eta/2}$ is the upper ηth percentile of the standard normal distribution.

2.3 Broyden-Fletcher-Goldfarb-Shanno Algorithm

When the maximum likelihood estimators for the parameters have no closed form, the system of equations are solved using numerical techniques. This study employed the BFGS method to solve such system of equations. The algorithm for the BFGS is an iterative technique for solving unconstrained optimization problem and was independently developed by Broyden (1970), Fletcher (1970), Goldfarb (1970) and Shanno (1970). To optimize a given function (ℓ), the process start with an initial guess say $\boldsymbol{\vartheta}_0$ and an approximate Hessian matrix $\boldsymbol{H}_0$. The following steps are then repeated as $\boldsymbol{\vartheta}_i$ converges to the solution.

1. First get a direction $\boldsymbol{a}_i$ by solving

$$\boldsymbol{H}_i \boldsymbol{a}_i + \nabla \ell(\boldsymbol{\vartheta}_i) = 0.$$

2. A one dimensional optimization is then performed to look for an acceptable step size γ_i in the direction found in step 1.

3. Set $\boldsymbol{b}_i = \gamma_i \boldsymbol{a}_i$ and update $\boldsymbol{\vartheta}_{i+1} = \boldsymbol{\vartheta}_i + \boldsymbol{b}_i$.

4. Let $\boldsymbol{y}_i = \nabla \ell(\boldsymbol{\vartheta}_{i+1}) - \nabla \ell(\boldsymbol{\vartheta}_i)$.

5. $\boldsymbol{H}_{i+1} = \boldsymbol{H}_0 + \frac{\boldsymbol{y}_i \boldsymbol{y}_i^{'}}{\boldsymbol{y}_i^{'} \boldsymbol{b}_i} - \frac{\boldsymbol{H}_i \boldsymbol{b}_i \boldsymbol{b}_i^{'} \boldsymbol{H}_i}{\boldsymbol{b}_i^{'} \boldsymbol{H}_i \boldsymbol{b}_i}$.

The algorithm's convergence is checked by observing the norm of the gradient, $|\nabla \ell(\boldsymbol{\vartheta}_i)|$. In practice, $\boldsymbol{H}_0$ can be initialized with the identity matrix, $\boldsymbol{H}_0 = \boldsymbol{I}$, to make the first step equivalent to a gradient descent, but additional steps are refined by the approximation of the Hessian, $\boldsymbol{H}_i$. Step one of the algorithm is performed using the inverse of $\boldsymbol{H}_i$, which can be efficiently obtained by applying the Sherman-Morrison formula to the fifth step of the algorithm. Hence,

$$\boldsymbol{H}_{i+1}^{-1} = \left(\boldsymbol{I} - \frac{\boldsymbol{b}_i \boldsymbol{y}_i^{'}}{\boldsymbol{y}_i^{'} \boldsymbol{b}_i} \right) \boldsymbol{H}_i^{-1} \left(\boldsymbol{I} - \frac{\boldsymbol{y}_i \boldsymbol{b}_i^{'}}{\boldsymbol{y}_i^{'} \boldsymbol{b}_i} \right) + \frac{\boldsymbol{b}_i \boldsymbol{b}_i^{'}}{\boldsymbol{y}_i^{'} \boldsymbol{b}_i}. \tag{2.10}$$

Since, $\boldsymbol{H}_{i+1}^{-1}$ is symmetric and the terms $\boldsymbol{y}_i^{'} \boldsymbol{H}_i^{-1} \boldsymbol{y}_i$ and $\boldsymbol{b}_i^{'} \boldsymbol{y}_i$ are scalar, equation (2.10) can be estimated more efficiently using the expansion

$$\boldsymbol{H}_{i+1}^{-1} = \boldsymbol{H}_i^{-1} + \frac{(\boldsymbol{b}_i^{'} \boldsymbol{y}_i + \boldsymbol{y}_i^{'} \boldsymbol{H}_i^{-1} \boldsymbol{y}_i)(\boldsymbol{b}_i \boldsymbol{b}_i^{'})}{(\boldsymbol{b}_i^{'} \boldsymbol{y}_i)^2} - \frac{\boldsymbol{H}_i^{-1} \boldsymbol{y}_i \boldsymbol{b}_i^{'} + \boldsymbol{b}_i \boldsymbol{y}_i^{'} \boldsymbol{H}_i^{-1}}{\boldsymbol{b}_i^{'} \boldsymbol{y}_i}. \tag{2.11}$$

In a classical estimation problem such as the maximum likelihood, confidence interval for the parameters can easily be obtained by inverting the final Hessian matrix.

2.4 Goodness of Fit Tests

Let X_1, X_2, $\ldots$, X_n be a random sample from a given distribution, a goodness-of-fit test is a method used to examine whether the random sample came from a specified distribution. In this section, three goodness-of-fit tests used in the study were discussed. These are the likelihood ratio test (LRT), Kolmogorov-Smirnov (K-S) test and Cramér-von Misses test.

2.4.1 Likelihood Ratio Test

The LRT is used to assess how well a model fits a given data set. The test is used to compare two models that are nested. Suppose that the random variable X has a PDF given by g$(x; \vartheta)$ with unknown parameter ϑ. The main goal is to test the following null and alternative hypotheses; $H_0 : \vartheta \in \boldsymbol{\vartheta}_0$ and $H_1 : \vartheta \in \boldsymbol{\vartheta}_1$, where $\boldsymbol{\vartheta}_0$ and $\boldsymbol{\vartheta}_1$ are parameter spaces of the reduced and full model respectively. The test statistic for the test is given by

$$\omega = -2 \log \left(\frac{L_0(\hat{\vartheta})}{L_1(\hat{\vartheta})} \right), \tag{2.12}$$

where L_0 and L_1 are the likelihood functions for the reduced and full model respectively. Under H_0, ω is asymptotically distributed as a chi-square random variable with degrees of freedom equal to the difference between the number of parameters of the two models. When the null hypothesis is rejected, it implies that the full model provides a good fit to the data than the reduced model.

2.4.2 Kolmogorov-Smirnov Test

The K-S test is used for testing whether a given random sample $X_1, X_2, \ldots, X_n$ belong to a population with a specific distribution. The test statistic measures the distance between the empirical distribution function of the given sample and the estimated CDF of the candidate distribution. The null and alternative hypotheses for the test are; H_0 : The sample follows the specific distribution and H_1 : The sample does not follows the specific distribution. If $G(x_i)$ is the value of the CDF of the candidate distribution at x_i and $\hat{G}(x_i)$ is the value of the empirical distribution at x_i. The value of the K-S test statistic is define by

$$\mathrm{K-S} = \max\left\{\left|G(x_i) - \hat{G}(x_i)\right|, \left|G(x_i) - \hat{G}(x_{i-1})\right|\right\}, i = 1, 2, \ldots, n, \tag{2.13}$$

where

$$\hat{G}(x_i) = \frac{\sharp\left\{x_j : x_j \leq x_i\right\}}{n},$$

and $\sharp\{\cdot\}$ is the number of points less than or equal to x_i when x_i are ordered from the smallest to the largest value. The computed value of test statistic is then compared with a tabulated K-S value at a given significance level to decide whether or not to reject the null hypothesis. If there are more than one distribution to be compared, the distribution with the smaller K-S value is the most appropriate to fit the given sample.

2.4.3 Cramér-von Mises Test

The Cramér-von Misses test statistic, W*, is a test based on the empirical distribution. Suppose $G(x_i; \boldsymbol{\vartheta})$ is the CDF such that the form of G is known but the k-dimensional parameter vector $\boldsymbol{\vartheta}$ is unknown. The test statistic, W*, is obtained as follows:

1. First arrange the x_i's in ascending order and estimate $G(x, \hat{\boldsymbol{\vartheta}}) = u_i$.

2. Estimate $z_i = \Phi^{-1}(u_i)$, where $\Phi^{-1}(\cdot)$ is the quantile of the standard normal distribution and $\Phi(\cdot)$ is the CDF.

3. Compute $\mathrm{W}^2 = \sum_{i=1}^{n}(z_i - \frac{(2i-1)}{2n})^2 + \frac{1}{12n}$. Transform W^2 into $\mathrm{W}^* = \mathrm{W}^2(1 + \frac{0.5}{n})$ to obtain the test statistic.

When comparing models, the one with the smallest value of the test statistic W* is the best.

2.5 Information Criteria

The consequences of increasing the number of parameters, usually improves the fit of a given model and of course the likelihood also increases irrespective of whether the additional parameter is important or not. When the models to be compared are not nested, the LRT is not the best option and therefore one has to employ other methods to compare the models. The information criteria enable us to do this comparison when the models are not nested. The most widely used information criteria are; the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICc) and Bayesian Information Criterion (BIC).

2.5.1 Akaike Information Criterion

The AIC was first introduced by Akaike (1973) and developed further in Akaike (1974). It is the most widely employed model selection tool used by researchers. To apply AIC, one starts with some optional models, which are regarded as proper models for certain data. The test statistic is given by

$$\text{AIC} = -2\log L(\hat{\theta}) + 2k, \tag{2.14}$$

where k is the number of estimated parameters for the model. The best model for the data set is the one with the smallest value of AIC compared to others. One of the advantages of the AIC is that it has the ability to penalize models with many parameters. For large sample, the AIC introduces good model selection. However, there are issues of bias associated with the AIC. The AICc was therefore developed to overcome this problem (Sugiura, 1978). Hurvich and Tsai (1989) proved that the AICc improved model selections also in small samples. Also, when the model has a large number of parameters then the AICc is preferred. The test statistic of the AICc is given by

$$\text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}. \tag{2.15}$$

2.5.2 Bayesian Information Criterion

The BIC also known as Schwarz Information Criterion (SIC) in literature was developed by Schwarz (1978). The main idea of BIC comes from approximating the Bayes factor with the assumption that the data is independent and identically distributed. The test statistics for the BIC is given by

$$\text{BIC} = -2\log L(\hat{\theta}) + k\log(n), \tag{2.16}$$

where n is the sample size and $\log L(\hat{\theta})$ is the natural logarithm of the likelihood function. The BIC has the power to penalize models with many parameters compared to the AIC and AICc in both large and small samples. It is therefore important to use the BIC together with the AIC and AICc when selecting a best model among competing models. Like the AIC, the appropriate model is the one with the minimum BIC value compared to others.

2.6 Total Time on Test

Researchers are often interested in how the shape of hazard rate function of a given data set looks like. The TTT transformation, usually written as TTT-transform provides researchers a graphical way of viewing the shape of the hazard rate. The method was developed by Barlow and Doksum (1972) for statistical inference problems under order restrictions. The technique was employed by Aarset (1987) to check if a random sample is from a family of life distributions with bathtub shaped hazard rate. If G is the CDF of a distribution, then the TTT-transform is defined as

$$H^{-1}(p) = \int_0^{G^{-1}(p)} S(u)\,du,\; p \in [0,\, 1], \tag{2.17}$$

where $S(u) = 1 - G(u)$ is the survival function. The scaled TTT-transform is computed using

$$\varphi_G(p) = \frac{H^{-1}(p)}{H^{-1}(1)}. \tag{2.18}$$

The curve of $\varphi_G(p)$ versus $0 \leq p \leq 1$ is the scaled TTT-transform curve. According to Barlow and Doksum (1972), the shape of the hazard rate function can be classified as one of the following using the scaled TTT-transform curve:

1. The hazard rate function is said to be increasing if the scaled TTT-transform curve is concave above the 45^o line.

2. The hazard rate function is decreasing if the scaled TTT-transform curve is convex below the 45^o line.

3. The hazard rate function exhibits a bathtub shape if the scaled TTT-transform curve is first convex below the 45^o line and then concave above the line.

4. The hazard rate function is upside down bathtub or unimodal if the scaled TTT-transform curve is first concave above 45^o line and then convex below the 45^o line.

Given an ordered sample $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, the TTT test statistics are computed using

$$\text{TTT}_i = \sum_{j=1}^{i} (n - j + 1)(x_{j:n} - x_{j-1:n}),\ i = 1, 2, \ldots, n. \tag{2.19}$$

The empirical scaled TTT-transform is given by

$$\text{TTT}_i^* = \frac{\text{TTT}_i}{\text{TTT}_n}, \tag{2.20}$$

where $0 \leq \text{TTT}_n \leq 1$. The empirical scaled TTT-transform curve is obtained by plotting $\frac{i}{n}$ against TTT_i^*.

2.7 Summary

The chapter gave detailed explanation of the various techniques used in order to achieve the objectives of the study. The method of maximum likelihood estimation and its properties were discussed. The BFGS algorithm for optimizing the likelihood function was also discussed. Measures of goodness-of-fit such as the LRT, K-S and the Cramér-von Misses tests were all explained in this chapter. In addition, the information criteria for model selection such as the AIC, AICc and BIC were discussed. Finally, the TTT-transform for determining the nature of the hazard rate function of a given data was also explained.

CHAPTER 3

EXPONENTIATED GENERALIZED *T-X* FAMILY OF DISTRIBUTIONS

3.1 Introduction

This chapter presents a new generalization of the T-X family of distributions called the exponentiated generalized (EG) T-X family that extends the works of Alzaatreh et al. (2013) and Alzaghal et al. (2013) on the T-X family of distributions.

3.2 Exponentiated Generalized *T-X*

Let $r(t)$ and $R(t)$ be the PDF and CDF respectively of a non-negative random variable T with support $[0, \infty)$. Let $-\log[1-(1-\bar{F}^d(x))^c]$ be a function of the CDF $F(x)$ of any random variable X such that:

$$\begin{cases} -\log[1-(1-\bar{F}^d(x))^c] \in [0, \infty) \\ -\log[1-(1-\bar{F}^d(x))^c] \text{ is differentiable and monotonically non-decreasing} \\ -\log[1-(1-\bar{F}^d(x))^c] \to 0 \quad \text{as } x \to -\infty \text{ and} \\ -\log[1-(1-\bar{F}^d(x))^c] \to \infty \text{ as } x \to \infty \end{cases}, \quad (3.1)$$

where $\bar{F}(x) = 1 - F(x)$ is the survival function of the random variable X and $c > 0, d > 0$ are shape parameters. The conditions stated in equation (3.1) are important for the CDF of the EG T-X family of distributions to satisfy the basic properties of probability distribution. The CDF for the EG T-X family for a random variable X is defined as

$$G(x) = \int_0^{-\log[1-(1-\bar{F}^d(x))^c]} r(t)dt = R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\},\ c,\ d\ > 0,\ x \in \mathbb{R}. \quad (3.2)$$

By differentiating equation (3.2), the corresponding PDF of the new class is given by

$$\text{g}(x) = cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}}{1-(1-\bar{F}^d(x))^c} r\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}. \quad (3.3)$$

Proposition 3.1. The EG T-X PDF is a well defined density function.

Proof. It is worth mentioning that $\text{g}(x)$ is nonnegative. Now, it is important to show that the integration over the support of the random variable is one. Choosing the support of $\text{g}(x)$ to be $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} \text{g}(x)dx = \int_{-\infty}^{\infty} cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}}{1-(1-\bar{F}^d(x))^c} r\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\} dx.$$

Letting $t = -\log[1-(1-\bar{F}^d(x))^c]$, as $x \longrightarrow -\infty$, $t \longrightarrow 0$ and when $x \longrightarrow \infty$, $t \longrightarrow \infty$. In addition,

$$dx = \frac{1-(1-\bar{F}^d(x))^c\ dt}{c\,d\,f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}}.$$

Hence,

$$\int_{-\infty}^{\infty} \text{g}(x)dx = \int_0^{\infty} r(t)\,dt = 1.$$

This completes the proof.

Employing similar naming convention as "T-X distribution", each member of the new family of distributions generated from (3.3) is named EG T-X distribution. When the parameter $d = 1$, the PDF in (3.3) reduces to

$$\mathrm{g}(x) = \frac{cf(x)F^{c-1}(x)}{1 - F^c(x)} r\left\{-\log(1 - F^c(x))\right\}. \tag{3.4}$$

The density function in (3.4) is the PDF of the exponentiated T-X developed by Alzaghal et al. (2013). When $c = d = 1$, equation (3.3) reduces to

$$\mathrm{g}(x) = \frac{f(x)}{1 - F(x)} r\left\{-\log(1 - F(x))\right\}, \tag{3.5}$$

which is the PDF of the T-X distribution developed by Alzaatreh et al. (2013). The CDF and PDF of the EG T-X distribution can be written as $G(x) = R\left\{-\log[1 - (1 - \bar{F}^d(x))^c]\right\}$ $= R(H(x))$ and $\mathrm{g}(x) = h(x)r(H(x))$, where $H(x)$ and $h(x)$ are respectively the cumulative hazard and hazard functions of the random variable X with CDF $[1 - (1 - F(x))^d]^c$. Thus, the EG T-X distribution can be described as a family of distributions arising from a weighted hazard function.

The hazard rate function plays an important role in describing the failure rate of a phenomenon. It is the instantaneous rate at which events occur given no previous events (instantaneous failure rate). Mathematically, the hazard rate function is expressed as

$$\tau(x) = \lim_{\Delta x \longrightarrow 0} \frac{\mathbb{P}(x < X \leq x + \Delta x | X > x)}{\Delta x} = \frac{\mathrm{g}(x)}{1 - G(x)}. \tag{3.6}$$

Hence, the hazard rate function of the EG T-X family is given by

$$\tau(x) = cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}r\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}}{\left(1-(1-\bar{F}^d(x))^c\right)\left(1-R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}\right)}. \tag{3.7}$$

Lemma 3.1. Let T be a random variable with PDF $r(t)$, then the random variable $X = Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, where $Q_X(\cdot) = F^{-1}(\cdot)$ is the quantile function of the random variable X with CDF $F(x)$, follows the EG T-X distribution.

Proof. Using the fact that $G(x) = R\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}$ gives the relationship between the random variable T and X as $T = -\log[1-(1-\bar{F}^d(X))^c]$. Thus, solving for X yields $X = Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$.

Lemma 3.1 makes it easy to simulate the random observation of X by first generating random samples from the distribution of the random variable T and then computing $X = Q_X\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, which has the CDF $G(x)$.

3.3 EG Families for Different *T*-Distributions

The EG T-X Family can be categorized into two broad sub-families. One sub-family has the same T distribution but different X distributions and the other sub-family has different T distributions but the same X distribution. Table 3.1 displays different EG T-X distributions with different T distributions but the same X distribution.

Table 3.1: **EG *T-X* Families from Different T Distributions**

Name	Density $r(t)$	EG T-X Family density g(x)
Exponential	$\lambda e^{-\lambda t}$	$\frac{cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}}{[1-(1-\bar{F}^d(x))^c]^{1-\lambda}}$
Beta-exponential	$\frac{\lambda e^{-\lambda\beta t}(1-e^{-\lambda t})^{\alpha-1}}{\mathrm{B}(\alpha,\beta)}$	$\frac{cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}[1-(1-\bar{F}^d(x))^c]^{\lambda\beta-1}}{\mathrm{B}(\alpha,\beta)\left\{1-[1-(1-\bar{F}^d(x))^c]^{\lambda}\right\}^{1-\alpha}}$
Exponentiated exponential	$\alpha\lambda e^{-\lambda t}(1-e^{-\lambda t})^{\alpha-1}$	$\frac{cd\alpha\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda-1}}{\left\{1-\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda}\right\}^{1-\alpha}}$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}t^{\alpha-1}e^{-\frac{t}{\beta}}$	$cd\frac{f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\frac{1}{\beta}-1}}{\Gamma(\alpha)\beta^{\alpha}\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}^{1-\alpha}}$
Gompertz	$\theta e^{\gamma t}e^{-\frac{\theta}{\gamma}(e^{\gamma t}-1)}$	$\frac{cd\theta f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\exp\left(\frac{\theta}{\gamma}\left\{1-\left[1-(1-\bar{F}^d(x))^c\right]^{-\gamma}\right\}\right)}{\left\{1-\left[1-(1-\bar{F}^d(x))^c\right]\right\}^{\gamma+1}}$
Half logistic	$\frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2}$	$\frac{2cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda-1}}{\left\{1+\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda}\right\}^2}$
Lomax	$\frac{\lambda k}{(1+\lambda t)^{k+1}}$	$\frac{cd\lambda kf(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left\{1-\lambda\log[1-(1-\bar{F}^d(x))^c]\right\}^{-k-1}}{[1-(1-\bar{F}^d(x))^c]}$
Burr XII	$\frac{\alpha kt^{\alpha-1}}{(1+t^{\alpha})^{k+1}}$	$\frac{cd\alpha kf(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left\{-\log[1-(1-\bar{F}^d(x))^c]\right\}^{\alpha-1}}{[1-(1-\bar{F}^d(x))^c]\left\{1+\left[-\log\left(1-(1-\bar{F}^d(x))^c\right)\right]^{\alpha}\right\}^{k+1}}$
Weibull	$\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}e^{-\left(\frac{t}{\gamma}\right)^{\alpha}}$	$\frac{cd\alpha f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\exp\left\{-\left[-\log\left(1-(1-\bar{F}^d(x))^c\right)^{\frac{1}{\gamma}}\right]^{\alpha}\right\}}{\gamma[1-(1-\bar{F}^d(x))^c]\left\{-\log[1-(1-\bar{F}^d(X))^c]\right\}^{1-\alpha}}$

The EG T-X family houses several families of distributions. These include: EG gamma-X, EG Weibull-X, EG beta-exponential-X, exponentiated gamma-X, exponentiated Weibull-X, exponentiated beta-exponential-X, beta family, Kumaraswamy family and exponentiated family. Figure 3.1 shows several sub-families of the EG T-X family of distributions.

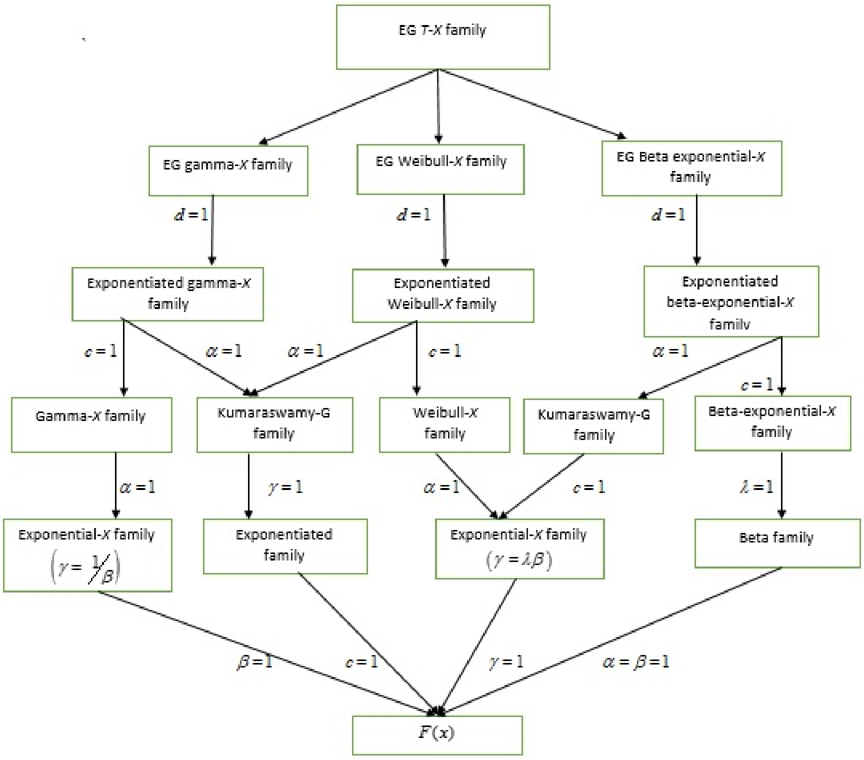

Figure 3.1: **Families of EG T-X distributions**

3.3.1 EG Half Logistic-X Family

If the random variable T follows the half logistic distribution with parameter λ, then $r(t) = \frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2}, t > 0, \lambda > 0$. Using (3.3), the PDF of the EG half logistic (EGHL)-X family is defined as

$$\mathrm{g(x)} = \frac{2cd\lambda f(x)(1-F(x))^{d-1}(1-\bar{F}^d(x))^{c-1}\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda-1}}{\left\{1+\left[1-(1-\bar{F}^d(x))^c\right]^{\lambda}\right\}^2}. \tag{3.8}$$

Using the CDF of the half logistic distribution, $R(t) = \frac{1-e^{-\lambda t}}{1+e^{-\lambda t}}$ and equation (3.2), the corresponding CDF of the EGHL-X family is given by

$$G(x) = \frac{1-\left[1-\left(1-\bar{F}^d(x)\right)^c\right]^\lambda}{1+\left[1-\left(1-\bar{F}^d(x)\right)^c\right]^\lambda}.$$

The EGHL-X family generalizes all half logistic families of Alzaghal et al. (2013) exponentiated T-X family and Alzaatreh et al. (2013) T-X family. If the random variable X follows a Frétchet distribution with CDF $F(x) = e^{-\left(\frac{a}{x}\right)^b}, x > 0, a > 0, b > 0$, then the CDF of the EGHL-Frétchet (EGHLF) distribution is given by

$$G(x) = \frac{1-\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^d\right]^c\right\}^\lambda}{1+\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^d\right]^c\right\}^\lambda}. \tag{3.9}$$

The corresponding PDF of the EGHLF distribution is obtained by differentiating (3.9) and is given by

$$\text{g(x)} = \frac{2a^b bcd\lambda\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^{d-1}\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^d\right]^{c-1}\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^d\right]^c\right\}^{\lambda-1}}{x^{b+1}e^{\left(\frac{a}{x}\right)^b}\left\{1+\left\{1-\left[1-\left(1-e^{-\left(\frac{a}{x}\right)^b}\right)^d\right]^c\right\}^\lambda\right\}^2}. \tag{3.10}$$

Some special cases of the EGHLF distribution are:

1. When $\lambda = 1$, the EGHLF distribution reduces to EG standardized half logistic Frétchet distribution.

2. When $b = 1$, the EGHLF distribution reduces to EGHL inverse exponential distribution.

3. When $c = d = 1$, the EGHLF distribution reduces to half logistic Frétchet distri-

bution.

4. When $c = d = b = 1$, the EGHLF distribution reduces to half logistic inverse exponential distribution.

Figure 3.2 displays the density plots of the EGHLF distribution for different parameter values. From Figure 3.2, it can be seen that the density of the EGHLF distribution exhibit unimodal shapes with small and large values of skewness and kurtosis measure.

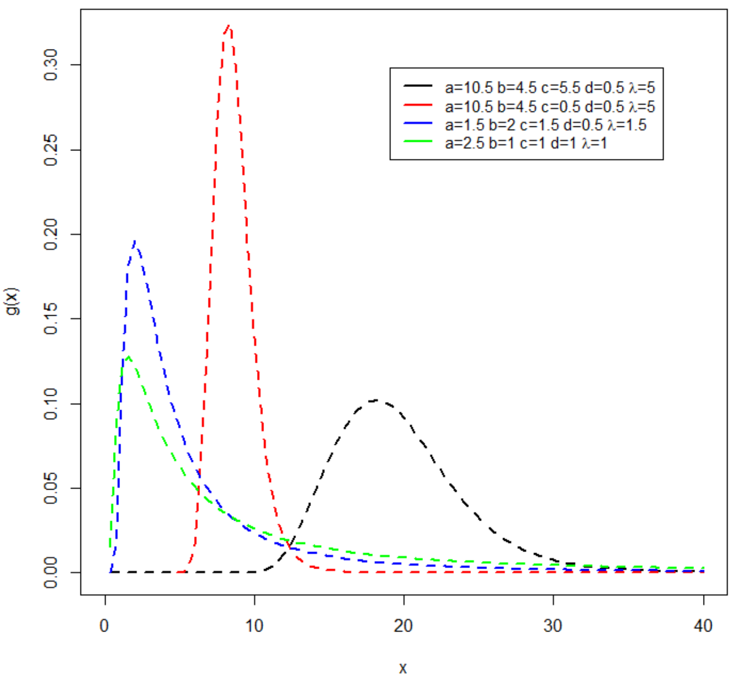

Figure 3.2: **Density function plots of EGHLF distribution for some parameter values**

3.4 Statistical Properties of EG *T-X* Family

When new families of distributions are developed it is often customary to establish some of the statistical properties of these new families. In this section, the quantile function, moments, moment generating function (MGF) and Shannon entropy of the EG *T-X* family of distributions were derived.

3.4.1 Quantile Function

The quantile function plays a key role in simulating random samples from a given distribution. The characteristics of a distribution such as the median, kurtosis and skewness can also be described using the quantile function.

Lemma 3.2. The quantile function of the EG T-X family for $p \in (0,1)$ is given by $Q(p) = Q_X \left\{1 - \left[1 - \left(1 - e^{-Q_T(p)}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, where $Q_X(\cdot) = F^{-1}(\cdot)$ is the quantile function of the random variable X with CDF $F(x)$ and $Q_T(\cdot) = R^{-1}(\cdot)$ is the quantile function of the random variable T with CDF $R(t)$.

Proof. Using the CDF of the EG T-X family defined in equation (3.2), the quantile function is obtained by solving the equation

$$R\left\{-\log\left[1 - \left(1 - \bar{F}^d(Q(p))\right)^c\right]\right\} = p,$$

for $Q(p)$. Thus, the proof is complete.

The median of the EG T-X family is obtained by substituting $p = 0.5$ into Lemma 3.2.

Corollary 3.1. Based on Lemma 3.2, the quantile function for EGHL-X family is given by, $Q(p) = Q_X \left\{1 - \left[1 - \left(1 - \left(\frac{1-p}{1+p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$.

3.4.2 Moments

Moments are essential in any statistical analysis, especially in applications. They are used for finding measures of central tendency, variation, kurtosis and skewness among others. The following proposition gives the r^{th} non-central moment of the EG T-X family.

Proposition 3.2. The r^{th} non-central moment of the EG T-X family of distributions is given by

$$\mu_r^{'} = \sum_{i,\,k,\,l,\,m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, l^m\, \delta_{r,i}\, \Gamma\,(i+1)\,\Gamma\left(\frac{j}{d}+1\right)\Gamma\left(\frac{k}{c}+1\right)}{j!\,k!\,l!\,m!\,\Gamma\,(i-j+1)\,\Gamma\left(\frac{j}{d}-k+1\right)\Gamma\left(\frac{k}{c}-l+1\right)} E\left(T^m\right), \tag{3.11}$$

where $\delta_{r,i} = (ih_0)^{-1} \sum_{s=1}^{i} \left[s\,(r+1) - i\right] h_s\, \delta_{r,i-s}$ with $\delta_{r,0} = h_0^r$, $h_i\,(i = 0, 1, \ldots)$ are suitably chosen real numbers that depend on the parameters of the $F(x)$ distribution, $E(T^m)$ is the m^{th} moment of the random variable T, $\Gamma(\cdot)$ is the gamma function and $r = 1, 2, \ldots$.

Proof. From Lemma 3.1, $X = Q_X\left\{1 - \left[1 - \left(1 - e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$, where $Q_X(\cdot) = F^{-1}(\cdot)$ is a quantile function. Thus, $Q_X(\cdot) = F^{-1}(\cdot)$ can be expressed in terms of power series using the following power series expansion of the quantile.

$$Q_X(u) = \sum_{i=0}^{\infty} h_i u^i, \tag{3.12}$$

where the coefficients are suitably chosen real numbers that depend on the parameters of the $F(x)$ distribution. For a power series raised to a positive integer r (for $r \geq 1$),

$$\left(Q_X(u)\right)^r = \left(\sum_{i=0}^{\infty} h_i u^i\right)^r = \sum_{i=0}^{\infty} \delta_{r,\,i}\, u^i, \tag{3.13}$$

where the coefficients $\delta_{r,\,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation $\delta_{r,i} = (ih_0)^{-1} \sum_{s=1}^{i} \left[s\,(r+1) - i\right] h_s\, \delta_{r,i-s}$ and $\delta_{r,0} = h_0^r$ (Gradshteyn and Ryzhik, 2007). Using equation (3.13), the r^{th} non-central moment of the EG T-X family of distributions can be expressed as

$$E\left(X^r\right) = \mu_r^{'} = E\left\{\sum_{i=0}^{\infty} \delta_{r,\,i} \left[1 - \left(1 - \left(1 - e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]^i\right\}. \tag{3.14}$$

Since $0 < \left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}} < 1$, for $T \in [0,\infty)$, applying the binomial series expansion

$$(1-z)^{\eta} = \sum_{j=0}^{\infty} \frac{(-1)^{j}\,\Gamma(\eta+1)}{j!\,\Gamma(\eta-j+1)} z^{\eta}, |\, z \,|< 1,$$

thrice,

$$\left[1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]^{i} = \sum_{k,\,l=0}^{\infty}\sum_{j=0}^{i} \frac{(-1)^{j+k+l}\,\Gamma(i+1)\,\Gamma\left(\frac{j}{d}+1\right)\Gamma\left(\frac{k}{c}+1\right)e^{-lT}}{j!\,k!\,l!\,\Gamma(i-j+1)\,\Gamma\left(\frac{j}{d}-k+1\right)\Gamma\left(\frac{k}{c}-l+1\right)}.$$

But the series expansion of e^{-lT} is given by

$$e^{-lT} = \sum_{m=0}^{\infty} \frac{(-1)^{m}\,l^{m}\,T^{m}}{m!}.$$

Thus,

$$\left[1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]^{i} = \sum_{k,\,l\,m=0}^{\infty}\sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\,\Gamma(i+1)\,\Gamma\left(\frac{j}{d}+1\right)\Gamma\left(\frac{k}{c}+1\right)l^{m}T^{m}}{j!\,k!\,l!\,m!\,\Gamma(i-j+1)\,\Gamma\left(\frac{j}{d}-k+1\right)\Gamma\left(\frac{k}{c}-l+1\right)}. \tag{3.15}$$

Substituting equation (3.15) into (3.14) and simplifying, yields

$$\mu_r^{'} = \sum_{i,\,k,\,l,\,m=0}^{\infty}\sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\,l^{m}\,\delta_{r,i}\,\Gamma(i+1)\,\Gamma\left(\frac{j}{d}+1\right)\Gamma\left(\frac{k}{c}+1\right)}{j!\,k!\,l!\,m!\,\Gamma(i-j+1)\,\Gamma\left(\frac{j}{d}-k+1\right)\Gamma\left(\frac{k}{c}-l+1\right)} E\left(T^{m}\right).$$

Using the m^{th} moment of the half logistic distribution, the r^{th} moment of the EGHL-X family is given by Corollary 3.2.

Corollary 3.2. Using Proposition 3.2, the r^{th} moment of the EGHL-X family is

$$\mu_r^{'} = \sum_{i,\,k,\,l,\,m=0}^{\infty}\sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\,l^{m}\,\delta_{r,i}\,\Gamma(i+1)\,\Gamma\left(\frac{j}{d}+1\right)\Gamma\left(\frac{k}{c}+1\right)}{j!\,k!\,l!\,m!\,\Gamma(i-j+1)\,\Gamma\left(\frac{j}{d}-k+1\right)\Gamma\left(\frac{k}{c}-l+1\right)} \left\{2\sum_{n=0}^{\infty} \frac{(-1)^{n}\,\Gamma(m+1)}{\lambda^{m}\,(n+1)^{m}}\right\},$$

for $m = 1, 2 \ldots$.

3.4.3 Moment Generating Function

The MGF of the EG T-X family of distributions is given by the following proposition.

Proposition 3.3. The MGF of the EG T-X family of distributions is given by

$$M_X(z) = \sum_{r,i,k,l,m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m} z^r l^m \delta_{r,i} \Gamma(i+1) \Gamma\left(\frac{j}{d}+1\right) \Gamma\left(\frac{k}{c}+1\right)}{r!\, j!\, k!\, l!\, m!\, \Gamma(i-j+1) \Gamma\left(\frac{j}{d}-k+1\right) \Gamma\left(\frac{k}{c}-l+1\right)} E(T^m). \tag{3.16}$$

Proof. By definition, the MGF is given by

$$M_X(z) = E\left(e^{zX}\right).$$

Using the series expansion of e^{zX}, gives

$$M_X(z) = \sum_{r=0}^{\infty} \frac{z^r \mu_r^{'}}{r!}. \tag{3.17}$$

Substituting $\mu_r^{'}$ into equation (3.17), yields

$$M_X(z) = \sum_{r,i,k,l,m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m} z^r l^m \delta_{r,i} \Gamma(i+1) \Gamma\left(\frac{j}{d}+1\right) \Gamma\left(\frac{k}{c}+1\right)}{r!\, j!\, k!\, l!\, m!\, \Gamma(i-j+1) \Gamma\left(\frac{j}{d}-k+1\right) \Gamma\left(\frac{k}{c}-l+1\right)} E(T^m),$$

which is the MGF.

Corollary 3.3. Based on Proposition 3.3, the MGF of the EGHL-X family is

$$M_X(z) = \sum_{r,\, i,\, k,\, l,\, m=0}^{\infty} \sum_{j=0}^{i} \frac{(-1)^{j+k+l+m}\, z^r\, l^m\, \delta_{r,i}\, \Gamma\left(i+1\right) \Gamma\left(\frac{j}{d}+1\right) \Gamma\left(\frac{k}{c}+1\right)}{r!\, j!\, k!\, l!\, m!\, \Gamma\left(i-j+1\right) \Gamma\left(\frac{j}{d}-k+1\right) \Gamma\left(\frac{k}{c}-l+1\right)} \times \left\{ 2 \sum_{n=0}^{\infty} \frac{(-1)^n\, \Gamma\left(m+1\right)}{\lambda^m\, (n+1)^m} \right\}.$$

3.4.4 Entropy

Entropy is a measure of variation or uncertainty of a random variable. Entropy has been used extensively in several fields such as engineering and information theory. According to Shannon (1948), the entropy of a random variable X with PDF g(x) is given by $\eta_X = -E\left\{\log\left(\mathrm{g}(X)\right)\right\}$.

Proposition 3.4. The Shannon entropy for the EG T-X family of distributions is given by

$$\eta_X = -\log\left(cd\right) - \mu_T + \eta_T - E\left\{\log f\left[F^{-1}\left(1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)\right]\right\} + \left(\frac{1-d}{d}\right) E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)\right] + \left(\frac{1-c}{c}\right) E\left[\log\left(1-e^{-T}\right)\right], \tag{3.18}$$

where μ_T and η_T are the mean and the Shannon entropy of the random variable T.

Proof. By definition

$$\eta_X = (1-c)E\left\{\log\left[1-\left(1-F(X)\right)^d\right]\right\} + (d-1)E\left[\log\left(1-F(X)\right)\right] - \log\left(cd\right) - E\left[\log f(X)\right] + E\left\{\log\left[1-\left(1-\left(1-F(X)\right)^d\right)^c\right]\right\} - E\left\{\log\left[r\left(-\log\left(1-\left(1-\bar{F}^d(x)\right)^c\right)\right)\right]\right\}. \tag{3.19}$$

From Lemma 3.1, $T = -\log[1-(1-\bar{F}^d(X))^c]$ and $X = F^{-1}\left\{1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}$.

Hence,

$$E\left[\log f(X)\right] = E\left\{\log f\left[F^{-1}\left(1-\left[1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right)\right]\right\}, \tag{3.20}$$

$$E\left[\log\left(1-F(X)\right)\right] = E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right], \tag{3.21}$$

$$E\left\{\log\left[1-\left(1-F(X)\right)^d\right]\right\} = E\left[\log\left(1-e^{-T}\right)^{\frac{1}{c}}\right], \tag{3.22}$$

$$E\left\{\log\left[1-\left(1-\left(1-F(X)\right)^{\frac{1}{d}}\right)^{\frac{1}{c}}\right]\right\} = E\left(-T\right), \tag{3.23}$$

and

$$E\left\{\log\left[r\left(-\log\left(1-\left(1-\bar{F}^d(x)\right)^c\right)\right)\right]\right\} = E\left[\log r\left(T\right)\right]. \tag{3.24}$$

Substituting (3.20) through (3.24) into (3.19) yields

$$\begin{aligned}\eta_X = -\log\left(cd\right) - \mu_T + \eta_T - E\left\{\log f\left[F^{-1}\left(1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)\right]\right\} + \\ \left(\frac{1-d}{d}\right) E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)\right] + \left(\frac{1-c}{c}\right) E\left[\log\left(1-e^{-T}\right)\right].\end{aligned}$$

Substituting the mean and the Shannon entropy of the half logistic distribution into (3.18), gives the Shannon entropy of the EGHL-X family.

Corollary 3.4. The Shannon entropy of the EGHL-X family is

$$\begin{aligned}\eta_X = 2 - \log\left(2cd\lambda\right) - \frac{2\log(2)}{\lambda} - E\left\{\log f\left[F^{-1}\left(1-\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)\right]\right\} + \\ \left(\frac{1-d}{d}\right) E\left[\log\left(1-\left(1-e^{-T}\right)^{\frac{1}{c}}\right)\right] + \left(\frac{1-c}{c}\right) E\left[\log\left(1-e^{-T}\right)\right].\end{aligned}$$

The mean of the half logistic distribution is $\mu_T = \frac{2\log(2)}{\lambda}$ and the Shannon entropy is $\eta_T = 2 - \log(2\lambda)$.

3.5 Summary

In this chapter, the EG T-X family which is an extension of the T-X and the exponentiated T-X families of distributions was proposed. The new family contains several generalized classes of distributions as shown in Figure 3.1. The extra shape parameters c and d provide greater flexibility for controlling skewness, kurtosis and possibly adding entropy to the center of the EG T-X density function. Specific example of this new family, namely EGHLF distribution was given and its relationship with other baseline distributions established. Some statistical properties of the family such as the quantile, moments, MGF, and Shannon entropy were derived.

CHAPTER 4

EXPONENTIATED GENERALIZED EXPONENTIAL DAGUM DISTRIBUTION

4.1 Introduction

The development of generalized classes of distributions have received much attention in recent times. This requires the use of different transformation techniques to modify existing statistical distributions to make them more flexible. The Dagum distribution (Dagum, 1977), like other existing statistical distributions have been modified using some of these methods. In this chapter, the CDF of the EG exponential (EGE)-X family was defined and used to generalize the Dagum distribution.

4.2 Generalized Exponential Dagum

Let T be a random variable with PDF $\lambda e^{-\lambda t}, t > 0, \lambda > 0$ and let X be a continuous random variable with CDF $F(x)$. The CDF of the EGE-X family of distributions is defined as

$$G(x) = \int_0^{-\log\left[1-\left(1-\bar{F}^d(x)\right)^c\right]} \lambda e^{-\lambda t} dt = 1 - \left\{1 - \left[1 - (1 - F(x))^d\right]^c\right\}^\lambda, \qquad (4.1)$$

where $\bar{F}(x) = 1 - F(x)$.

For positive integers λ and c, a physical interpretation of the EGE-X family of distributions CDF is given as follows. Equation (4.1) represents the CDF of the lifetime of a series-parallel system consisting of independent components with the CDF $1-(1-F(x))^d$ corresponding to the Lehman type II distribution. Given that a system is formed by λ independent component series subsystems and that each of the subsystems is made up of c independent parallel components. Suppose $X_{ij} \sim 1-(1-F(x))^d$, for $1 \leq i \leq c$ and $1 \leq j \leq \lambda$, represents the lifetime of the i^{th} component in the j^{th} subsystem and X is the lifetime of the entire system. Then,

$$\begin{aligned} \mathbb{P}(X \leq x) &= 1 - \left[1 - \mathbb{P}(X_{11} \leq x, \, \ldots, \, X_{1c} \leq x)\right]^{\lambda} \\ &= 1 - \left[1 - \mathbb{P}^{c}(X_{11} \leq x)\right]^{\lambda}, \end{aligned}$$

and X has the CDF defined in equation (4.1).

Suppose the random variable X follows a type I Dagum distribution with CDF $(1+\alpha x^{-\theta})^{-\beta}, x > 0, \alpha > 0, \beta > 0, \theta > 0$, then the CDF of the EGE-Dagum (EGED) distribution is given by

$$G(x) = 1 - \left\{1 - \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c}\right\}^{\lambda}, \, x > 0, \tag{4.2}$$

where the parameters α, β, θ, λ, c and d are non-negative, with β, θ, λ, c and d being shape parameters and α being a scale parameter. The corresponding PDF of the EGED distribution is given by

$$\mathrm{g}(x) = \frac{\alpha\beta\lambda\theta c d x^{-\theta-1} K \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c-1}}{\left\{1 - \left[1 - \left(1 - (1 + \alpha x^{-\theta})^{-\beta}\right)^{d}\right]^{c}\right\}^{1-\lambda}}, \, x > 0, \tag{4.3}$$

where

$$K = \left(1+\alpha x^{-\theta}\right)^{-\beta-1}\left(1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right)^{d-1}.$$

Lemma 4.1. The PDF of the EGED distribution can be expressed in terms of the density function of the Dagum distribution as

$$\text{g}(x) = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\omega_{ijk}\, f_D(x;\, \alpha,\, \theta,\, \beta_{k+1}),\ x > 0, \tag{4.4}$$

where $f_D(x;\, \alpha,\, \theta,\, \beta_{k+1})$ is the PDF of the Dagum distribution with parameters α, θ and $\beta_{k+1} = \beta(k+1)$ and

$$\omega_{ijk} = \frac{(-1)^{i+j+k}\,\Gamma(\lambda)\,\Gamma(c(i+1))\,\Gamma(d(j+1))}{i!\,j!\,(k+1)!\,\Gamma(\lambda-i)\,\Gamma(c(i+1)-j)\,\Gamma(d(j+1)-k)},\ \Gamma(a+1) = a!.$$

Proof. For a real non-integer $\eta > 0$, a series expansion for $(1-z)^{\eta-1}$, for $|z| < 1$ is

$$(1-z)^{\eta-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(\eta)}{i!\,\Gamma(\eta-i)}z^i. \tag{4.5}$$

Applying the series expansion in equation (4.5) twice and the fact that $0 < (1+\alpha x^{-\theta})^{-\beta} < 1$, implies that

$$\left[1-\left(1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c-1}\left\{1-\left[1-\left(1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c}\right\}^{\lambda-1} =$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\Gamma(\lambda)\,\Gamma(c(i+1))\left(1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right)^{dj}}{i!\,j!\,\Gamma(\lambda-i)\,\Gamma(c(i+1)-j)}. \tag{4.6}$$

Substituting equation (4.6) into equation (4.3) yields

$$\mathrm{g}(x) = \lambda c d \alpha \beta \theta x^{-\theta-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}\, \Gamma(\lambda)\, \Gamma(c(i+1))\, K^{*}}{i!\, j!\, \Gamma(\lambda - i)\, \Gamma(c(i+1) - j)},$$

where

$$K^{*} = \left(1 + \alpha x^{-\theta}\right)^{-\beta-1} \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d(j+1)-1}.$$

Applying the series expansion again to $\left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d(j+1)-1}$ gives the expansion of the density as

$$\mathrm{g}(x) = \lambda c d \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \omega_{ijk}\, f_D(x;\, \alpha,\, \theta,\, \beta_{k+1}),\; x > 0.$$

Equation (4.4) revealed that the PDF of the EGED distribution can be written as a linear combination of the Dagum distribution with different shape parameters. The expansion of the PDF is important in providing the statistical properties of the EGED distribution. The PDF of EGED distribution can be symmetric, left skewed, right skewed, reversed J-shape or unimodal with small and large values of skewness and kurtosis for different parameter values. Figure (4.1) displays the different shapes of the EGED distribution density function.

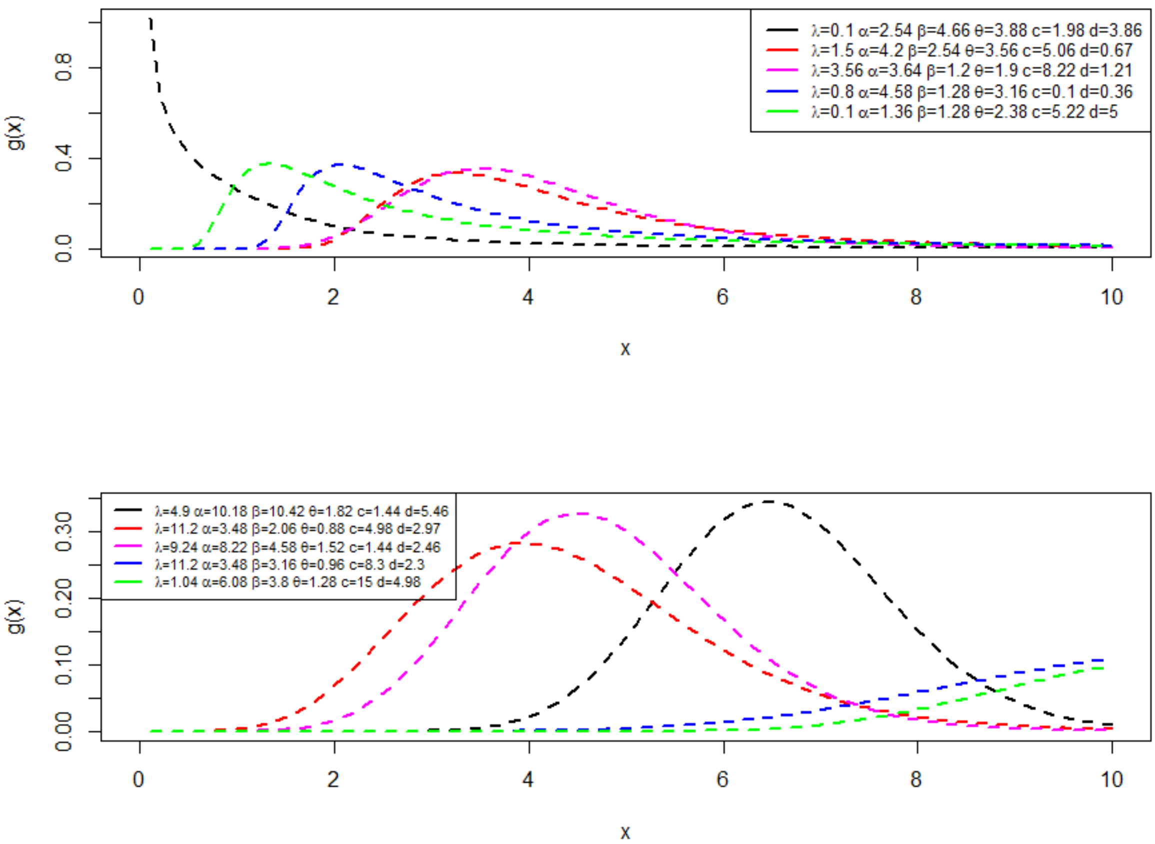

Figure 4.1: **EGED density function for some parameter values**

The survival function of this distribution is

$$S(x) = \left\{1 - \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c}\right\}^{\lambda},\ x > 0, \tag{4.7}$$

and the hazard function is

$$\tau(x) = \frac{\alpha\beta\lambda\theta c d x^{-\theta-1} K \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c-1}}{\left\{1 - \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c}\right\}},\ x > 0. \tag{4.8}$$

The plots of the hazard function displays various attractive shapes such as monotonically decreasing, monotonically increasing, upside down bathtub, bathtub and bathtub followed by upside down bathtub shapes for different combination of the values of the parameters. These features makes the EGED distribution suitable for modeling monotonic and non-monotonic failure rates that are more likely to be encountered in real life situation. Figure (4.2) displays the various shapes of the hazard function.

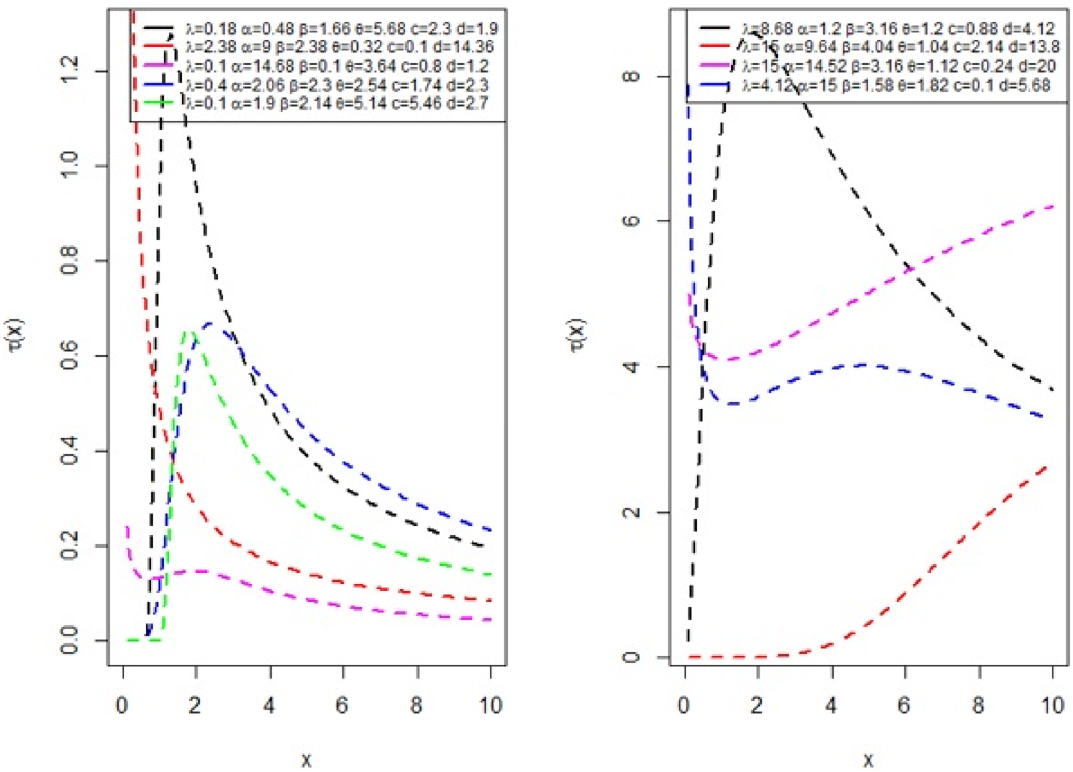

Figure 4.2: **Plots of the EGED hazard function for some parameter values**

4.3 Sub-models

The EGED distribution consists of a number of important sub-models that are widely used in lifetime modeling. These include:

1. **The Exponentiated Generalized Dagum Distribution**

   When $\lambda = 1$, the EGED reduces to the exponentiated generalized Dagum distribution (EGDD) with the following CDF:

$$G(x) = \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c},$$

for α, β, θ, d, $c > 0$ and $x > 0$.

2. **The Dagum Distribution**

When $\lambda = c = d = 1$, the EGED reduces to the Dagum distribution (DD) with the following CDF:

$$G(x) = \left(1 + \alpha x^{-\theta}\right)^{-\beta},$$

for α, β, θ, > 0 and $x > 0$.

3. **The Exponentiated Generalized Exponential Burr III Distribution**

When $\alpha = 1$, the EGED reduces to the exponentiated generalized exponential Burr III distribution (EGEBD) with the following CDF:

$$G(x) = 1 - \left\{1 - \left[1 - \left(1 - \left(1 + x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c}\right\}^{\lambda},$$

for λ, β, θ, d, $c > 0$ and $x > 0$.

4. **The Burr III Distribution**

When $\alpha = \lambda = c = d = 1$, the EGED reduces to the Burr III distribution (BD) with the following CDF:

$$G(x) = \left(1 + x^{-\theta}\right)^{-\beta},$$

for β, θ, > 0 and $x > 0$.

5. **The Exponentiated Generalized Burr III Distribution**

When $\alpha = \lambda = 1$, the EGED reduces to the exponentiated generalized Burr III distribution (EGBD) with the following CDF:

$$G(x) = \left[1 - \left(1 - \left(1 + x^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c},$$

for β, θ, d, c > 0 and x > 0.

6. **The Exponentiated Generalized Exponential Fisk Distribution**

When $\beta = 1$, the EGED reduces to the exponentiated generalized exponential Fisk distribution (EGEFD) with the following CDF:

$$G(x) = 1 - \left\{1 - \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-1}\right)^{d}\right]^{c}\right\}^{\lambda},$$

for α, λ, θ, d, c > 0 and x > 0.

7. **The Exponentiated Generalized Fisk Distribution**

When $\lambda = \beta = 1$, the EGED reduces to the exponentiated generalized Fisk distribution (EGFD) with the following CDF:

$$G(x) = \left[1 - \left(1 - \left(1 + \alpha x^{-\theta}\right)^{-1}\right)^{d}\right]^{c},$$

for α, θ, d, c > 0 and x > 0.

8. **The Fisk Distribution**

When $\lambda = \beta = c = d = 1$, the EGED reduces to the Fisk distribution (FD) with the following CDF:

$$G(x) = \left(1 + \alpha x^{-\theta}\right)^{-1},$$

for α, θ, > 0 and x > 0.

Table 4.1 displays a list of models that can be derived from the EGED distribution.

Table 4.1: **Summary of sub-models from the EGED distribution**

Distribution	α	λ	β	θ	c	d
EGDD	α	1	β	θ	c	d
DD	α	1	β	θ	1	1
EGEBD	1	λ	β	θ	c	d
BD	1	1	β	θ	1	1
EGBD	1	1	β	θ	c	d
EGEFD	α	λ	1	θ	c	d
EGFD	α	1	1	θ	c	d
FD	α	1	1	θ	1	1

4.4 Statistical Properties

In this section, various statistical properties of the EGED distribution such as the quantile, moments, MGF, incomplete moment, mean deviation, median deviation, inequality measures, reliability measure, entropy and order statistics were derived.

4.4.1 Quantile Function

The distribution of a random variable can be described using its quantile function. The quantile function is useful in computing the median, kurtosis and skewness of the distribution of a random variable.

Lemma 4.2. The quantile function of the EGED distribution for $p \in (0, 1)$ is given by

$$Q_X(p) = \left\{ \frac{1}{\alpha} \left[\left(1 - \left(1 - \left(1 - (1-p)^{\frac{1}{\lambda}} \right)^{\frac{1}{c}} \right)^{\frac{1}{d}} \right)^{-\frac{1}{\beta}} - 1 \right] \right\}^{-\frac{1}{\theta}}. \tag{4.9}$$

Proof. By definition, the quantile function returns the value x such that

$$G(x_p) = \mathbb{P}(X \leq x_p) = p.$$

Thus,

$$1 - \left\{1 - \left[1 - \left(1 - \left(1 + \alpha x_p^{-\theta}\right)^{-\beta}\right)^{d}\right]^{c}\right\}^{\lambda} = p. \tag{4.10}$$

Letting $x_p = Q_X(p)$ in equation (4.10) and solving for $Q_X(p)$ using inverse transformation yields

$$Q_X(p) = \left\{\frac{1}{\alpha}\left[\left(1 - \left(1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)^{-\frac{1}{\beta}} - 1\right]\right\}^{-\frac{1}{\theta}}.$$

When $p = 0.25$, 0.5 and 0.75, we obtain the first quartile, the median and the third quartile of the EGED distribution respectively. The closed form expression of the quantile makes it easy to simulate EGED random observations using the relation

$$x_p = \left\{\frac{1}{\alpha}\left[\left(1 - \left(1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right)^{-\frac{1}{\beta}} - 1\right]\right\}^{-\frac{1}{\theta}}.$$

4.4.2 Moments

It is imperative to derive the moments when a new distribution is proposed. They play a significant role in statistical analysis, particularly in applications. Moments are used in computing measures of central tendency, dispersion and shapes among others.

Proposition 4.1. The r^{th} non-central moment of the EGED distribution is given by

$$\mu_r^{'} = \lambda c d \alpha^{\frac{r}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left(\beta_{k+1} + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right), r < \theta, \tag{4.11}$$

where $B(\cdot,\cdot)$ is the beta function and $r = 1, 2, \ldots$.

Proof. By definition

$$\begin{aligned}
\mu_r^{'} &= \int_0^\infty x^r \mathrm{g}(x)dx \\
&= \int_0^\infty x^r \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \omega_{ijk}\, f_D(x;\, \alpha,\, \theta,\, \beta_{k+1})dx \\
&= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \omega_{ijk} \int_0^\infty x^r f_D(x;\, \alpha,\, \theta,\, \beta_{k+1})dx,
\end{aligned}$$

where $f_D(x;\, \alpha,\, \theta,\, \beta_{k+1}) = \alpha\theta\beta_{k+1}x^{-\theta-1}(1+\alpha x^{-\theta})^{-\beta_{k+1}-1}$. Letting $y = (1+\alpha x^{-\theta})^{-1}$ implies that if $x \to 0$, $y \to 0$ and if $x \to \infty$, $y \to 1$. Also, $dy = \alpha\theta x^{-\theta-1}(1+\alpha x^{-\theta})^{-2}dx$ and $x = (\alpha y)^{\frac{1}{\theta}}(1-y)^{-\frac{1}{\theta}}$. Thus,

$$\begin{aligned}
\mu_r^{'} &= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \omega_{ijk}\, \beta_{k+1} \int_0^1 \alpha^{\frac{r}{\theta}} y^{\beta_{k+1}+\frac{r}{\theta}-1} \,(1-y)^{(1-\frac{r}{\theta})-1}\, dy \\
&= \lambda cd\alpha^{\frac{r}{\theta}} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \omega_{ijk}\, \beta_{k+1}\, B\left(\beta_{k+1}+\frac{r}{\theta}, 1-\frac{r}{\theta}\right),\ r < \theta,
\end{aligned}$$

where $B(a,b) = \int_0^1 y^{a-1}(1-y)^{b-1}dy$.

The values for the first six moments of the EGED distribution for selected values of the parameters are displayed in Table 4.2. By fixing $\alpha = 5.0$, $\beta = 2.5$ and $\theta = 10.5$, the values of the first six moments are obtained using numerical integration.

Table 4.2: **First six moments of EGED distribution**

r	$\lambda = 1.5$, $c = 8.5$, $d = 7.5$	$\lambda = 4.5$, $c = 6.5$, $d = 3.5$	$\lambda = 7.5$, $c = 4.5$, $d = 1.5$
$\mu_1^{'}$	1.203056	1.228397	1.276269
$\mu_2^{'}$	1.448742	1.510330	1.631464
$\mu_3^{'}$	1.746298	1.858654	2.088834
$\mu_4^{'}$	2.107028	2.289379	2.678664
$\mu_5^{'}$	2.544777	2.822464	3.440476
$\mu_6^{'}$	3.076518	3.482813	4.425910

4.4.3 Moment Generating Function

The MGF of the random variable X having the EGED distribution if it exist is given by the following proposition.

Proposition 4.2. The MGF of the random variable X is given by

$$M_X(z) = \lambda cd \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\omega_{ijk}\, \beta_{k+1}\, \alpha^{\frac{r}{\theta}}\, z^r}{r!}\, B\left(\beta_{k+1} + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right), r < \theta, \tag{4.12}$$

where $B(\cdot, \cdot)$ is the beta function and $r = 1, 2, \ldots$.

Proof. Using the identity

$$e^{zX} = \sum_{r=0}^{\infty} \frac{z^r\, X^r}{r!},$$

the MGF of the EGED distribution is

$$\begin{aligned} M_X(z) &= E\left(e^{zX}\right) \\ &= \sum_{r=0}^{\infty} \frac{z^r\, E(X^r)}{r!} \\ &= \sum_{r=0}^{\infty} \frac{z^r\, \mu_r^{'}}{r!} \\ &= \lambda cd \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\omega_{ijk}\, \beta_{k+1}\, \alpha^{\frac{r}{\theta}}\, z^r}{r!}\, B\left(\beta_{k+1} + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right), r < \theta. \end{aligned}$$

4.4.4 Incomplete Moment

The incomplete moment plays an important role in computing the mean deviation, median deviation and measures of inequalities such as the Lorenz and Bonferroni curves.

Proposition 4.3. The r^{th} incomplete moment of the EGED distribution is given by

$$M_r(x) = \lambda c d \alpha^{\frac{r}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left((1+\alpha x^{-\theta})^{-1};\, \beta_{k+1} + \frac{r}{\theta},\, 1 - \frac{r}{\theta}\right), r < \theta, \quad (4.13)$$

where $B(\cdot;\cdot,\cdot)$ is the incomplete beta function and $r = 1, 2, \ldots$.

Proof. Using the identity

$$B(q; a, b) = \int_0^q y^{a-1}(1-y)^{b-1} dy,$$

and the concepts for proving the moment, the incomplete moment of the EGED distribution is

$$\begin{aligned} M_r(x) &= E\left(X^r \mid X \le x\right) \\ &= \int_0^x u^r \mathrm{g}(u) du \\ &= \lambda c d \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1} \int_0^{(1+\alpha x^{-\theta})^{-1}} \alpha^{\frac{r}{\theta}} y^{\beta_{k+1}+\frac{r}{\theta}-1} (1-y)^{(1-\frac{r}{\theta})-1} dy \\ &= \lambda c d \alpha^{\frac{r}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left((1+\alpha x^{-\theta})^{-1};\, \beta_{k+1} + \frac{r}{\theta},\, 1 - \frac{r}{\theta}\right), r < \theta. \end{aligned}$$

4.4.5 Mean and Median Deviations

The variation in a population can be measured to some degree by the totality of deviations from the mean and the median. If the random variable X follows the EGED distribution, then the mean and the median deviations are given by the following propositions.

Proposition 4.4. The expected value of the absolute deviation of a random variable X having the EGED distribution from its mean is

$$\delta_1(x) = 2\mu G(\mu) - 2\lambda c d \alpha^{\frac{1}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left(q^*;\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1, \quad (4.14)$$

where $q^* = (1 + \alpha\mu^{-\theta})^{-1}$ and $\mu = \mu_1^{'}$ is the mean of X.

Proof. By definition

$$\begin{aligned}
\delta_1(x) &= \int_0^{\infty} |x - \mu|\, \mathrm{g}(x)dx \\
&= \int_0^{\mu} (\mu - x)\mathrm{g}(x)dx + \int_{\mu}^{\infty} (x - \mu)\mathrm{g}(x)dx \\
&= 2\mu G(\mu) - 2\int_0^{\mu} x\mathrm{g}(x)dx \\
&= 2\mu G(\mu) - 2\lambda c d \alpha^{\frac{1}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left(q^*;\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1,
\end{aligned}$$

where $\int_0^{\mu} x\mathrm{g}(x)dx$ is simplified using the first incomplete moment.

Proposition 4.5. The expected value of the absolute deviation of a random variable X having the EGED distribution from its median is

$$\delta_2(x) = \mu - 2\lambda c d \alpha^{\frac{1}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left(q^{**};\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1, \quad (4.15)$$

where $q^{**} = (1 + \alpha M^{-\theta})^{-1}$ and M is the median of X.

Proof. By definition

$$\begin{aligned}
\delta_2(x) &= \int_0^\infty |x - M| \, \mathrm{g}(x)dx \\
&= \int_0^M (M - x)\mathrm{g}(x)dx + \int_M^\infty (x - M)\mathrm{g}(x)dx \\
&= \mu - 2\int_0^M x\mathrm{g}(x)dx \\
&= \mu - 2\lambda cd\alpha^{\frac{1}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left(q^{**};\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1,
\end{aligned}$$

where $\int_0^M x\mathrm{g}(x)dx$ is simplified using the first incomplete moment.

4.4.6 Inequality Measures

The most widely used approach for measuring the income inequality of given population are the Lorenz and Bonferroni curves. The Lorenz curve, $L_G(x)$ represents the fraction of total income volume accumulated by those units with income lower than or equal to the volume x, and the Bonferroni curve, $B_G(x)$ is the scaled conditional mean curve, that is the ratio of group mean income of the population.

Proposition 4.6. If $X \sim \mathrm{EGED}(\alpha, \lambda, \beta, \theta, c, d)$, then the Lorenz curve $L_G(x)$ is given by

$$L_G(x) = \frac{\lambda cd\alpha^{\frac{1}{\theta}}}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left((1 + \alpha x^{-\theta})^{-1};\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1. \quad (4.16)$$

Proof. By definition

$$\begin{aligned}
L_G(x) &= \frac{1}{\mu}\int_0^x u\mathrm{g}(u)du \\
&= \frac{\lambda cd\alpha^{\frac{1}{\theta}}}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left((1 + \alpha x^{-\theta})^{-1};\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1.
\end{aligned}$$

Proposition 4.7. If $X \sim \text{EGED}(\alpha, \lambda, \beta, \theta, c, d)$, then the Bonferroni curve $B_G(x)$ is given by

$$B_G(x) = \frac{\lambda c d \alpha^{\frac{1}{\theta}}}{\mu G(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left((1+\alpha x^{-\theta})^{-1};\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1. \quad (4.17)$$

Proof.

$$\begin{aligned} B_G(x) &= \frac{L_G(x)}{G(x)} \\ &= \frac{\lambda c d \alpha^{\frac{1}{\theta}}}{\mu G(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \omega_{ijk}\, \beta_{k+1}\, B\left((1+\alpha x^{-\theta})^{-1};\, \beta_{k+1} + \frac{1}{\theta},\, 1 - \frac{1}{\theta}\right), \theta > 1. \end{aligned}$$

4.4.7 Entropy

Entropy plays a vital role in science, engineering and probability theory, and has been used in various situations as a measure of variation or uncertainty of a random variable (Rényi, 1961). The Rényi entropy of a random X having the EGED distribution is given by the following proposition.

Proposition 4.8. If $X \sim \text{EGED}(\alpha, \lambda, \beta, \theta, c, d)$, then the Rényi entropy is given by

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[(\lambda \beta c d)^{\delta} \alpha^{\frac{1-\delta}{\theta}} \theta^{\delta-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varpi_{ijk} B\left(\beta(\delta+k) + \frac{1-\delta}{\theta}, \delta + \frac{\delta-1}{\theta}\right)\right], \quad (4.18)$$

where $\delta \neq 1$, $\delta > 0$, $\beta(\delta+k) + \frac{1-\delta}{\theta} > 0$, $\delta + \frac{\delta-1}{\theta} > 0$ and

$$\varpi_{ijk} = \frac{(-1)^{i+j+k}\Gamma(\delta(\lambda-1)+1)\Gamma(c(\delta+i)-\delta+1)\Gamma(d(\delta+j)-\delta+1)}{i!j!k!\Gamma(\delta(\lambda-1)-i+1)\Gamma(c(\delta+i)-\delta-j+1)\Gamma(d(\delta+j)-\delta-k+1)}.$$

Proof. The Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \log \left[\int_0^\infty g^\delta(x) dx \right], \ \delta \neq 1, \ \delta > 0.$$

Using the same approach for expanding the density,

$$g^\delta(x) = (\alpha\lambda\beta\theta cd)^\delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \varpi_{ijk} x^{-\delta(\theta+1)} \left(1 + \alpha x^{-\theta}\right)^{-\beta(\delta+k)-\delta}.$$

Thus,

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log \left[\int_0^\infty (\alpha\lambda\beta\theta cd)^\delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \varpi_{ijk} x^{-\delta(\theta+1)} \left(1 + \alpha x^{-\theta}\right)^{-\beta(\delta+k)-\delta} dx \right] \\ &= \frac{1}{1-\delta} \log \left[(\alpha\lambda\beta\theta cd)^\delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \varpi_{ijk} \int_0^\infty x^{-\delta(\theta+1)} \left(1 + \alpha x^{-\theta}\right)^{-\beta(\delta+K)-\delta} dx \right]. \end{aligned}$$

Letting $y = (1 + \alpha x^{-\theta})^{-1}$, when $x \to \infty, y \to 1$ and when $x \to 0, y \to 0$. Also, $dy = \alpha\theta x^{-\theta-1}(1 + \alpha x^{-\theta})^{-2} dx$ and $x = (\alpha y)^{\frac{1}{\theta}}(1 - y)^{-\frac{1}{\theta}}$. Hence,

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log \left[(\alpha\lambda\beta\theta cd)^\delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \varpi_{ijk} \int_0^1 y^{\beta(\delta+k)+\delta-2} \left((\alpha y)^{\frac{1}{\theta}} (1-y)^{-\frac{1}{\theta}} \right)^{-\delta(\theta+1)+\theta+1} dy \right] \\ &= \frac{1}{1-\delta} \log \left[(\lambda\beta cd)^\delta \alpha^{\frac{1-\delta}{\theta}} \theta^{\delta-1} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \varpi_{ijk} B\left(\beta(\delta+k) + \frac{1-\delta}{\theta}, \delta + \frac{\delta-1}{\theta} \right) \right], \end{aligned}$$

where $\delta \neq 1$, $\delta > 0$, $\beta(\delta + k) + \frac{1-\delta}{\theta} > 0$ and $\delta + \frac{\delta-1}{\theta} > 0$.

The Rényi entropy tends to Shannon entropy as $\delta \to 1$.

4.4.8 Stress-Strength Reliability

The estimation of reliability is vital in stress-strength models. If X_1 is the strength of a component and X_2 is the stress, the component fails when $X_1 \leq X_2$. Then the estimate of the stress-strength reliability of the component R is $\mathbb{P}(X_2 < X_1)$.

Proposition 4.9. If $X_1 \sim \text{EGED}(\alpha, \lambda, \beta, \theta, c, d)$ and $X_2 \sim \text{EGED}(\alpha, \lambda, \beta, \theta, c, d)$, then the estimation of stress-strength reliability R is given by

$$R = 1 - \lambda cd \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\nu_{ijk}}{(k+1)}, \tag{4.19}$$

where

$$\nu_{ijk} = \frac{(-1)^{i+j+k}\Gamma(2\lambda)\Gamma(c(i+1))\Gamma(d(j+1))}{i!j!k!\Gamma(2\lambda - i)\Gamma(c(i+1) - j)\Gamma(d(j+1) - k)}.$$

Proof. By definition

$$\begin{aligned} R &= \int_0^{\infty} \mathrm{g}(x) G(x) dx \\ &= 1 - \int_0^{\infty} \mathrm{g}(x) S(x) dx \\ &= 1 - \int_0^{\infty} \alpha\lambda\beta\theta cd \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \nu_{ijk} x^{-\theta-1} \left(1 + \alpha x^{-\theta}\right)^{-\beta(k+1)-1} dx \\ &= 1 - \alpha\lambda\beta\theta cd \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \nu_{ijk} \int_0^{\infty} x^{-\theta-1} \left(1 + \alpha x^{-\theta}\right)^{-\beta(k+1)-1} dx \\ &= 1 - \lambda cd \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\nu_{ijk}}{(k+1)}. \end{aligned}$$

4.4.9 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the EGED distribution and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are order statistics obtained from the sample. Then the PDF, $\mathrm{g}_{p:n}(x)$, of the p^{th} order statistic $X_{p:n}$ is given by

$$\mathrm{g}_{p:n}(x) = \frac{1}{B(p,\, n-p+1)} \left[G(x)\right]^{p-1} \left[1 - G(x)\right]^{n-p} \mathrm{g}(x),$$

where $G(x)$ and $\mathrm{g}(x)$ are the CDF and PDF of the EGED distribution respectively, and $B(\cdot,\cdot)$ is the beta function. Since $0 < G(x) < 1$ for $x > 0$, using the binomial series expansion of $[1 - G(x)]^{n-p}$, which is given by

$$[1 - G(x)]^{n-p} = \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} [G(x)]^l \,,$$

we have

$$\mathrm{g}_{p:n}(x) = \frac{1}{B(p,\, n-p+1)} \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} [G(x)]^{p+l-1} \, \mathrm{g}(x). \tag{4.20}$$

Therefore, substituting the CDF and PDF of the EGED distribution into equation (4.20) yields

$$\mathrm{g}_{p:n}(x) = \sum_{l=0}^{n-p} \sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\,(p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} \mathrm{g}(x;\alpha,\, \lambda_{m+1},\, \beta,\, \theta,\, c,\, d), \tag{4.21}$$

where $\mathrm{g}(x;\alpha,\, \lambda_{m+1},\, \beta,\, \theta,\, c,\, d)$ is the PDF of the EGED distribution with parameters $\alpha,\, \beta,\, \theta,\, c,\, d$ and $\lambda_{m+1} = \lambda(m+1)$. It is obvious that the density of the p^{th} order statistic given in equation (4.21) is a weighted function of the EGED distribution with different shape parameters.

Proposition 4.10. The r^{th} non-central moment of the p^{th} order statistic is given by

$$\mu_r^{\prime(p:n)} = \lambda\beta c d \alpha^{\frac{r}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \omega^*_{ijklm}\, B\left(\beta_{k+1} + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right), r < \theta, \tag{4.22}$$

where $r = 1, 2, \ldots$ and

$$\omega^*_{ijklm} = \frac{(-1)^{i+j+k+l+m}\Gamma(n+1)\Gamma(p+l)\Gamma(\lambda(m+1))\Gamma(c(i+1))\Gamma(d(j+1))}{i!j!k!l!m!(p-1)!(n-p-l)!\Gamma(p+l-m)\Gamma(\lambda(m+1)-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned}
\mu_r^{\prime(p:n)} &= \int_0^\infty x^r \mathrm{g}_{p:n}(x)dx \\
&= \int_0^\infty x^r \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\, (p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} \mathrm{g}(x;\alpha,\,\lambda_{m+1},\,\beta,\,\theta,\,c,\,d)dx \\
&= \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\, (p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} \int_0^\infty x^r \mathrm{g}(x;\alpha,\,\lambda_{m+1},\,\beta,\,\theta,\,c,\,d)dx.
\end{aligned}$$

Using the same method for deriving the non-central moment, we obtain

$$\mu_r^{\prime(p:n)} = \lambda\beta c d \alpha^{\frac{r}{\theta}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \omega^*_{ijklm}\, B\left(\beta_{k+1} + \frac{r}{\theta}, 1 - \frac{r}{\theta}\right), r < \theta.$$

4.5 Parameter Estimation

In this section, the maximum likelihood estimators for the unknown parameters of the EGED distribution were derived. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the EGED distribution. Let $z_i = (1 + \alpha x_i^{-\theta})$, then the log-likelihood function is given by

$$\begin{aligned}
\ell = n\log(\alpha\lambda\beta\theta c d) - (\theta+1)\sum_{i=1}^{n}\log(x_i) - (\beta+1)\sum_{i=1}^{n}\log(z_i) + (d-1)\sum_{i=0}^{n}\log\left(1 - z_i^{-\beta}\right) \\
+(c-1)\sum_{i=1}^{n}\log\left[1 - \left(1 - z_i^{-\beta}\right)^d\right] + (\lambda-1)\sum_{i=1}^{n}\log\left\{1 - \left[1 - \left(1 - z_i^{-\beta}\right)^d\right]^c\right\}.
\end{aligned} \quad (4.23)$$

Taking the first partial derivatives of the log-likelihood function in equation (4.23) with respect to the parameters α, λ, β, θ, c and d, the score functions are:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left\{1 - [1 - (1 - z_i^{-\beta})^d]^c\right\}, \quad (4.24)$$

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \log[1-(1-z_i^{-\beta})^d] - (\lambda-1)\sum_{i=1}^{n} \frac{[1-(1-z_i^{-\beta})^d]^c \log[1-(1-z_i^{-\beta})^d]}{1-[1-(1-z_i^{-\beta})^d]^c}, \tag{4.25}$$

$$\frac{\partial \ell}{\partial d} = \frac{n}{d} + \sum_{i=1}^{n} \log(1-z_i^{-\beta}) + (\lambda-1)\sum_{i=1}^{n} \frac{c(1-z_i^{-\beta})^d \left[1-(1-z_i^{-\beta})^d\right]^{c-1} \log(1-z_i^{-\beta})}{1-\left[1-(1-z_i^{-\beta})^d\right]^c} - (c-1)\sum_{i=1}^{n} \frac{(1-z_i^{-\beta})^d \log(1-z_i^{-\beta})}{1-(1-z_i^{-\beta})^d}, \tag{4.26}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log(z_i) + (d-1)\sum_{i=1}^{n} \frac{z_i^{-\beta}\log(z_i)}{1-z_i^{-\beta}} - (c-1)\sum_{i=1}^{n} \frac{dz_i^{-\beta}(1-z_i^{-\beta})^{d-1}\log(z_i)}{1-(1-z_i^{-\beta})^d} + (\lambda-1)\sum_{i=1}^{n} \frac{cdz_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c}, \tag{4.27}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \log(x_i) + (\beta+1)\sum_{i=1}^{n} \frac{\alpha x_i^{-\theta}\log(x_i)}{z_i} - (d-1)\sum_{i=1}^{n} \frac{\alpha\beta x_i^{-\theta} z_i^{-\beta-1}\log(x_i)}{1-z_i^{-\beta}} - (\lambda-1)\sum_{i=1}^{n} \frac{\alpha\beta c d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c} + (c-1)\sum_{i=1}^{n} \frac{\alpha\beta d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(x_i)}{1-(1-z_i^{-\beta})^d}, \tag{4.28}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - (\beta+1)\sum_{i=1}^{n} \frac{x_i^{-\theta}}{z_i} + (d-1)\sum_{i=1}^{n} \frac{\beta x_i^{-\theta} z_i^{-\beta-1}}{1-z_i^{-\beta}} - (c-1)\sum_{i=1}^{n} \frac{\beta d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}}{1-(1-z_i^{-\beta})^d} + (\lambda-1)\sum_{i=1}^{n} \frac{\beta c d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}}{1-[1-(1-z_i^{-\beta})^d]^c}. \tag{4.29}$$

The estimates for the parameters α, λ, β, θ, c and d are obtained by equating the score functions to zero and solving the system of nonlinear equations numerically. In order to construct confidence intervals for the parameters, the observed information matrix $J(\boldsymbol{\vartheta})$ is used since the expected information matrix is complicated. The observed information matrix is given by

$$J(\boldsymbol{\vartheta}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial c} & \frac{\partial^2 \ell}{\partial \lambda \partial d} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} & \frac{\partial^2 \ell}{\partial \lambda \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} \\ & \frac{\partial^2 \ell}{\partial c^2} & \frac{\partial^2 \ell}{\partial c \partial d} & \frac{\partial^2 \ell}{\partial c \partial \beta} & \frac{\partial^2 \ell}{\partial c \partial \theta} & \frac{\partial^2 \ell}{\partial c \partial \alpha} \\ & & \frac{\partial^2 \ell}{\partial d^2} & \frac{\partial^2 \ell}{\partial d \partial \beta} & \frac{\partial^2 \ell}{\partial d \partial \theta} & \frac{\partial^2 \ell}{\partial d \partial \alpha} \\ & & & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \theta} & \frac{\partial^2 \ell}{\partial \beta \partial \alpha} \\ & & & & \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \alpha} \\ & & & & & \frac{\partial^2 \ell}{\partial \alpha^2} \end{bmatrix},$$

where $\boldsymbol{\vartheta} = (\alpha,\, \lambda,\, \beta,\, \theta,\, c,\, d)'$. The expression for the elements of the observed information matrix are given in Appendix A1. When the usual regularity conditions are fulfilled and that the parameters are within the interior of the parameter space, that is not on the boundary, $\sqrt{n}(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}) \xrightarrow{\text{Dist}} N_6(\mathbf{0}, I^{-1}(\boldsymbol{\vartheta}))$, where $I(\boldsymbol{\vartheta})$ is the expected information matrix. The asymptotic behavior is still valid when $I(\boldsymbol{\vartheta})$ is replaced by the observed information matrix evaluated at $J(\hat{\boldsymbol{\vartheta}})$. The asymptotic multivariate normal distribution $N_6(\mathbf{0}, J^{-1}(\hat{\boldsymbol{\vartheta}}))$ can be used to construct an approximate $100(1-\eta)\%$ two-sided confidence intervals for the model parameters, where η is the significance level.

4.6 Monte Carlo Simulation

In this section, a simulation study was carried out to examine the average bias (AB) and root mean square error (RMSE) of the maximum likelihood estimators for the parameters of the EGED distribution. The quantile function given in equation (4.9) was used to generate random samples from the EGED distribution. The simulation experiment was repeated for $N = 1,000$ times each with sample sizes n = 25, 50, 75, 100, 200, 300, 600 and parameter values (α, λ, β, θ, c, d) = (2.5, 1.5, 0.4, 0.5, 1.0, 0.2) and (0.3, 0.5, 0.8, 0.2, 0.7, 1.5). The AB and RMSE for the estimators of the parameters were computed using the following relations:

$$\text{AB} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\vartheta}_i - \vartheta\right),$$

and

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\vartheta}_i - \vartheta\right)^2}.$$

Figure 4.3 and 4.4 respectively show the AB and RMSE of the maximum likelihood estimators of (α, λ, β, θ, c, d) = (2.5, 1.5, 0.4, 0.5, 1.0, 0.2) for n = 25, 50, 75, 100, 200, 300, 600. The AB for some estimators of the parameters decreases as the sample size increases while it fluctuates for others. The RMSE for the estimators of all the parameters showed a decreasing pattern.

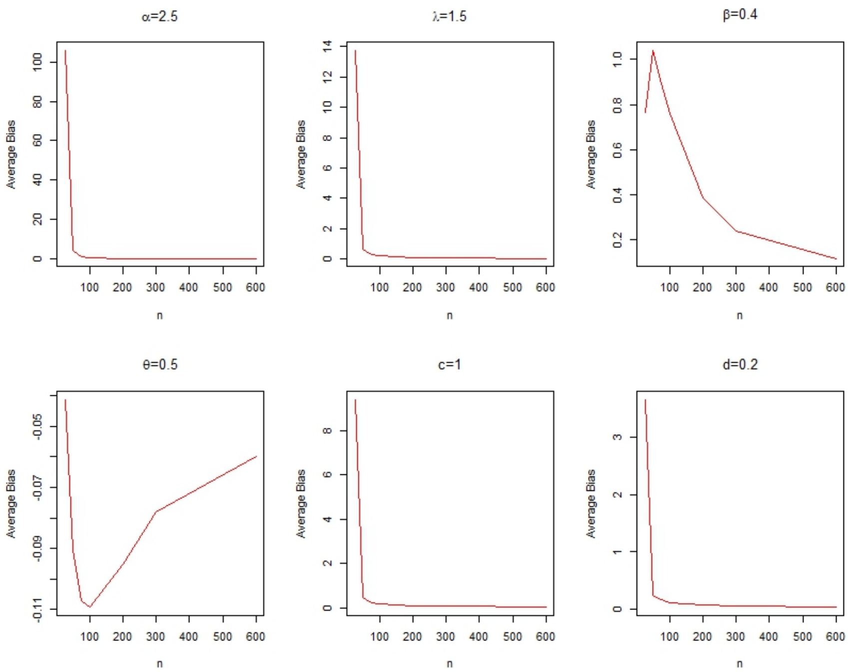

Figure 4.3: **AB for Estimators**

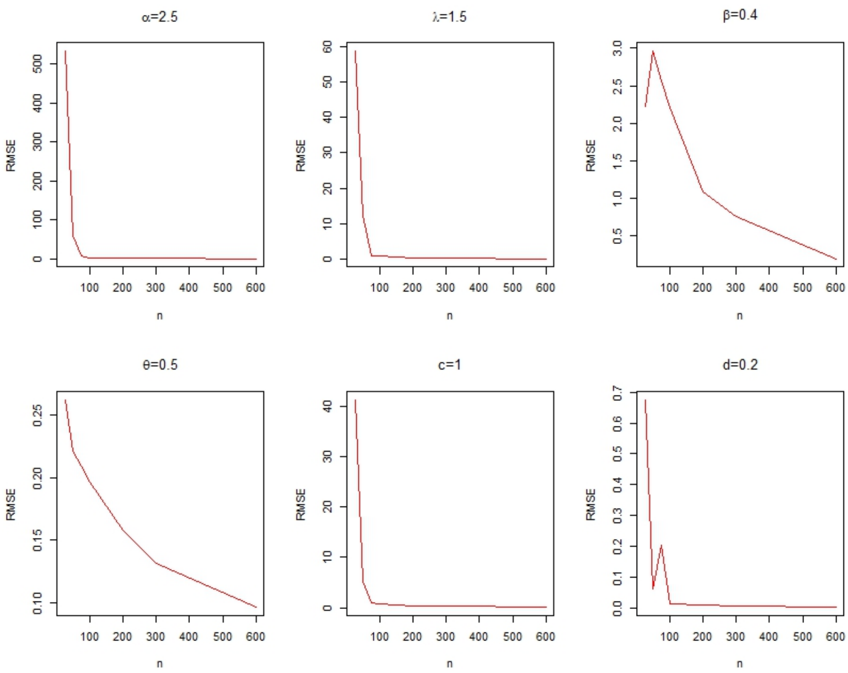

Figure 4.4: **RMSE for Estimators**

Figure 4.5 and 4.6 respectively shows the AB and RMSE for the maximum likelihood estimators of (α, λ, β, θ, c, d) = (0.3, 0.5, 0.8, 0.2, 0.7, 1.5) for n = 25, 50, 75, 100, 200, 300, 600. Just like the first case the AB decreases for estimators of some parameters as the sample size increases whiles for others it fluctuates. The RMSE for the estimators on average decreases for all parameters as the sample size increases.

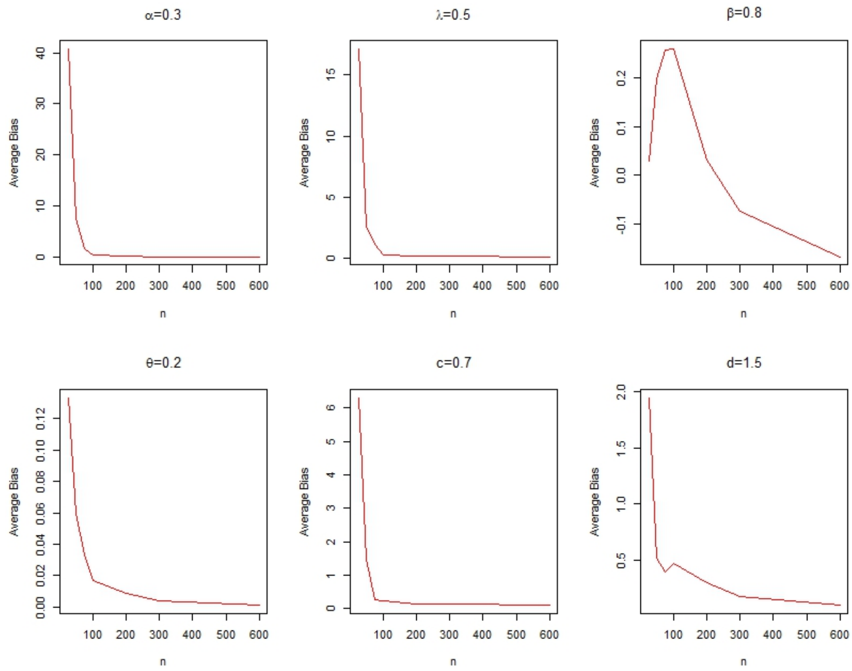

Figure 4.5: **AB for Estimators**

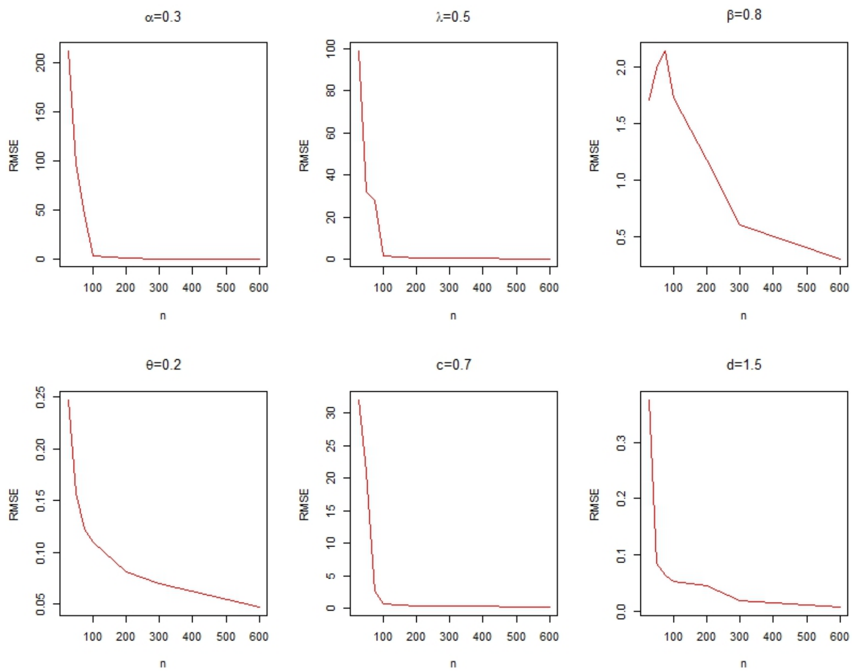

Figure 4.6: **RMSE for Estimators**

4.7 Applications

In this section, the applications of the EGED distribution were demonstrated using two real data sets. The goodness-of-fit of the EGED distribution was compared with that of its sub-models, the exponentiated Kumaraswamy Dagum (EKD) distribution and the Mc-Dagum (McD) distribution using K-S statistic and Cramér-von (W^*) Misses distance values, as well as AIC, AICc and BIC. The PDF of EKD distribution is given by

$$\mathrm{g}(x) = \alpha\lambda\delta\phi\theta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\alpha-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi-1}\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi}\right\}^{\theta-1}, \tag{4.30}$$

for $\alpha > 0$, $\lambda > 0$, $\delta > 0$, $\phi > 0$, $\theta > 0$, $x > 0$, and that of McD distribution is

$$\mathrm{g}(x) = \frac{c\beta\lambda\delta x^{-\delta-1}}{B(a,\, b)}(1+\lambda x^{-\delta})^{-\beta ac-1}[1-(1+\lambda x^{-\delta})^{-c\beta}]^{b-1}, \tag{4.31}$$

for $a > 0$, $b > 0$, $c > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $x > 0$.

4.7.1 Yarn Data

The data in Table 4.3 represents the time to failure of a 100 cm polyster/viscose yarn subjected to 2.3% strain level in textile experiment in order to assess the tensile fatigue characteristics of the yarn. The data set can be found in Quesenberry and Kent (1982) and Pal and Tiensuwan (2014).

Table 4.3: **Failure time data on 100 cm yarn subjected to 2.3% strain level**

86	146	251	653	98	249	400	292	131	169
175	176	76	264	15	364	195	262	88	264
157	220	42	321	180	198	38	20	61	121
282	224	149	180	325	250	196	90	229	166
38	337	65	151	341	40	40	135	597	246
211	180	93	315	353	571	124	279	81	186
497	182	423	185	229	400	338	290	398	71
246	185	188	568	55	55	61	244	20	289
393	396	203	829	239	236	286	194	277	143
198	264	105	203	124	137	135	350	193	188

The data set have an increasing failure rate as displayed by the TTT transform plot, which has a concave shape as shown in Figure 4.7.

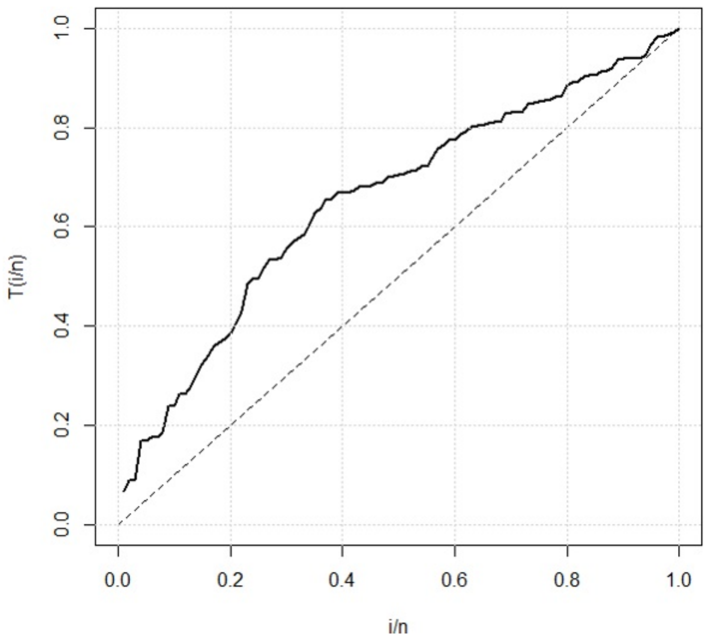

Figure 4.7: **TTT-transform plot for yarn data**

The maximum likelihood estimates for the parameters of the fitted models with their corresponding standard errors in brackets are given in Table 4.4. The parameters of most of the fitted distributions were significant at the 5% significance level. This can be verified by using the standard error test which states that for a parameter to be significant at the 5% significance level the standard error should be less than half the parameter value.

Table 4.4: **Maximum likelihood estimates of parameters and standard errors for yarn**

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
EGED	0.026	75.310	0.017	3.513	45.692	0.090
	(0.007)	(0.007)	(0.005)	(0.631)	(0.036)	(0.011)
EGDD	1.992		10.480	4.733	75.487	0.223
	(0.251)		(13.022)	(0.587)	(27.669)	(0.032)
DD	19.749		11.599	1.126		
	(10.814)		(5.008)	(0.069)		
EGEBD		35.463	35.965	4.859	15.667	0.070
		(0.271)	(0.120)	(0.666)	(2.714)	(0.011)
EGBD			24.801	4.196	73.9120	0.258
			(15.068)	(1.808)	(22.832)	(0.112)
EGEFD	20.662	34.477		5.217	16.438	0.65
	(2.365)	(0.278)		(0.578)	(2.708)	(0.009)
EGFD	10.537			5.239	21.341	0.140
	(1.115)			(0.429)	(4.089)	(0.015)
	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
McD	0.027	0.600	98.780	0.333	25.042	46.276
	(1.848×10^{-2})	(9.647×10^{-2})	(2.180×10^{-5})	(1.504×10^{-1})	(4.507×10^{-4})	(4.654×10^{-5})
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\theta}$	
EKD	46.109	39.413	5.188	0.203	31.169	
	(1.295)	(5.006)	(0.961)	(0.040)	(11.023)	

The EGED distribution provides a better fit to the yarn data than its sub-models, the McD and the EKD distribution. From Table 4.5, the EGED distribution has the highest log-likelihood and the smallest K-S, W*, AIC, AICc, and BIC values compared to the other models. Although the EGED distribution provides the best fit to the data, the McD distribution, EGEBD and EGEFD are alternatively good models for the data since their measures of fit values are close to that of the EGED distribution.

Table 4.5: **Log-likelihood, goodness-of-fit statistics and information criteria for yarn**

Model	ℓ	AIC	AICc	BIC	K-S	W*
EGED	**-628.170**	**1268.336**	**1269.553**	**1283.967**	**0.124**	**0.249**
EGDD	-653.070	1316.137	1317.040	1329.163	0.172	0.948
DD	-649.260	1304.517	1304.938	1312.333	0.164	0.821
EGEBD	-630.870	1271.745	1272.648	1284.771	0.136	0.340
EGBD	-653.030	1314.056	1314.694	1324.447	0.174	0.969
EGEFD	-630.760	1271.523	1272.426	1284.549	0.139	0.339
EGFD	-666.880	1341.757	1342.395	1352.177	0.236	0.760
McD	-628.200	1268.399	1269.616	1284.030	0.128	0.285
EKD	-653.960	1317.913	1318.816	1330.938	0.178	0.985

The LRT was performed to compare the EGED distribution with its sub-models. The LRT statistics and their corresponding P-values in Table 4.6 revealed that the EGED distribution provides a good fit than its sub-models.

Table 4.6: **Likelihood ratio test statistic for yarn**

Model	Hypotheses	LRT	P-values
EGDD	$H_0 : \lambda = 1$ vs $H_1 : H_0$ is false	49.801	< 0.001
DD	$H_0 : \lambda = c = d = 1$ vs $H_1 : H_0$ is false	42.181	< 0.001
EGEBD	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	5.409	0.020
EGBD	$H_0 : \lambda = \alpha = 1$ vs $H_1 : H_0$ is false	49.721	< 0.001
EGEFD	$H_0 : \beta = 1$ vs $H_1 : H_0$ is false	5.187	0.023
EGFD	$H_0 : \lambda = \beta = 1$ vs $H_1 : H_0$ is false	77.421	< 0.001

The asymptotic variance-covariance matrix for the estimated parameters of the EGED distribution for the yarn data is given by

$$J^{-1} = \begin{bmatrix} 5.0338\times10^{-5} & 2.1232\times10^{-5} & 1.3887\times10^{-5} & 0.0045 & 2.5246\times10^{-4} & -7.5812\times10^{-5} \\ 2.1232\times10^{-5} & 5.1601\times10^{-5} & 1.0316\times10^{-5} & 0.0019 & 1.0648\times10^{-4} & -2.5628\times10^{-5} \\ 1.3887\times10^{-5} & 1.0316\times10^{-5} & 2.2466\times10^{-5} & 0.0012 & 6.9642\times10^{-5} & -1.6719\times10^{-5} \\ 4.4786\times10^{-3} & 1.8887\times10^{-3} & 1.2354\times10^{-3} & 0.3985 & 2.2462\times10^{-2} & -6.7451\times10^{-3} \\ 2.5246\times10^{-4} & 1.0648\times10^{-4} & 6.9642\times10^{-5} & 0.0225 & 1.2662\times10^{-3} & -3.8023\times10^{-4} \\ -7.5812\times10^{-5} & -2.5628\times10^{-5} & -1.6719\times10^{-5} & -0.0067 & -3.8023\times10^{-4} & 1.1654\times10^{-4} \end{bmatrix}.$$

The approximate 95% confidence interval for the parameters α, λ, β, θ, c and d of the EGED distribution are [0.012, 0.040], [75.296, 75.324], [0.007, 0.027], [2.276, 4.750], [45.621, 45.763] and [0.068, 0.111] respectively. It can be seen that the confidence intervals for the parameters do not contain zero. This is an indication that all the parameters of the EGED distribution were significant at the 5% significance level. The plots of the empirical density and the densities of the fitted distributions are shown in Figure 4.8.

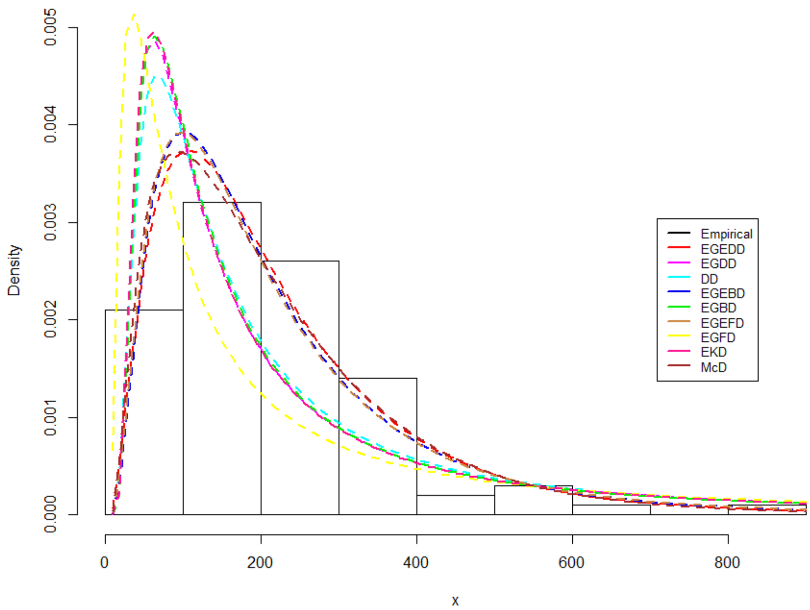

Figure 4.8: **Empirical and fitted densities plot for yarn data**

4.7.2 Appliances Data

The appliances data was obtained from Lawless (1982). The data set consists of failure times for 36 appliances subjected to an automatic life test. The data set are given in Table 4.7.

Table 4.7: **Failure Times for 36 appliances subjected to an automatic life test**

11	35	49	170	329	381	708	958	1062	1167	1594	1925
1990	2223	2327	2400	2451	2471	2551	2565	2568	2694	2702	2761
2831	3034	3059	3112	3214	3478	3504	4329	6367	6976	7846	13403

The TTT transform curve of the data set displays a convex shape and then concave shape followed by convex shape as shown in Figure 4.9. Thus, the failure rate function of the data set has a modified bathtub shape.

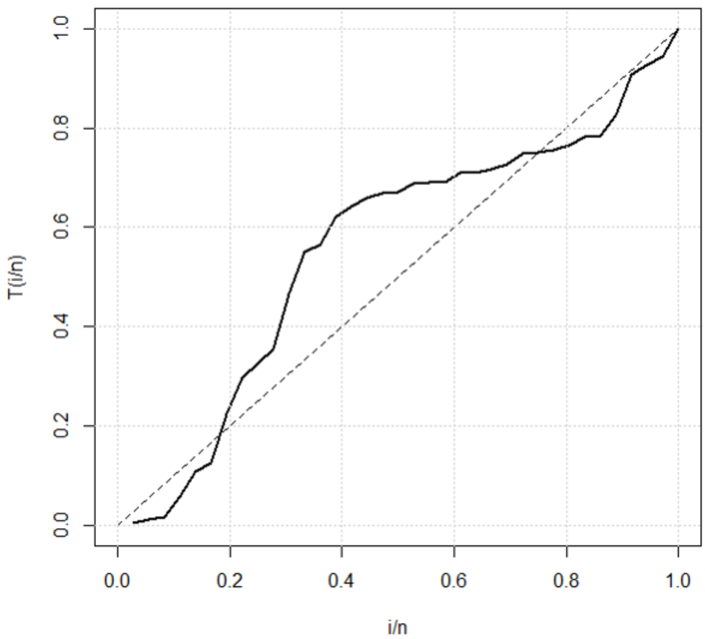

Figure 4.9: **TTT-transform plot for appliances data**

Table 4.8 provides the maximum likelihood estimates for the parameters with their corresponding standard errors in brackets for the models fitted to the appliances data. Using the standard error approach for testing for the significance of the parameters, it can be seen from Table 4.8 that most of the parameters for the various estimated models are significant at the 5% significance level.

Table 4.8: **Maximum likelihood estimates of parameters and standard errors for appliances**

Model	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
EGED	0.001	27.198	4.560	2.838	20.866	0.070
	(1.000×10^{-4})	(0.001)	(0.847)	(0.123)	(0.010)	(0.003)
EGDD	7.977		0.404	3.570	15.862	0.130
	(0.651)		(0.044)	(0.391)	(5.196)	(0.021)
DD	0.018		1495.519	0.509		
	(0.0062)		(1.058×10^{-7})	(0.056)		
EGEBD		25.705	14.152	3.412	8.332	0.047
		(0.514)	(0.110)	(0.247)	(1.934)	(0.009)
EGBD			9.504	3.392	11.226	0.129
			(3.205)	(0.388)	(3.440)	(0.022)
EGEFD	13.048	27.555		3.561	9.084	0.047
	(1.817)	(0.071)		(0.392)	(2.186)	(0.009)
EGFD	8.4843			3.429	16.533	0.143
	(1.550)			(0.711)	(5.833)	(0.034)
	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
McD	1.427	3.455	1.275	10.505	0.064	500.556
	(0.092)	(0.212)	(6.875)	(56.906)	(0.012)	(6.796)
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\theta}$	
EKD	5.562	12.683	3.716	0.128	11.609	
	(1.517)	(2.158)	(0.755)	(0.029)	(3.922)	

From Table 4.9, it was clear that the EGED distribution provided a better fit to the appliances data than the other models. It has the highest log-likelihood and the smallest K-S, W*, AIC, AICc and BIC values. Alternatively, the EGEBD and EGEFD are good models since their goodness-of-fit measures are close to that of the EGED distribution.

Table 4.9: **Log-likelihood, goodness-of-fit statistics and information criteria for appliances**

Model	ℓ	AIC	AICc	BIC	K-S	W*
EGED	**-328.870**	**669.740**	**670.957**	**679.241**	**0.253**	**0.569**
EGDD	-340.910	691.818	692.721	699.736	0.264	0.882
DD	-339.610	685.225	685.646	689.976	0.257	0.858
EGEBD	-330.910	671.823	672.726	679.741	0.272	0.634
EGBD	-341.520	691.037	691.675	697.371	0.268	0.881
EGEFD	-330.730	671.460	672.363	679.377	0.269	0.625
EGFD	-341.030	690.054	690.692	696.388	0.269	0.907
McD	-356.480	724.955	728.950	734.456	0.347	0.986
EKD	-341.650	693.295	694.198	701.213	0.269	0.925

The LRT was performed in order to compare the EGED distribution with its sub-models. From Table 4.10, the LRT revealed that the EGED distribution provides a better fit to the appliances data than its sub-models. Although the LRT favored the EGEFD at the 5% level of significance, the EGED distribution was better than it at the 10% level of significance.

Table 4.10: **Likelihood ratio test statistic for appliances**

Model	Hypotheses	LRT	P-values
EGDD	$H_0 : \lambda = 1$ vs $H_1 : H_0$ is false	24.078	< 0.001
DD	$H_0 : \lambda = c = d = 1$ vs $H_1 : H_0$ is false	21.486	< 0.001
EGEBD	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	4.084	0.043
EGBD	$H_0 : \lambda = \alpha = 1$ vs $H_1 : H_0$ is false	25.297	< 0.001
EGEFD	$H_0 : \beta = 1$ vs $H_1 : H_0$ is false	3.720	0.054
EGFD	$H_0 : \lambda = \beta = 1$ vs $H_1 : H_0$ is false	24.315	< 0.001

The asymptotic variance-covariance matrix for the estimated parameters of the EGED distribution for the appliances data is given by

$$J^{-1} = \begin{bmatrix} 1.7033 \times 10^{-6} & 1.5346 \times 10^{-8} & 1.1045 \times 10^{-3} & 3.7492 \times 10^{-5} & 1.2695 \times 10^{5} & -6.6696 \times 10^{-8} \\ 1.5346 \times 10^{-8} & 1.4494 \times 10^{-8} & 8.8310 \times 10^{-6} & 5.7406 \times 10^{-6} & 1.1008 \times 10^{-7} & -8.3473 \times 10^{-8} \\ 1.1045 \times 10^{-3} & 8.8310 \times 10^{-6} & 7.1688 \times 10^{-1} & 2.1348 \times 10^{-2} & 8.2348 \times 10^{-3} & 1.3547 \times 10^{-5} \\ 3.7492 \times 10^{-5} & 5.7406 \times 10^{-6} & 2.1348 \times 10^{-2} & 1.5185 \times 10^{-2} & 2.6827 \times 10^{-4} & -2.8002 \times 10^{-4} \\ 1.2695 \times 10^{-5} & 1.1008 \times 10^{-7} & 8.2348 \times 10^{-3} & 2.6827 \times 10^{-4} & 9.4629 \times 10^{-5} & -2.9359 \times 10^{-7} \\ -6.6696 \times 10^{-8} & -8.3473 \times 10^{-8} & 1.3547 \times 10^{-5} & -2.8002 \times 10^{-4} & -2.9359 \times 10^{-7} & 8.4565 \times 10^{-6} \end{bmatrix}.$$

Thus, the approximate 95% confidence interval for the parameters α, λ, β, θ, c and d of the EGED distribution are [0.0008, 0.0012], [27.1955, 27.2005], [2.9005, 6.2195], [2.5965, 3.0795], [20.8470, 20.8850] and [0.0643, 0.0757] respectively. The confidence intervals for the parameters do not contain zero. This implies that the estimated parameters of the EGED distribution were all significant at the 5% significance level. Figure 4.10

displays the empirical density and the fitted densities of the fitted distributions.

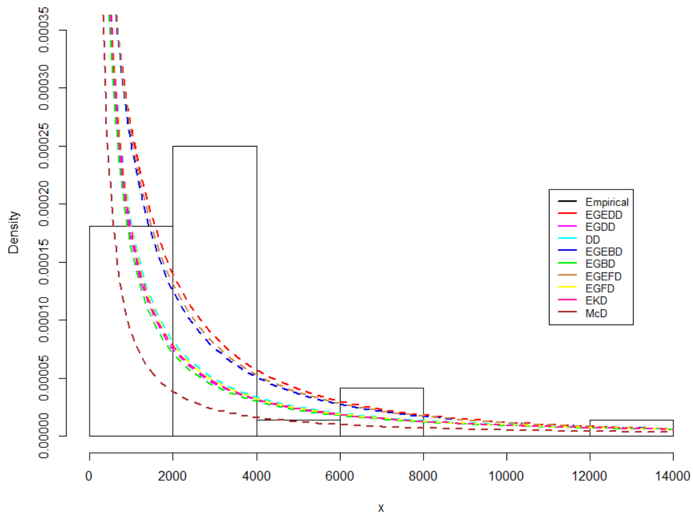

Figure 4.10: **Empirical and fitted densities plot for appliances data**

4.8 Summary

In this chapter, the EGED distribution was proposed and studied. The distribution contains a number of sub-models with potential applications to a wide area of probability and statistics. Statistical properties such as the quantile function, moment, MGF, incomplete moment, mean and median deviations, inequality measures, entropy, reliability and order statistics were derived. The estimation of the parameters of the model was done using maximum likelihood estimation and simulation experiments were performed to investigate the statistical properties of the estimators. Finally, the usefulness of the EGED distribution was demonstrated using two data sets.

CHAPTER 5

NEW EXPONENTIATED GENERALIZED MODIFIED INVERSE RAYLEIGH DISTRIBUTION

5.1 Introduction

Recently, Khan (2014) proposed the modified inverse Rayleigh (MIR) distribution and studied its theoretical properties. The CDF of this distribution is given by

$$F(x) = e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}, \ \alpha, \ \theta, \ x > 0, \tag{5.1}$$

where the two parameters of the distribution are scale parameters. However, to control skewness, kurtosis, model data with heavy tails and non-monotonic failure rates there is the need for a distribution to have shape parameters. In this chapter, the CDF of the EGE-X family was used to develop and investigate the theoretical properties of a new model called the new exponentiated generalized MIR (NEGMIR) distribution.

5.2 Generalized Modified Inverse Rayleigh

Suppose the random variable X has the CDF defined in equation (5.1), then the CDF of the NEGMIR distribution is given by

$$G(x) = 1 - \left\{ 1 - \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)} \right)^d \right]^c \right\}^\lambda, x > 0, \tag{5.2}$$

where $\alpha \geq 0$, $\theta \geq 0$, $(\alpha + \theta > 0)$ are scale parameters and $\lambda > 0$, $c > 0$, $d > 0$ are shape parameters. Finding the first derivative of equation (5.2), the PDF of the NEGMIR distribution is given by

$$\begin{aligned} \mathrm{g}(x) = \lambda cd \left(\frac{\alpha}{x^2} + \frac{\theta}{x^3} \right) e^{-\left(\frac{\alpha}{x} + \frac{2\theta}{x^2}\right)} \left(1 - e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)} \right)^{d-1} \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)} \right)^d \right]^{c-1} \times \\ \left\{ 1 - \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)} \right)^d \right]^c \right\}^{\lambda - 1}, x > 0. \end{aligned} \tag{5.3}$$

Lemma 5.1. The NEGMIR distribution PDF can be written in a mixture form as

$$\mathrm{g}(x) = \lambda cd \left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3} \right) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \xi_{ijkm}\, x^{-2m} e^{-(k+1)\left(\frac{\alpha}{x}\right)}, x > 0, \tag{5.4}$$

where

$$\xi_{ijkm} = \frac{(-1)^{i+j+k+m}\,(k+1)^m\,\theta^m\,\Gamma(\lambda)\,\Gamma(c(i+1))\,\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,\Gamma(\lambda - i)\,\Gamma(c(i+1) - j)\,\Gamma(d(j+1) - k)}, \Gamma(a+1) = a!$$

Proof. For a real non-integer $\eta > 0$, a series representation for $(1 - z)^{\eta - 1}$, for $|z| < 1$ is

$$(1 - z)^{\eta - 1} = \sum_{i=0}^{\infty} \frac{(-1)^i\,\Gamma(\eta)}{i!\,\Gamma(\eta - i)} z^i. \tag{5.5}$$

Using the series expansion in equation (5.5) thrice and the fact that $0 < 1 - e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)} < 1$, yields,

$$
\mathrm{g}(x) = \lambda cd \left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3}\right) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j+k}\,\Gamma(\lambda)\,\Gamma(c(i+1))\,\Gamma(d(j+1))}{i!\,j!\,k!\,\Gamma(\lambda-i)\,\Gamma(c(i+1)-j)\,\Gamma(d(j+1)-k)} e^{-(k+1)\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}. \tag{5.6}
$$

But

$$
e^{-(k+1)\left(\frac{\theta}{x^2}\right)} = \sum_{m=0}^{\infty} \frac{(-1)^m\,(k+1)^m\,\theta^m\,x^{-2m}}{m!}. \tag{5.7}
$$

Substituting equation (5.7) into equation (5.6), the mixture representation of the PDF of the NEGMIR distribution is obtained as

$$
\mathrm{g}(x) = \lambda cd \left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3}\right) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}\,x^{-2m} e^{-(k+1)\left(\frac{\alpha}{x}\right)},\; x > 0.
$$

The PDF of NEGMIR distribution can be symmetric, left skewed, right skewed, J-shape, reverse J-shape or unimodal with small and large values of skewness and kurtosis for different parameter values. Figure 5.1 displays the different shapes of the NEGMIR distribution density function.

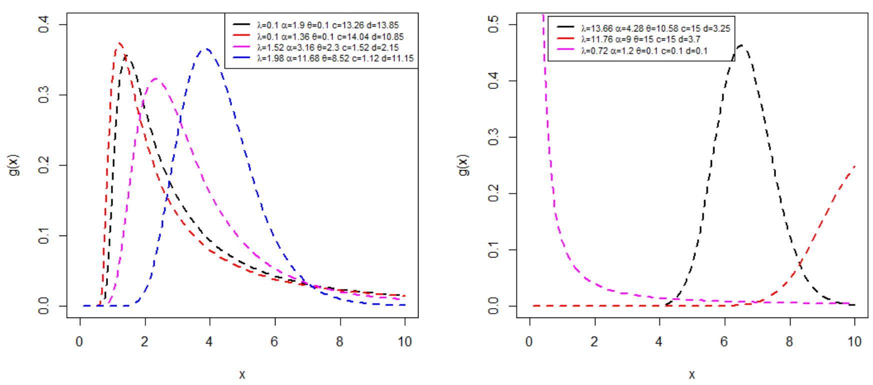

Figure 5.1: **NEGMIR density function for some parameter values**

The survival function of the NEGMIR distribution is

$$S(x) = \left\{1 - \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^c\right\}^\lambda, x > 0, \tag{5.8}$$

and the hazard function is given by

$$\tau(x) = \frac{\lambda cd\left(\frac{\alpha}{x^2}+\frac{2\theta}{x^3}\right)e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^{d-1}\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^{c-1}}{1-\left[1-\left(1-e^{-\left(\frac{\alpha}{x}+\frac{\theta}{x^2}\right)}\right)^d\right]^c}, x > 0 \tag{5.9}$$

The plots of the hazard function reveals different shapes such as monotonically decreasing, monotonically increasing or unimodal for different combination of the values of the parameters. These features make the NEGMIR distribution suitable for modeling different failure rates that are more likely to be encountered in real life situation. Figure 5.2 displays the various shapes of the hazard function.

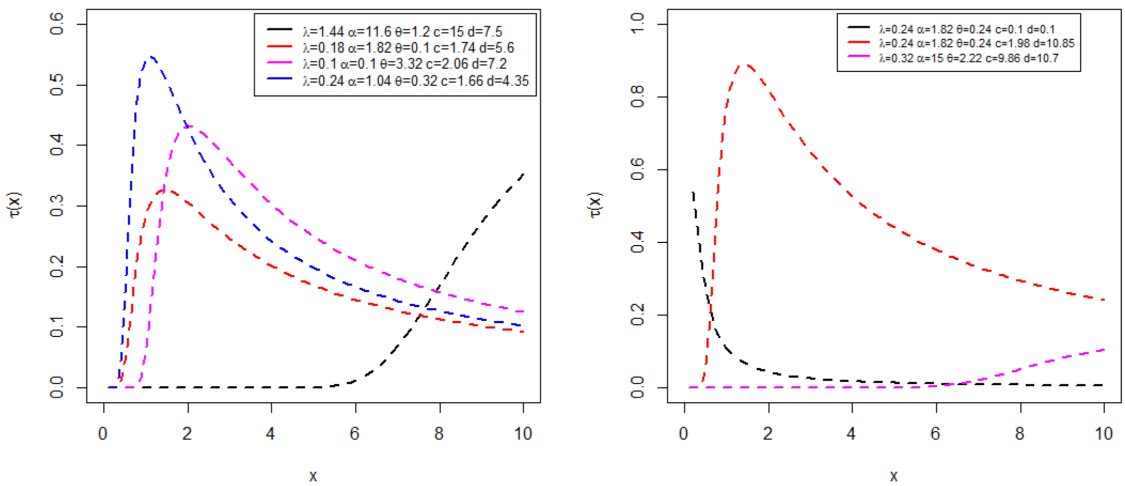

Figure 5.2: **Plots of the NEGMIR hazard function for some parameter values**

5.3 Sub-models

The NEGMIR distribution houses a number of sub-models that can be used in different fields for modeling data sets. These include:

1. **Exponentiated Generalized Modified Inverse Rayleigh Distribution**

   When $\lambda = 1$, the NEGMIR reduces to the exponentiated generalized modified inverse Rayleigh (EGMIR) distribution with the following CDF:

$$G(x) = \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)}\right)^d\right]^c,$$

   for α, θ, c, $d > 0$ and $x > 0$.

2. **Exponentiated Generalized Exponential Inverse Rayleigh Distribution**

   When $\alpha = 0$, the NEGMIR reduces to the exponentiated generalized exponential inverse Rayleigh (EGEIR) distribution with the following CDF:

$$G(x) = 1 - \left\{1 - \left[1 - \left(1 - e^{-\left(\frac{\theta}{x^2}\right)}\right)^d\right]^c\right\}^\lambda,$$

   for λ, θ, c, $d > 0$ and $x > 0$.

3. **Exponentiated Generalized Inverse Rayleigh Distribution**

   When $\alpha = 0$ and $\lambda = 1$, the NEGMIR reduces to the exponentiated generalized inverse Rayleigh (EGIR) distribution with CDF:

$$G(x) = \left[1 - \left(1 - e^{-\left(\frac{\theta}{x^2}\right)}\right)^d\right]^c,$$

   for θ, c, $d > 0$ and $x > 0$.

4. **Exponentiated Generalized Exponential Inverse Exponential Distribution**

When $\theta = 0$, the NEGMIR reduces to the exponentiated generalized exponential inverse exponential (EGEIE) distribution with CDF:

$$G(x) = 1 - \left\{1 - \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x}\right)}\right)^d\right]^c\right\}^\lambda,$$

for λ, α, c, $d > 0$ and $x > 0$.

5. **Exponentiated Generalized Inverse Exponential Distribution**

When $\theta = 0$ and $\lambda = 1$, the NEGMIR reduces to the exponentiated generalized inverse exponential (EGIE) distribution with CDF:

$$G(x) = \left[1 - \left(1 - e^{-\left(\frac{\alpha}{x}\right)}\right)^d\right]^c,$$

for α, c, $d > 0$ and $x > 0$.

6. **Modified Inverse Rayleigh Distribution**

When $\lambda = c = d = 1$, the NEGMIR reduces to the MIR distribution with CDF:

$$G(x) = e^{-\left(\frac{\alpha}{x} + \frac{\theta}{x^2}\right)},$$

for α, θ, > 0 and $x > 0$.

7. **Inverse Rayleigh Distribution**

When $\alpha = 0$ and $\lambda = c = d = 1$, the NEGMIR reduces to the inverse Rayleigh (IR) distribution with CDF:

$$G(x) = e^{-\frac{\theta}{x^2}},$$

for $\theta, > 0$ and $x > 0$.

8. **Inverse Exponential Distribution**

When $\theta = 0$ and $\lambda = c = d = 1$, the NEGMIR reduces to the inverse exponential (IE) distribution with CDF:

$$G(x) = e^{-\frac{\alpha}{x}},$$

for $\alpha, > 0$ and $x > 0$.

A summary of the various sub-models of the NEGMIR distribution are given in Table 5.1.

Table 5.1: **Summary of sub-models from the NEGMIR distribution**

Distribution	λ	α	θ	c	d
EGMIR	1	α	θ	c	d
EGEIR	λ	0	θ	c	d
EGIR	1	0	θ	c	d
EGEIE	λ	α	0	c	d
EGIE	1	α	0	c	d
MIR	1	α	θ	1	1
IR	1	0	θ	1	1
IE	1	α	0	1	1

5.4 Statistical Properties

In this section, the quantile, moments, moment generating function, incomplete moment, mean deviation, median deviation, inequality measures, reliability measure, entropy and order statistics were derived. Apart from the quantile function, all other statistical properties were derived using the parameter conditions $\alpha > 0$, $\theta > 0$, $\lambda > 0$, $c > 0$ and $d > 0$.

5.4.1 Quantile Function

In order to simulate random samples from the NEGMIR distribution, it is important to develop its quantile function.

Lemma 5.2. The quantile function of the NEGMIR distribution for $p \in (0,\,1)$ is

$$Q_X(p) = \begin{cases} \dfrac{2\theta}{-\alpha + \sqrt{\alpha^2 - 4\theta \log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}, & \alpha > 0,\ \theta > 0, \\ \sqrt{\dfrac{\theta}{-\log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}, & \alpha = 0,\ \theta > 0, \\ \dfrac{\alpha}{-\log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}, & \alpha > 0,\ \theta = 0. \end{cases} \tag{5.10}$$

For the case of $\alpha > 0$ and $\theta > 0$, the proof of the quantile is as follows.

Proof. By definition, the quantile function is given by

$$G(x_p) = \mathbb{P}(X \leq x_p) = p.$$

Hence,

$$\frac{\theta}{x^2} + \frac{\alpha}{x} + \log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\} = 0. \tag{5.11}$$

Letting $x_p = Q_X(p)$ in equation (5.11) and solving for $Q_X(p)$ gives

$$Q_X(p) = \frac{2\theta}{-\alpha + \sqrt{\alpha^2 - 4\theta \log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}.$$

Putting $p = 0.25$, 0.5 and 0.75, gives the first quartile, the median and the third quartile of the NGMIR distribution respectively. Since the quantile of the NEGMIR distribution is tractable, random observations from the distribution can easily be simulated using the relation

$$x_p = \frac{2\theta}{-\alpha + \sqrt{\alpha^2 - 4\theta \log\left\{1 - \left[1 - \left(1 - (1-p)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}}}, \ \alpha > 0, \ \theta > 0.$$

5.4.2 Moments

Proposition 5.1. The r^{th} non-central moment of the NEGMIR distribution is given by

$$\mu_r^{'} = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{*} \left[\Gamma(2m - r + 1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m - r + 2)\right], \qquad (5.12)$$

where r=1,2,... and

$$\xi_{ijkm}^{*} = \frac{(-1)^{i+j+k+m}\,(k+1)^{r-m-1}\,\theta^m\,\alpha^{r-2m}\,\Gamma(\lambda)\,\Gamma(c(i+1))\,\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,\Gamma(\lambda - i)\,\Gamma(c(i+1) - j)\,\Gamma(d(j+1) - k)}.$$

Proof. By definition

$$
\begin{aligned}
\mu_r^{'} &= \int_0^\infty x^r \text{g}(x)dx \\
&= \int_0^\infty x^r \lambda cd \left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3}\right) \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm}\, x^{-2m} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx \\
&= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm} \int_0^\infty x^r \left(\frac{\alpha}{x^2} + \frac{2\theta}{x^3}\right) x^{-2m} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx \\
&= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm} \int_0^\infty \left(\alpha x^{r-2m-2} + 2\theta x^{r-2m-3}\right) e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx \\
&= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm} \left[\int_0^\infty \alpha x^{r-2m-2} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx + \int_0^\infty 2\theta x^{r-2m-3} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx\right]
\end{aligned}
$$

Letting $y = \frac{\alpha(k+1)}{x}$ implies that if $x \to 0$, $y \to \infty$ and if $x \to \infty$, $y \to 0$. Also, $x = \frac{\alpha(k+1)}{y}$ and $dx = -\frac{x^2 dy}{\alpha(k+1)}$. Using the identity $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$,

$$
\begin{aligned}
\mu_r^{'} &= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm} \times \\
&\left[\int_0^\infty \frac{1}{(k+1)} \left(\frac{\alpha(k+1)}{y}\right)^{r-2m} e^{-y} dy + \int_0^\infty \frac{2\theta}{\alpha(k+1)} \left(\frac{\alpha(k+1)}{y}\right)^{r-2m-1} e^{-y} dy\right] \\
&= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm} \times \\
&\left[\alpha^{r-2m}(k+1)^{r-2m-1}\Gamma(2m-r+1) + 2\theta\alpha^{r-2m-2}(k+1)^{r-2m-2}\Gamma(2m-r+2)\right] \\
&= \lambda cd \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{m=0}^\infty \xi_{ijkm}^{*} \left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right].
\end{aligned}
$$

The values for the first six moments of the NEGMIR distribution for selected values of the parameters are displayed in Table 5.2. The values for the first six moments are obtained using numerical integration. The following parameter values were used for the computation. I : $\lambda = 8.4$, $\alpha = 11.6$, $\theta = 1.2$, $c = 15.0$, $d = 7.5$; II : $\lambda = 3.7$, $\alpha = 4.28$, $\theta = 10.58$, $c = 5.0$, $d = 3.25$, and III : $\lambda = 4.5$, $\alpha = 10.36$, $\theta = 5.1$, $c = 14.04$, $d = 10.85$.

Table 5.2: **First six moments of NEGMIR distribution**

r	I	II	III
$\mu_1^{'}$	7.79800	5.54376	6.386897
$\mu_2^{'}$	61.74888	32.40517	41.401826
$\mu_3^{'}$	496.43131	200.51399	272.412977
$\mu_4^{'}$	4051.39777	1320.04281	1819.550601
$\mu_5^{'}$	33558.88419	9307.36739	12339.260225
$\mu_6^{'}$	282109.99686	70922.61523	84971.961184

5.4.3 Moment Generating Function

Proposition 5.2. The MGF of the NEGMIR distribution is

$$M_X(z) = \lambda cd \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{**} \left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right], \tag{5.13}$$

where

$$\xi_{ijkm}^{**} = \frac{(-1)^{i+j+k+m}\, z^r\, (k+1)^{r-m-1}\, \theta^m\, \alpha^{r-2m}\, \Gamma(\lambda)\, \Gamma(c(i+1))\, \Gamma(d(j+1))}{i!\, j!\, k!\, m!\, r!\, \Gamma(\lambda-i)\, \Gamma(c(i+1)-j)\, \Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned} M_X(z) &= \int_0^{\infty} e^{zx} \mathrm{g}(x) dx \\ &= \sum_{r=0}^{\infty} \frac{z^r}{r!} \int_0^{\infty} x^r \mathrm{g}(x) dx \\ &= \lambda cd \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{**} \left[\Gamma(2m-r+1) + \frac{2\theta}{\alpha^2(k+1)}\Gamma(2m-r+2)\right]. \end{aligned}$$

Note that the following series expansion $e^{zx} = \sum_{r=0}^{\infty} \frac{z^r x^r}{r!}$ was employed in the proof.

5.4.4 Incomplete Moment

Proposition 5.3. The r^{th} incomplete moment of the NEGMIR distribution is

$$M_r(x) = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^{*}_{ijkm} \times$$
$$\left[\Gamma\left(2m - r + 1,\, \frac{\alpha(k+1)}{x}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m - r + 2,\, \frac{\alpha(k+1)}{x}\right)\right], \qquad (5.14)$$

where $r = 1, 2, \ldots$ and $\Gamma(s,\, q) = \int_q^\infty u^{s-1}e^{-u}du$ is the lower incomplete gamma function.

Proof. Using the definition of incomplete moment of a random variable and the approach for proving the moment of the NEGMIR distribution,

$$\begin{aligned}
M_r(x) &= E(X^r|X \le x)\\
&= \int_0^x u^r \mathrm{g}(u)du\\
&= \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{x}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm} \left[\int_0^\infty \alpha u^{r-2m-2} e^{-(k+1)\left(\frac{\alpha}{u}\right)}du + \int_0^\infty 2\theta u^{r-2m-3} e^{-(k+1)\left(\frac{\alpha}{u}\right)}du\right]\\
&= \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm} \times\\
&\left[\int_{\frac{\alpha(k+1)}{x}}^\infty \frac{1}{(k+1)}\left(\frac{\alpha(k+1)}{y}\right)^{r-2m} e^{-y}dy + \int_{\frac{\alpha(k+1)}{x}}^\infty \frac{2\theta}{\alpha(k+1)}\left(\frac{\alpha(k+1)}{y}\right)^{r-2m-1} e^{-y}dy\right]\\
&= \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^{*}_{ijkm} \times\\
&\left[\Gamma\left(2m - r + 1,\, \frac{\alpha(k+1)}{x}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m - r + 2,\, \frac{\alpha(k+1)}{x}\right)\right].
\end{aligned}$$

5.4.5 Mean and Median Deviations

Proposition 5.4. The mean deviation of a random variable X having the NEGMIR distribution is

$$\delta_1(x) = 2\mu G(\mu) - 2\lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^{*1}_{ijkm} \left[\Gamma\left(2m, \frac{\alpha(k+1)}{\mu}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{\mu}\right)\right], \tag{5.15}$$

where $\mu = \mu_1^{'}$ is the mean of X and

$$\xi^{*1}_{ijkm} = \frac{(-1)^{i+j+k+m}\,(k+1)^{-m}\,\theta^m\,\alpha^{1-2m}\,\Gamma(\lambda)\,\Gamma(c(i+1))\,\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,\Gamma(\lambda-i)\,\Gamma(c(i+1)-j)\,\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned}
\delta_1(x) &= \int_0^{\infty} |x-\mu|\,\mathrm{g}(x)dx \\
&= \int_0^{\mu} (\mu - x)\mathrm{g}(x)dx + \int_{\mu}^{\infty} (x-\mu)\mathrm{g}(x)dx \\
&= 2\mu G(\mu) - 2\int_0^{\mu} x\mathrm{g}(x)dx \\
&= 2\mu G(\mu) - 2\lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^{*1}_{ijkm} \left[\Gamma\left(2m, \frac{\alpha(k+1)}{\mu}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{\mu}\right)\right],
\end{aligned}$$

where $\int_0^{\mu} x\mathrm{g}(x)dx$ is simplified using the first incomplete moment.

Proposition 5.5. The median deviation of a random variable X having the NEGMIR distribution is

$$\delta_2(x) = \mu - 2\lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^{*1}_{ijkm} \left[\Gamma\left(2m, \frac{\alpha(k+1)}{M}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{M}\right)\right], \tag{5.16}$$

where M is the median of X.

Proof. By definition

$$\begin{aligned}
\delta_2(x) &= \int_0^\infty |x - M|\, \mathrm{g}(x)dx \\
&= \int_0^M (M - x)\mathrm{g}(x)dx + \int_M^\infty (x - M)\mathrm{g}(x)dx \\
&= \mu - 2\int_0^M x\mathrm{g}(x)dx \\
&= \mu - 2\lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{*1} \left[\Gamma\left(2m, \frac{\alpha(k+1)}{M}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{M}\right)\right],
\end{aligned}$$

where $\int_0^M x\mathrm{g}(x)dx$ is simplified using the first incomplete moment.

5.4.6 Inequality Measures

Proposition 5.6. The Lorenz curve, $L_G(x)$ is given by

$$L_G(x) = \frac{\lambda cd}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{*} \left[\Gamma\left(2m, \frac{\alpha(k+1)}{x}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{x}\right)\right]. \tag{5.17}$$

Proof. By definition

$$\begin{aligned}
L_G(x) &= \frac{1}{\mu}\int_0^x u\mathrm{g}(u)du \\
&= \frac{\lambda cd}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi_{ijkm}^{*} \left[\Gamma\left(2m, \frac{\alpha(k+1)}{x}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{x}\right)\right].
\end{aligned}$$

Proposition 5.7. The Bonferroni curve, $B_G(x)$ is given by

$$B_G(x) = \frac{\lambda cd}{\mu G(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^*_{ijkm} \times$$
$$\left[\Gamma\left(2m, \frac{\alpha(k+1)}{x}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{x}\right)\right]. \qquad (5.18)$$

Proof. By definition

$$\begin{aligned} B_G(x) &= \frac{L_G(x)}{G(x)} \\ &= \frac{\lambda cd}{\mu G(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \xi^*_{ijkm} \times \\ &\left[\Gamma\left(2m, \frac{\alpha(k+1)}{x}\right) + \frac{2\theta}{\alpha^2(k+1)}\Gamma\left(2m+1, \frac{\alpha(k+1)}{x}\right)\right]. \end{aligned}$$

5.4.7 Entropy

In this subsection, the Rényi entropy (Rényi, 1961) of the NEGMIR random variable X was derived .

Proposition 5.8. The Rényi entropy of a random variable X having the NEGMIR distribution is

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[(\alpha\lambda cd)^{\delta} \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijkmn} \frac{\Gamma(2(\delta+m)+n-1)}{[\alpha(\delta+k)]^{2(\delta+m)+n-1}}\right], \qquad (5.19)$$

where $\delta \neq 1$, $\delta > 0$ and

$$\zeta_{ijkmn} = \frac{(-1)^{i+j+k+m}\theta^m(\delta+k)^m\left(\frac{2\theta}{\alpha}\right)^m\Gamma(\delta+1)\Gamma(\delta(\lambda-1)+1)\Gamma(c(\delta+i)-\delta+1)\Gamma(d(\delta+j)-\delta+1)}{i!j!k!m!\Gamma(\delta-n+1)\Gamma(\delta(\lambda-1)-i+1)\Gamma(c(\delta+i)-\delta-j+1)\Gamma(d(\delta+j)-\delta-k+1)}.$$

Proof. The Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \log \left[\int_0^\infty \mathrm{g}^\delta(x) dx \right], \; \delta \neq 1, \; \delta > 0.$$

Using the same method for expanding the density,

$$\mathrm{g}^\delta(x) = (\alpha\lambda cd)^\delta \sum_{i=}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \zeta_{ijkmn} \left(\frac{1}{x}\right)^{2(\delta+m)+n} e^{-(\delta+k)\left(\frac{\alpha}{x}\right)}.$$

Hence,

$$I_R(\delta) = \frac{1}{1-\delta} \log \left[(\alpha\lambda cd)^\delta \sum_{i=}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \zeta_{ijkmn} \int_0^\infty \left(\frac{1}{x}\right)^{2(\delta+m)+n} e^{-(\delta+k)\left(\frac{\alpha}{x}\right)} dx \right].$$

Letting $y = \frac{\alpha(\delta+k)}{x}$, when $x \to 0$, $y \to \infty$ and when $x \to \infty$, $y \to 0$. Also, $\frac{1}{x} = \frac{y}{\alpha(\delta+k)}$ and $dx = \frac{-x^2 dy}{\alpha(\delta+k)}$. Thus,

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log \left[(\alpha\lambda cd)^\delta \sum_{i=}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \zeta_{ijkmn} \int_0^\infty \frac{y^{2(\delta+m)+n-2}}{[\alpha(\delta+k)]^{2(\delta+m)+n-1}} e^{-y} dy \right] \\ &= \frac{1}{1-\delta} \log \left[(\alpha\lambda cd)^\delta \sum_{i=}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \zeta_{ijkmn} \frac{\Gamma(2(\delta+m)+n-1)}{[\alpha(\delta+k)]^{2(\delta+m)+n-1}} \right], \end{aligned}$$

where $\delta \neq 1$ and $\delta > 0$.

The Rényi entropy tends to Shannon entropy as $\delta \to 1$.

5.4.8 Stress-Strength Reliability

Proposition 5.9. If X_1 is the strength of a component and X_2 is the stress, such that both follow the NEGMIR distribution with the same parameters, then the stress-strength reliability is given by

$$R = 1 - \alpha\lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\frac{\Gamma(2(m+1)-1)}{[\alpha(k+1)]^{2(m+1)-1}} + \frac{2\theta\,\Gamma(2(m+1))}{\alpha\,[\alpha(k+1)]^{2(m+1)}} \right], \quad (5.20)$$

where

$$\varsigma_{ijkm} = \frac{(-1)^{i+j+k+m}\,\theta^m\,(k+1)^m\,\Gamma(\lambda+1)\,\Gamma(c(i+1))\,\Gamma(d(j+1))}{i!\,j!\,k!\,m!\,\Gamma(\lambda-i+1)\,\Gamma(c(i+1)-j)\,\Gamma(d(j+1)-k)}.$$

Proof. By definition

$$\begin{aligned}
R &= \mathbb{P}(X_2 < X_1) \\
&= \int_0^{\infty} \mathrm{g}(x)G(x)dx \\
&= 1 - \int_0^{\infty} \mathrm{g}(x)S(x)dx \\
&= 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\int_0^{\infty} \alpha x^{-2(m+1)} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx + \int_0^{\infty} 2\theta x^{-(3+2m)} e^{-(k+1)\left(\frac{\alpha}{x}\right)} dx \right]
\end{aligned}$$

Letting $y = \frac{\alpha(k+1)}{x}$, when $x \to 0$, $y \to \infty$ and when $x \to \infty$, $y \to 0$. Also, $x = \frac{\alpha(k+1)}{y}$ and $dx = \frac{-x^2 dy}{\alpha(k+1)}$. Thus

$$\begin{aligned}
R &= 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\int_0^{\infty} \frac{\alpha y^{2(m+1)-2}}{[\alpha(k+1)]^{2(m+1)-1}} e^{-y} dy + \int_0^{\infty} \frac{2\theta y^{2m+1}}{[\alpha(k+1)]^{2(m+1)}} e^{-y} dy \right] \\
&= 1 - \lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\frac{\alpha\Gamma(2(m+1)-1)}{[\alpha(k+1)]^{2(m+1)-1}} + \frac{2\theta\Gamma(2(m+1))}{[\alpha(k+1)]^{2(m+1)}} \right] \\
&= 1 - \alpha\lambda cd \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \varsigma_{ijkm} \left[\frac{\Gamma(2(m+1)-1)}{[\alpha(k+1)]^{2(m+1)-1}} + \frac{2\theta\,\Gamma(2(m+1))}{\alpha\,[\alpha(k+1)]^{2(m+1)}} \right].
\end{aligned}$$

5.4.9 Order Statistics

Order statistics have a very useful role in statistics and probability. Hence, in this section the p^{th} order statistics of the NEGMIR distribution was derived. Suppose $X_1,\, X_2,\, \ldots,\, X_n$ is a random sample having the NEGMIR distribution and $X_{1:\,n} < X_{2:\,n} < \ldots < X_{n:\,n}$ are order statistics obtained from the sample. The PDF, $\mathrm{g}_{p:\,n}(x)$, of the p^{th} order statistic $X_{p:\,n}$ is

$$\mathrm{g}_{p:n}(x) = \frac{1}{B(p,\, n-p+1)} \sum_{l=0}^{n-p} (-1)^l \binom{n-p}{l} \left[G(x)\right]^{p+l-1} \mathrm{g}(x). \tag{5.21}$$

Substituting the CDF and PDF of the NEGMIR distribution into equation (5.21) gives

$$\mathrm{g}_{p:n}(x) = \sum_{l=0}^{n-p} \sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\, (p+l-1)!}{l!\, (m+1)!\, (p-1)!\, (n-p-l)!\, (p+l-m-1)!} \mathrm{g}(x; \alpha,\, \lambda_{m+1},\, \theta,\, c,\, d), \tag{5.22}$$

where $\mathrm{g}(x; \alpha,\, \lambda_{m+1},\, \theta,\, c,\, d)$ is the PDF of the NEGMIR distribution with parameters $\alpha,\, \theta,\, c,\, d$ and $\lambda_{m+1} = \lambda(m+1)$. It is clear that the density of the p^{th} order statistic given in equation (5.22) is a weighted function of the NEGMIR distribution with different shape parameters.

Proposition 5.10. The r^{th} non-central moment of the p^{th} order statistic is given by

$$\mu_r^{\prime (p:n)} = \lambda c d \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \sum_{l=0}^{n-p} \sum_{m=0}^{p+l-1} \varphi_{ijklmq} \left[\Gamma(2q-r+1) + \frac{2\theta\, \Gamma(2q-r+2)}{\alpha^2 (k+1)} \right], \tag{5.23}$$

where $r = 1, 2, \ldots$ and

$$\varphi_{ijklmq} = \frac{(-1)^{i+j+k+l+m+q} (k+1)^{r-q-1} \theta^q \alpha^{r-2q} \Gamma(n+1) \Gamma(p+l) \Gamma(\lambda(m+1)) \Gamma(c(i+1)) \Gamma(d(j+1))}{i! j! k! l! m! q! (p-1)! (n-p-l)! \Gamma(p+l-m) \Gamma(\lambda(m+1)-i) \Gamma(c(i+1)-j) \Gamma(d(j+1)-k)}.$$

Proof. By definition

$$
\begin{aligned}
\mu_r^{\prime(p:n)} &= \int_0^\infty x^r \mathrm{g}_{p:n}(x)dx \\
&= \int_0^\infty x^r \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\,(p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} \mathrm{g}(x;\alpha,\,\lambda_{m+1},\,\theta,\,c,\,d)dx \\
&= \sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \frac{(-1)^{l+m}\, n!\,(p+l-1)!}{l!\,(m+1)!\,(p-1)!\,(n-p-l)!\,(p+l-m-1)!} \int_0^\infty x^r \mathrm{g}(x;\alpha,\,\lambda_{m+1},\,\theta,\,c,\,d)dx.
\end{aligned}
$$

Employing the same method for deriving the non-central moment,

$$
\mu_r^{\prime(p:n)} = \lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{n-p}\sum_{m=0}^{p+l-1} \varphi_{ijklmq} \left[\Gamma(2q-r+1) + \frac{2\theta\,\Gamma(2q-r+2)}{\alpha^2(k+1)}\right].
$$

5.5 Parameter Estimation

In this section, the estimation of the unknown parameter vector $\boldsymbol{\vartheta} = (\lambda,\,\alpha,\,\theta,\,c,\,d)'$ using the method of maximum likelihood estimation was carried out. Let $X_1,\,X_2,\ldots,\,X_n$ be a random sample of size n from NEGMIR distribution. Let $z_i = e^{-\left(\frac{\alpha}{x_i}+\frac{\theta}{x_i^2}\right)}$ and $\bar{z}_i = 1 - e^{-\left(\frac{\alpha}{x_i}+\frac{\theta}{x_i^2}\right)}$, then the log-likelihood function is given by

$$
\begin{aligned}
\ell =& n\log(cd\lambda) + (d-1)\sum_{i=1}^{n}\log(\bar{z}_i) + (c-1)\sum_{i=1}^{n}\log(1-\bar{z}_i^d) + (\lambda-1)\sum_{i=1}^{n}\log\left[1-(1-\bar{z}_i^d)^c\right] \\
&+ \sum_{i=1}^{n}\log\left(\frac{\alpha}{x_i^2}+\frac{\theta}{x_i^3}\right) - \sum_{i=1}^{n}\left(\frac{\alpha}{x_i}+\frac{\theta}{x_i^2}\right). \qquad (5.24)
\end{aligned}
$$

By differentiating the log-likelihood function with respect to the parameters λ, c, d, α and θ, the score functions are obtained as:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left[1 - (1 - \bar{z}_i^d)^c\right], \tag{5.25}$$

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \log\left(1 - \bar{z}_i^d\right) - (\lambda - 1)\sum_{i=1}^{n} \frac{(1 - \bar{z}_i^d)^c \log(1 - \bar{z}_i^d)}{1 - (1 - \bar{z}_i^d)^c}, \tag{5.26}$$

$$\frac{\partial \ell}{\partial d} = \frac{n}{d} + \sum_{i=1}^{n} \log(\bar{z}_i) - (c - 1)\sum_{i=1}^{n} \frac{\bar{z}_i^d \log(\bar{z}_i)}{1 - \bar{z}_i^d} + (\lambda - 1)\sum_{i=1}^{n} \frac{c\bar{z}_i^d (1 - \bar{z}_i^d)^{c-1} \log(\bar{z}_i)}{1 - (1 - \bar{z}_i^d)^c}, \tag{5.27}$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{x_i^2\left(\frac{\alpha}{x_i^2} + \frac{\theta}{x_i^3}\right)} - \sum_{i=1}^{n} \frac{1}{x_i} + (d - 1)\sum_{i=1}^{n} \frac{z_i}{x_i \bar{z}_i} - (c - 1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i(1 - \bar{z}_i^d)} + (\lambda - 1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1 - \bar{z}_i^d)^{c-1}}{x_i\left[1 - (1 - \bar{z}_i^d)^c\right]}, \tag{5.28}$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n} \frac{1}{x_i^3\left(\frac{\alpha}{x_i^2} + \frac{\theta}{x_i^3}\right)} - \sum_{i=1}^{n} \frac{1}{x_i^2} + (d - 1)\sum_{i=1}^{n} \frac{z_i}{x_i^2 \bar{z}_i} - (c - 1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^2(1 - \bar{z}_i^d)} + (\lambda - 1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1}(1 - \bar{z}_i^d)^{c-1}}{x_i^2\left[1 - (1 - \bar{z}_i^d)^c\right]}. \tag{5.29}$$

Equating the score functions to zero and solving for the unknown parameters in the system of nonlinear equations numerically yields the maximum likelihood estimates of the parameters. For the purpose of constructing confidence intervals for the parameters, the observed information matrix $J(\boldsymbol{\vartheta})$ is used due to the complex nature of the expected information matrix. The observed information matrix is given by

$$J(\boldsymbol{\vartheta}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial c} & \frac{\partial^2 \ell}{\partial \lambda \partial d} & \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda \partial \theta} \\ & \frac{\partial^2 \ell}{\partial c^2} & \frac{\partial^2 \ell}{\partial c \partial d} & \frac{\partial^2 \ell}{\partial c \partial \alpha} & \frac{\partial^2 \ell}{\partial c \partial \theta} \\ & & \frac{\partial^2 \ell}{\partial d^2} & \frac{\partial^2 \ell}{\partial d \partial \alpha} & \frac{\partial^2 \ell}{\partial d \partial \theta} \\ & & & \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \theta} \\ & & & & \frac{\partial^2 \ell}{\partial \theta^2} \end{bmatrix}.$$

The elements of the observed information matrix are given in Appendix A2. When the usual regularity conditions are satisfied and that the parameters are within the interior of the parameter space, but not on the boundary, the distribution of $\sqrt{n}(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta})$ converges to the multivariate normal distribution $N_5(\mathbf{0}, I^{-1}(\boldsymbol{\vartheta}))$, where $I(\boldsymbol{\vartheta})$ is the expected information matrix. The asymptotic behavior remains valid when $I(\boldsymbol{\vartheta})$ is replaced by the observed information matrix evaluated at $J(\hat{\boldsymbol{\vartheta}})$. The asymptotic multivariate normal distribution $N_5(\mathbf{0}, J^{-1}(\hat{\boldsymbol{\vartheta}}))$ is a very useful tool for constructing an approximate $100(1-\eta)\%$ two-sided confidence intervals for the model parameters.

5.6 Monte Carlo Simulation

In this section, the properties of the maximum likelihood estimators for the parameters of the NEGMIR distribution were examined using simulation. The AB and RMSE of the parameters were observed. The quantile function given in equation (5.10) was used to generate random samples from the NEGMIR distribution. The simulation experiment was repeated for $N = 1,000$ times each with sample sizes $n =$ 25, 50, 75, 100, 200, 300, 600 and parameter values (λ, α, θ, c, d) = (0.5, 0.1, 0.8, 0.4, 0.5) and (0.4, 0.5, 0.5, 2.5, 1.5). Figure 5.3 and 5.4 respectively shows the AB and RMSE for the maximum likelihood estimators of $(\lambda, \alpha, \theta, c, d) = (0.5, 0.1, 0.8, 0.4, 0.5)$ for $n = 25, 50, 75, 100, 200, 300, 600$. The AB for the estimators of the parameters fluctuates upward and downward as the sample size increases. However, the RMSE for the estimators of the parameters showed decreasing pattern as the sample size increases.

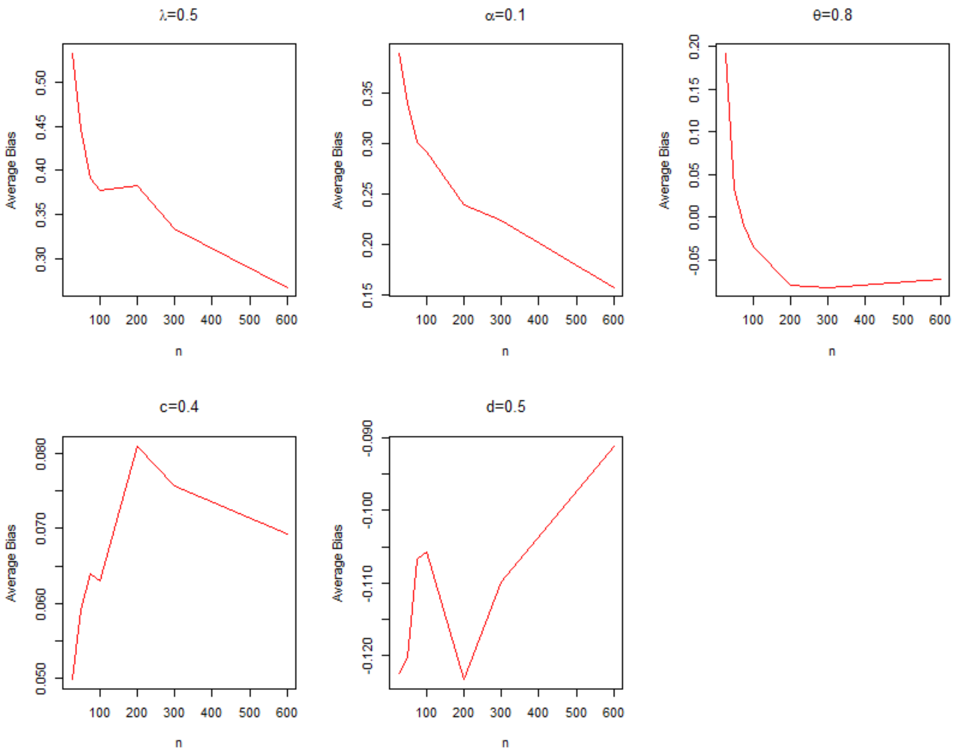

Figure 5.3: **AB for Estimators**

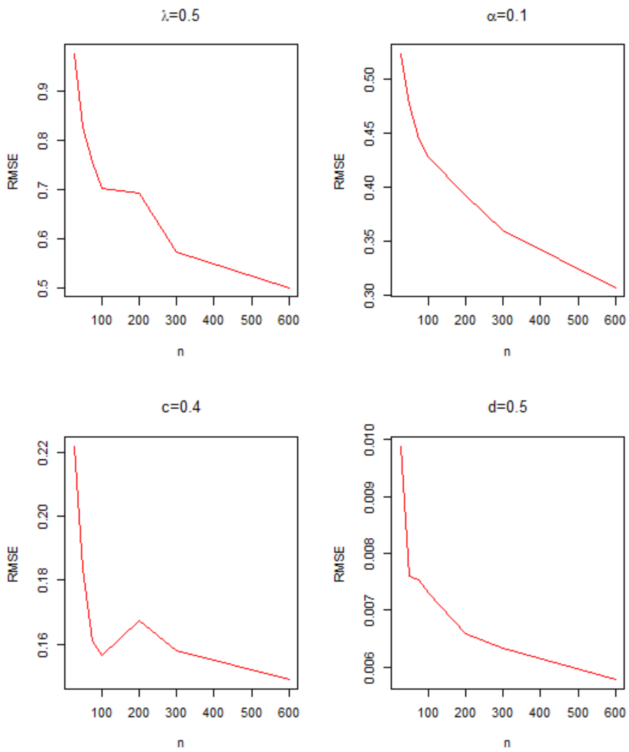

Figure 5.4: **RMSE for Estimators**

Figure 5.5 and 5.6 respectively shows the AB and RMSE for the maximum likelihood estimators of $(\lambda, \alpha, \theta, c, d) = (0.4, 0.5, 0.5, 2.5, 1.5)$ for $n = 25, 50, 75, 100, 200, 300, 600$. The AB for the estimators again exhibit an upward and downward pattern. The RMSE for the estimators decreases in general.

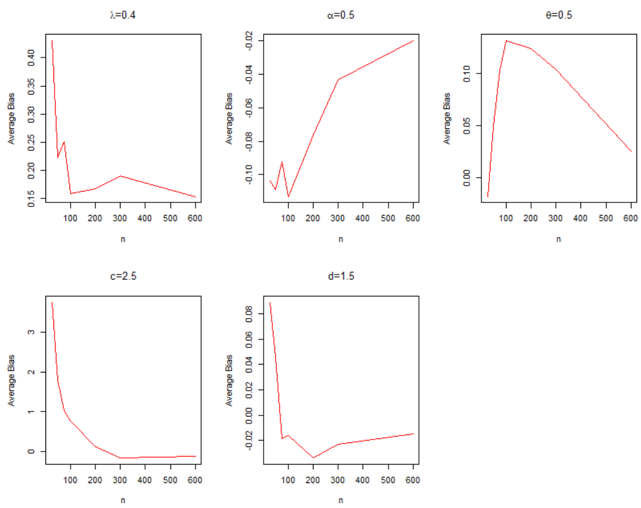

Figure 5.5: **AB for Estimators**

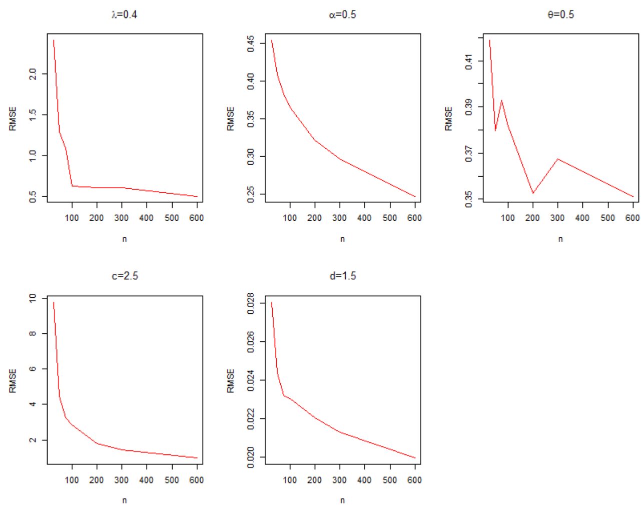

Figure 5.6: **RMSE for Estimators**

5.7 Applications

In this section, the applications of the NEGMIR distribution were demonstrated using real data sets. The goodness-of-fit of the NEGMIR distribution was compared with that of its sub-models and the new generalized inverse Weibull (NGIW) distribution. The PDF of the NGIW distribution is given by

$$\mathrm{g}(x) = \beta\left(\alpha + \eta\theta\left(\frac{1}{x}\right)^{\eta-1}\right)\left(\frac{1}{x}\right)^{2} e^{\left(-\frac{\alpha}{x}-\theta\left(\frac{1}{x}\right)^{\eta}\right)}\left(1-e^{\left(-\frac{\alpha}{x}-\theta\left(\frac{1}{x}\right)^{\eta}\right)}\right)^{\beta-1}, \; x>0, \qquad (5.30)$$

where $\eta > 0$, $\beta > 0$ are the shape parameters and $\alpha > 0$, $\theta > 0$ are scale parameters of the distribution.

5.7.1 Aircraft Data

The data comprises failure times for the air conditioning system of an aircraft from a random sample of 30 observations. The data set can be found in Linhart and Zucchini (1986) and Khan and King (2016). The data set is given in Table 5.3.

Table 5.3: **Failure times data for the air conditioning system of an aircraft**

23	261	87	7	120	14	62	47	225	71
246	21	42	20	5	12	120	11	3	14
71	11	14	11	16	90	1	16	52	95

The data set exhibit a bathtub failure rate since the TTT transform plot is first convex below the 45 degrees line and then followed by a concave shape above it as shown in Figure 5.7.

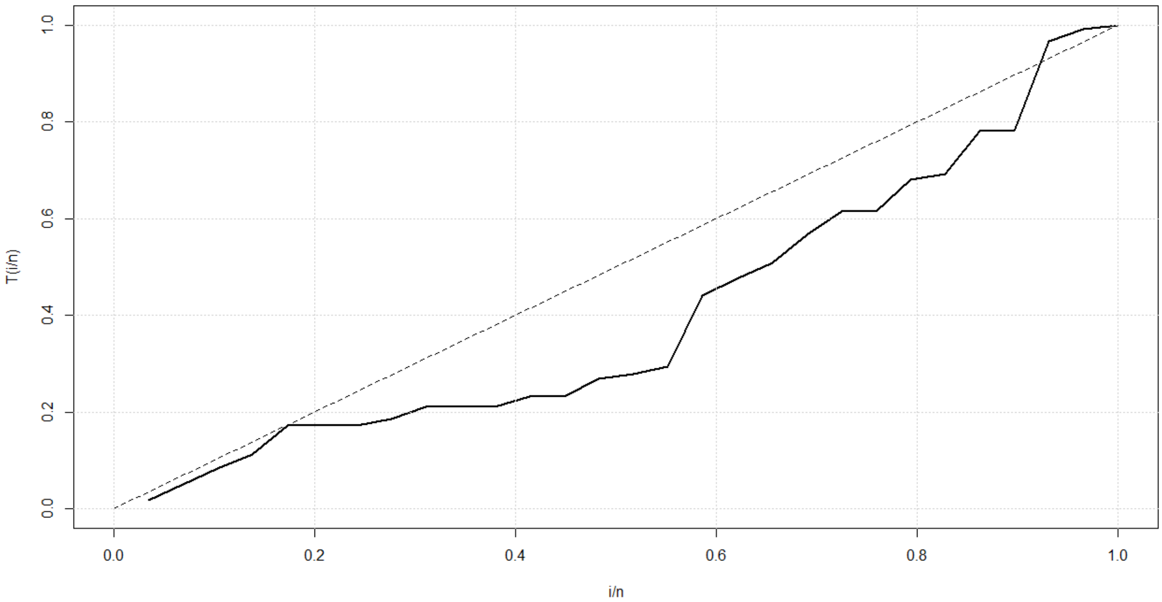

Figure 5.7: **TTT-transform plot for aircraft data**

The maximum likelihood estimates for the parameters and their corresponding standard errors in bracket are given in Table 5.4. Some of the parameters of the fitted distribution were significant at the 5% significance level. This can be confirmed using the standard error test.

Table 5.4: **Maximum likelihood estimates of parameters and standard errors for aircraft data**

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
NEGMIR	0.082	18.949	3.736	0.132	11.356
	(0.018)	(2.491)	(0.851)	(0.025)	(1.309)
EGMIR		29.072	1.569	0.326	0.674
		(12.559)	(0.566)	(0.133)	(0.153)
NEGIR	47.262		10.089	0.897	0.003
	(0.00016)		(0.0058)	(0.1642)	(0.00254)
NEGIE	0.062	1.734		13.278	6.537
	(0.016)	(0.234)		(14.581)	(1.014)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	
NGIW	7.312	0.628	0.944	150.959	
	(2.226)	(0.150)	(0.994)	(158.932)	

The NEGMIR distribution provides a better fit to the data set than its sub-models and the NGIW distribution. From Table 5.5, the NEGMIR distribution has the highest log-likelihood and the smallest K-S, W*, AIC, AICc, and BIC values compared to the other fitted models.

Table 5.5: **Log-likelihood, goodness-of-fit statistics and information criteria for aircraft data**

Model	ℓ	AIC	AICc	BIC	K-S	W*
NEGMIR	**-146.52**	**303.046**	**306.698**	**309.882**	**0.1490**	**0.0701**
EGMIR	-151.92	311.842	314.342	317.312	0.2336	0.1636
NEGIR	-158.36	324.723	327.223	330.192	0.3111	0.5359
NEGIE	-156.42	320.840	323.340	326.309	0.2816	0.5021
NGIW	-148.50	304.993	307.493	310.462	0.2270	0.1538

The LRT was performed to compare the NEGMIR distribution with its sub-models. The LRT statistics and their corresponding P-values in Table 5.6 revealed that the NEGMIR distribution provides a good fit than its sub-models.

Table 5.6: **Likelihood ratio test statistic for aircraft data**

Model	Hypotheses	LRT	P-values
EGMIR	$H_0 : \lambda = 1$ vs $H_1 : H_0$ is false	10.797	0.001
NEGIR	$H_0 : \alpha = 0$ vs $H_1 : H_0$ is false	23.677	< 0.001
NEGIE	$H_0 : \theta = 0$ vs $H_1 : H_0$ is false	19.794	< 0.001

The asymptotic variance-covariance matrix for the estimated parameters of the NEGMIR distribution is given by

$$J^{-1} = \begin{bmatrix} 3.1913 \times 10^{-4} & 1.3474 \times 10^{-2} & 8.2089 \times 10^{-4} & 6.003 \times 10^{-5} & -8.4330 \times 10^{-3} \\ 1.3474 \times 10^{-2} & 6.20503 & 1.2358 & 3.7437 \times 10^{-2} & 0.6235 \\ 8.2089 \times 10^{-4} & 1.2358 & 0.7243 & 1.4159 \times 10^{-2} & 0.4879 \\ 6.003 \times 10^{-5} & 3.7437 \times 10^{-2} & 1.4159 \times 10^{-2} & 6.3063 \times 10^{-4} & 1.0517 \times 10^{-2} \\ -8.4330 \times 10^{-3} & 0.6235 & 0.4879 & 1.0517 \times 10^{-2} & 1.7138 \end{bmatrix}.$$

Hence, the approximate 95% confidence interval for the parameters λ, α, θ, c and d are [0.0468, 0.1168], [14.0665, 23.8311], [2.0681, 5.4043], [0.0827, 0.1811] and [8.7900, 13.9218] respectively. From the estimated confidence intervals, it can be seen that none of them contains zero. Thus, the estimated parameters of the NEGMIR distribution were all significant at the 5% confidence interval. Figure 5.8 displays the empirical density and the fitted densities of the distributions.

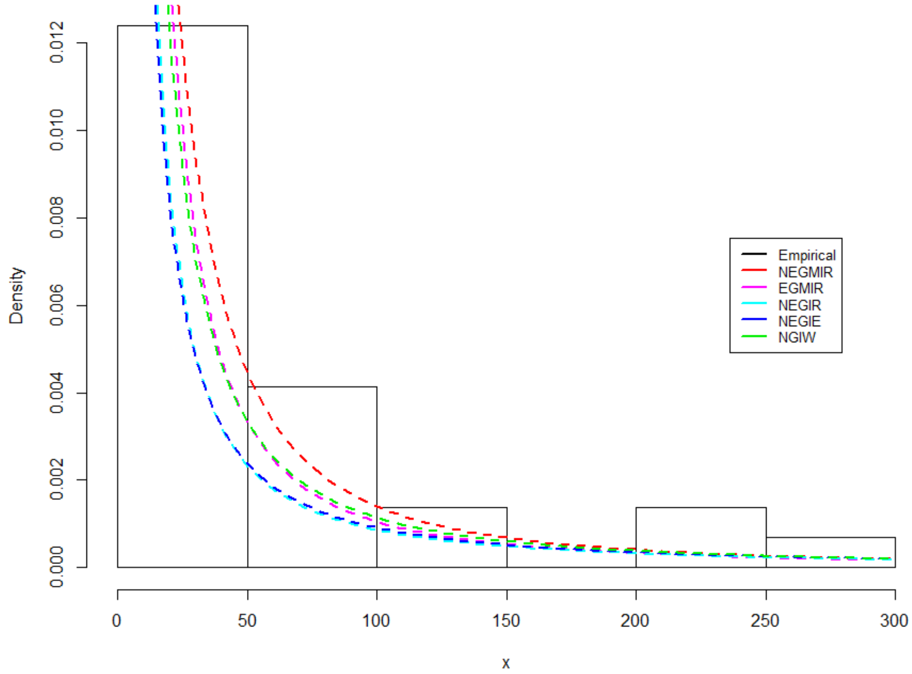

Figure 5.8: **Empirical and fitted densities plot of aircraft data**

5.7.2 Precipitation Data

The data was first reported by Hinkley (1977) and consists of 30 observations of March precipitation (in inches) in Minneapolis/ St Paul. The data set is given in Table 5.7.

Table 5.7: **March precipitation in Minneapolis/St Paul**

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20
3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81
2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

The precipitation data shows an increasing failure rate since the TTT transform plot is concave above the 45 degrees line as shown in Figure 5.9.

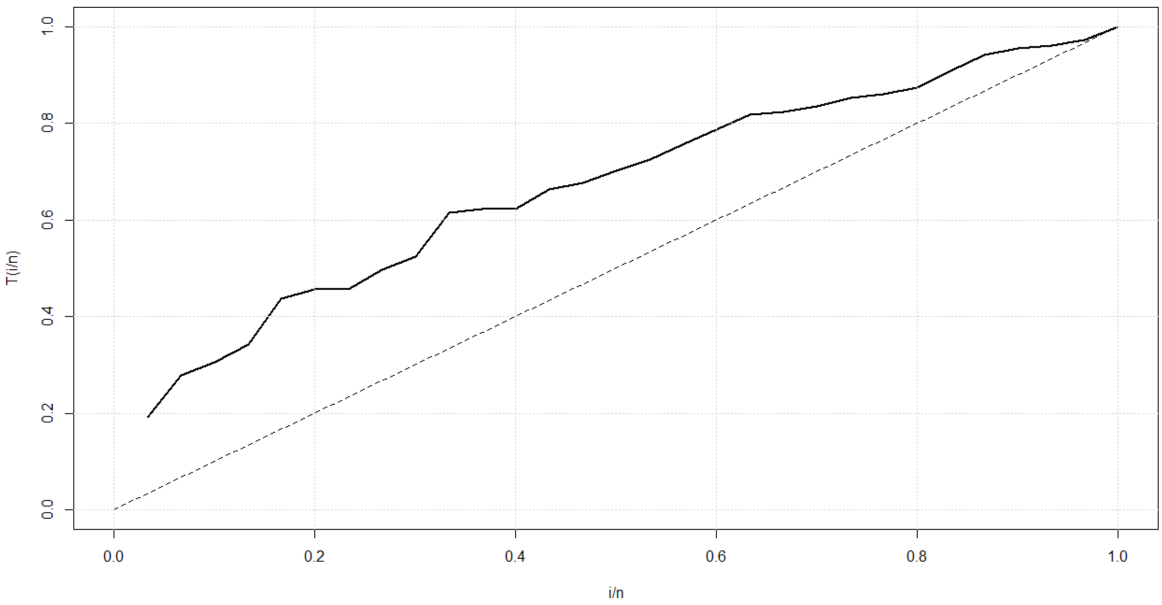

Figure 5.9: **TTT-transform plot for precipitation data**

The maximum likelihood estimates for the parameters of the fitted distributions and their corresponding standard errors in brackets are shown in Table 5.8. The NEGMIR distribution had all its parameters to be significant at the 5% significance level except d which was significant at 10%. The parameters of the EGMIR, NEGIR and NEGIE distributions were all significant. The parameters of the NGIW distribution were also significant, except θ.

Table 5.8: **Maximum likelihood estimates of parameters and standard errors for precipitation data**

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
NEGMIR	0.225	3.022	2.246	0.112	24.039
	(0.102)	(0.515)	(0.281)	(0.052)	(12.399)
EGMIR		1.658	2.918	0.235	1.877
		(0.138)	(0.355)	(0.051)	(0.146)
NEGIR	0.087		1.305	0.219	10.813
	(0.018)		(0.181)	(0.028)	(1.555)
NEGIE	8.228	9.708		0.258	0.092
	(4.261)	(2.387)		(0.086)	(0.022)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$	
NGIW	2.202	3.292	4.635×10^{-5}	5.822	
	(0.448)	(1.087)	(0.002)	(0.014)	

Table 5.9 revealed that the NEGMIR distribution provides a better fit to the precipitation data compared to its sub-models and the NGIW distribution since it has the highest log-likelihood, smallest K-S, W*, AIC, AICc and BIC values.

Table 5.9: **Log-likelihood, goodness-of-fit statistics and information criteria for precipitation data**

Model	ℓ	AIC	AICc	BIC	K-S	W*
NEGMIR	**-37.870**	**85.738**	**89.390**	**92.744**	**0.076**	**0.014**
EGMIR	-42.750	93.492	96.101	99.097	0.208	0.138
NEGIR	-40.210	88.421	91.030	94.025	0.282	0.071
NEGIE	-40.460	88.912	91.521	94.517	0.140	0.070
NGIW	-39.66	87.326	89.935	92.931	0.125	0.066

The LRT was performed to compare the NEGMIR distribution with its sub-models. The results as shown in Table 5.10 revealed the NEGMIR distribution provides a better fit to the precipitation data than its sub-models.

Table 5.10: **Likelihood ratio test statistic for precipitation data**

Model	Hypotheses	LRT	P-values
EGMIR	$H_0 : \lambda = 1$ vs $H_1 : H_0$ is false	9.754	0.002
NEGIR	$H_0 : \alpha = 0$ vs $H_1 : H_0$ is false	4.682	0.030
NEGIE	$H_0 : \theta = 0$ vs $H_1 : H_0$ is false	5.174	0.023

The estimated asymptotic variance-covariance matrix of the NEGMIR distribution for the precipitation data is given by

$$J^{-1} = \begin{bmatrix} 0.010 & 0.003 & -1.033 & 0.017 & 0.004 \\ 0.003 & 0.003 & -0.307 & -0.001 & -0.003 \\ -1.033 & -0.307 & 153.725 & 0.287 & 0.099 \\ 0.017 & -0.001 & 0.287 & 0.265 & 0.046 \\ 0.004 & -0.003 & 0.099 & 0.046 & 0.079 \end{bmatrix}.$$

The approximate 95% confidence interval for the parameters λ, α, θ, c and d are [0.025, 0.424], [2.012, 4.032], [1.696, 2.797], [0.011, 0.214] and [0, 48.340] respectively. The confidence intervals for the parameters λ, α, θ and c do not contain zero. However, the confidence interval for the parameter d contains zero. Thus, all the estimated parameters of the NEGMIR distribution were significant at the 5% significance level with the exception of the parameter d. Figure 5.10 displays the empirical density and the fitted densities of the distributions.

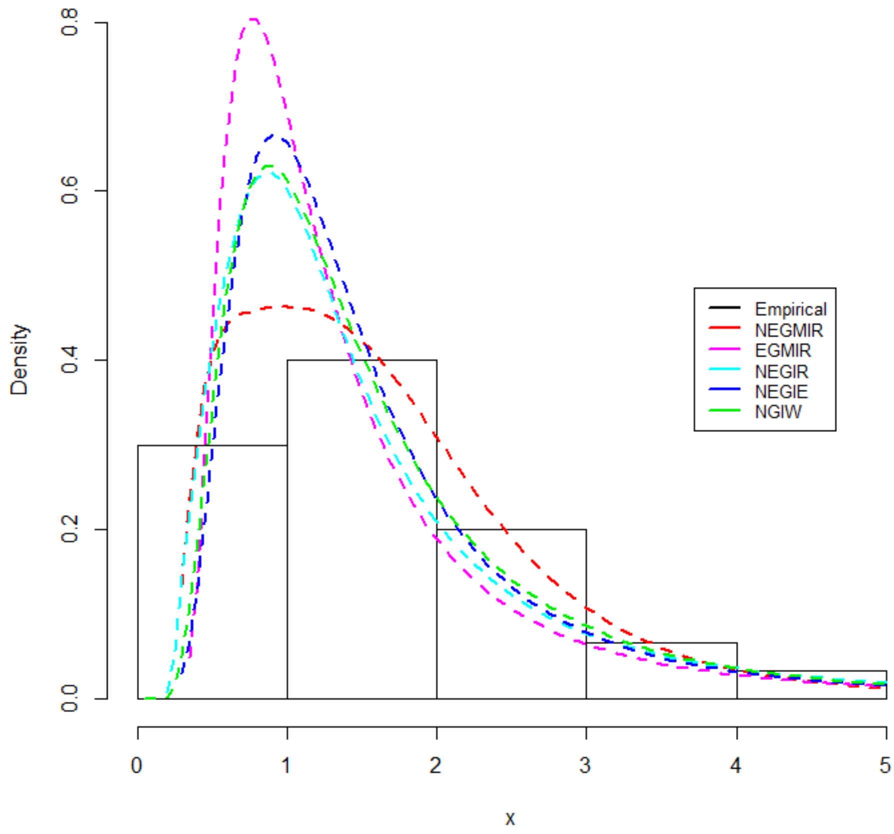

Figure 5.10: **Empirical and fitted densities plot for precipitation data**

5.8 Summary

In this chapter, a five-parameter distribution called NEGMIR distribution was proposed. The parameters of the model were estimated using the method of maximum likelihood estimation and simulation studies were performed to examine the statistical properties of the estimators. The applications of the model were demonstrated using real data sets and the empirical results showed that the NEGMIR distribution provided a better fit to the data compared to other candidate models as revealed by the various goodness-of-fit tests and model selection criteria.

CHAPTER 6

EXPONENTIATED GENERALIZED HALF LOGISTIC BURR X DISTRIBUTION

6.1 Introduction

The Burr X distribution is a member of the classical system of distributions developed by Burr (1942). The CDF of the Burr X distribution is given by

$$F(x) = (1 - e^{-(\alpha x)^2})^{\beta},\ x > 0,\ \alpha > 0,\ \beta > 0, \tag{6.1}$$

where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters respectively. In this chapter, a generalization of the Burr X distribution called the EGHL Burr X (EGHLBX) distribution was developed and studied.

6.2 Generalized Half Logistic Burr X

Suppose the random variable X has the CDF defined in equation (6.1), then the CDF of the EGHLBX distribution is given by

$$G(x) = \frac{1 - \left\{1 - \left[1 - \left(1 - \left(1 - e^{-(\alpha x)^2}\right)^{\beta}\right)^{d}\right]^{c}\right\}^{\lambda}}{1 + \left\{1 - \left[1 - \left(1 - (1 - e^{-(\alpha x)^2})^{\beta}\right)^{d}\right]^{c}\right\}^{\lambda}},\ x > 0, \tag{6.2}$$

where $\alpha > 0$ is a scale parameter and λ, β, c, $d > 0$ are shape parameters. The corresponding PDF is given by

$$\mathrm{g}(x) = \frac{4\lambda\alpha^2\beta c d x A e^{-(\alpha x)^2}\left[1-\left[1-\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\beta}\right)^{d}\right]^{c}\right]^{\lambda-1}}{\left\{1+\left[1-\left[1-\left(1-(1-e^{-(\alpha x)^2})^{\beta}\right)^{d}\right]^{c}\right]^{\lambda}\right\}^{2}},\ x > 0, \tag{6.3}$$

where

$$A = \left(1-e^{-(\alpha x)^2}\right)^{\beta-1}\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\beta}\right)^{d-1}\left[1-\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\beta}\right)^{d}\right]^{c-1}.$$

Lemma 6.1. The EGHLBX density function has a mixture representation of the form

$$\mathrm{g}(x) = 2\lambda c d \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_{ijkm} f_{BX}(x;\, \alpha,\, \beta_{m+1}),\ x > 0, \tag{6.4}$$

where $f_{BX}(x;\, \alpha,\, \beta_{m+1})$ is the PDF of the Burr X distribution with parameters α and $\beta_{m+1} = \beta(m+1)$ and

$$\omega_{ijkm} = \frac{(-1)^{i+j+k+m}\Gamma(i+2)\Gamma(\lambda(i+1))\Gamma(c(j+1))\Gamma(d(k+1))}{i!\,j!\,k!\,(m+1)!\,\Gamma(\lambda(i+1)-j)\Gamma(c(j+1)-k)\Gamma(d(k+1)-m)}.$$

Proof. For a real non-integer $\eta > 0$, the following identities hold:

$$(1-z)^{\eta-1} = \sum_{i=0}^{\infty} \frac{(-1)^i\,\Gamma(\eta)}{i!\,\Gamma(\eta-i)} z^i,\ |z| < 1, \tag{6.5}$$

and

$$(1+z)^{-\eta} = \sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(\eta+k)}{k!\,\Gamma(\eta)} z^k,\ |z| < 1. \tag{6.6}$$

Using equations (6.5) and (6.6), and the fact that $0 < (1 - e^{-(\alpha x)^2})^\beta < 1$, the PDF of the EGHLBX distribution can be written as

$$\mathrm{g}(x) = 2\lambda cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \omega_{ijkm} f_{BX}(x;\, \alpha,\, \beta_{m+1}),\; x > 0.$$

Equation (6.4) revealed that the PDF of the EGHLBX distribution can be written as a mixture of the Burr X distribution with different shape parameters. The PDF of the EGHLBX distribution can be symmetric, left skewed, right skewed, J-shape, reversed J-shape or unimodal with small and large values of skewness and kurtosis for different parameter values. Plot of PDF of the EGHLBX density function is shown in Figure 6.1.

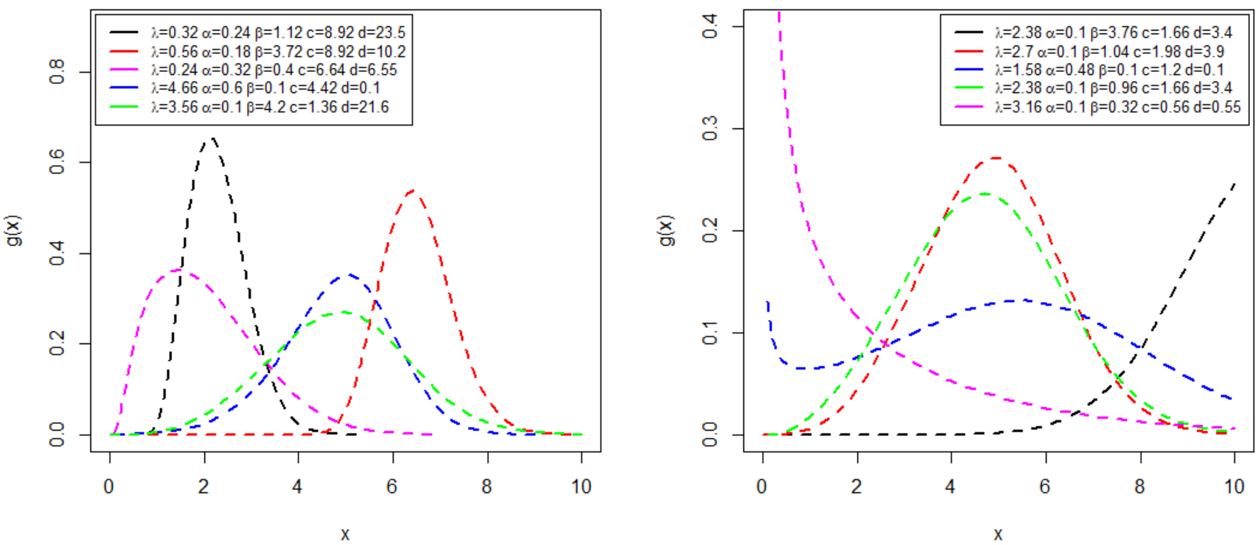

Figure 6.1: **EGHLBX density function for some parameter values**

The survival function of the EGHLBX distribution is

$$S(x) = \frac{2\left\{1 - \left[1 - \left(1 - \left(1 - e^{-(\alpha x)^2}\right)^\beta\right)^d\right]^c\right\}^\lambda}{1 + \left\{1 - \left[1 - \left(1 - (1 - e^{-(\alpha x)^2})^\beta\right)^d\right]^c\right\}^\lambda},\; x > 0, \tag{6.7}$$

and the hazard function is given by

$$\tau(x)=\frac{2\lambda\alpha^2\beta c d x A e^{-(\alpha x)^2}\left[1-\left[1-\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\beta}\right)^{d}\right]^{c}\right]^{-1}}{\left\{1+\left[1-\left[1-\left(1-(1-e^{-(\alpha x)^2})^{\beta}\right)^{d}\right]^{c}\right]^{\lambda}\right\}},\ x>0. \tag{6.8}$$

The hazard function of the EGHLBX distribution exhibit different shapes such as bathtub, monotonically increasing or monotonically decreasing for different combination of the parameter values. Figure 6.2 displays the various shapes of the hazard function of the EGHLBX distribution.

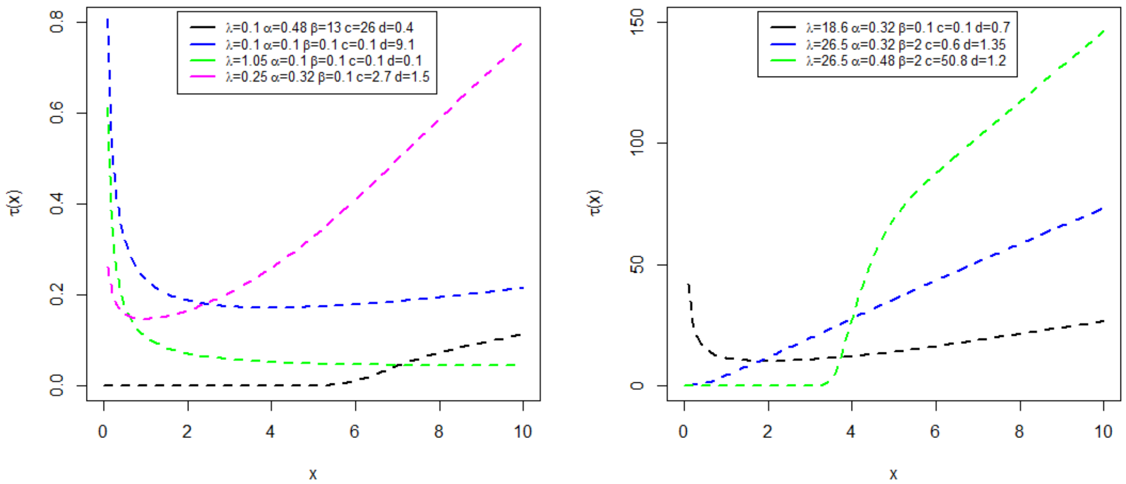

Figure 6.2: **Plots of the EGHLBX hazard function for some parameter values**

6.3 Sub-models

The sub-models of the EGHLBX distribution were discussed in this section.

1. **Exponentiated Generalized Standardized Half Logistic Burr X Distribution**

When $\lambda = 1$, the EGHLBX distribution reduces to the exponentiated generalized standardized half logistic Burr X (EGSHLBX) distribution with the following CDF:

$$G(x) = \frac{\left[1 - \left(1 - \left(1 - e^{-(\alpha x)^2}\right)^{\beta}\right)^{d}\right]^{c}}{2 - \left[1 - \left(1 - (1 - e^{-(\alpha x)^2})^{\beta}\right)^{d}\right]^{c}},$$

for α, β, c, d > 0 and $x > 0$.

2. **Exponentiated Half Logistic Burr X Distribution**

When $d = 1$, the EGHLBX distribution reduces to the exponentiated half logistic Burr X (EHLBX) distribution with the following CDF:

$$G(x) = \frac{1 - \left[1 - \left(1 - e^{-(\alpha x)^2}\right)^{c\beta}\right]^{\lambda}}{1 + \left[1 - (1 - e^{-(\alpha x)^2})^{c\beta}\right]^{\lambda}},$$

for α, β, λ, c > 0 and $x > 0$.

3. **Half Logistic Burr X Distribution**

When $c = d = 1$, the EGHLBX distribution reduces to the half logistic Burr X (HLBX) distribution with the following CDF:

$$G(x) = \frac{1 - \left[1 - \left(1 - e^{-(\alpha x)^2}\right)^{\beta}\right]^{\lambda}}{1 + \left[1 - (1 - e^{-(\alpha x)^2})^{\beta}\right]^{\lambda}},$$

for α, β, λ > 0 and $x > 0$.

4. **Standardized Half Logistic Burr X Distribution**

When $\lambda = c = d = 1$, the EGHLBX distribution reduces to the standardized half logistic Burr X (SHLBX) distribution with the following CDF:

$$G(x) = \frac{\left(1 - e^{-(\alpha x)^2}\right)^{\beta}}{2 - (1 - e^{-(\alpha x)^2})^{\beta}},$$

for α, β, > 0 and $x > 0$.

Table 6.1 shows the summary of sub-models that can be derived from the EGHLBX distribution

Table 6.1: **Summary of sub-models from the EGHLBX distribution**

Distribution	λ	α	β	c	d
EGSHLBX	1	α	β	c	d
EHLBX	λ	α	β	c	1
HLBX	λ	α	β	1	1
SHLBX	1	α	β	1	1

6.4 Statistical Properties

In this section, the quantile function, moments, MGF , incomplete moment, mean deviation, median deviation, inequality measures, reliability measure, entropy and order statistics were derived.

6.4.1 Quantile Function

The quantile function is a useful measure for describing the distribution of a random variable. It plays a key role when simulating random numbers and can also be used to compute the median, kurtosis and skewness of the distribution of a random variable.

Lemma 6.2. The quantile function of the EGHLBX distribution for $p \in (0,\ 1)$ is

$$Q_X(p) = \frac{1}{\alpha}\sqrt{-\log\left\{1 - \left[1 - \left(1 - \left(1 - \left(\frac{1-p}{1+p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)\right]^{\frac{1}{d}}\right\}^{\frac{1}{\beta}}}. \tag{6.9}$$

Proof. By definition, the quantile function is given by

$$G(x_p) = \mathbb{P}(X \leq x_p) = p.$$

Thus,

$$x_p^2 = -\frac{1}{\alpha^2}\log\left\{1 - \left[1 - \left(1 - \left(1 - \left(\frac{1-p}{1+p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)\right]^{\frac{1}{d}}\right\}^{\frac{1}{\beta}} \tag{6.10}$$

Letting $x_p = Q_X(p)$ in equation (6.10) and solving for $Q_X(p)$ gives

$$Q_X(p) = \frac{1}{\alpha}\sqrt{-\log\left\{1 - \left[1 - \left(1 - \left(1 - \left(\frac{1-p}{1+p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)\right]^{\frac{1}{d}}\right\}^{\frac{1}{\beta}}}.$$

By substituting $p = 0.25,\ 0.5$ and 0.75, the first quartile, the median and the third quartile of the EGHLBX distribution were obtained respectively. The closed form expression of the quantile function makes it easy to simulate the EGHLBX random variable using the relation

$$x_p = \frac{1}{\alpha}\sqrt{-\log\left\{1 - \left[1 - \left(1 - \left(1 - \left(\frac{1-p}{1+p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{c}}\right)\right]^{\frac{1}{d}}\right\}^{\frac{1}{\beta}}}.$$

6.4.2 Moments

Proposition 6.1. The r^{th} non-central moment of the EGHLBX distribution is given by

$$\mu_r^{'} = 2\lambda\beta cd\alpha^{-r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\omega_{ijkml}^{*}\,\Gamma(\frac{r}{2}+1)}{(l+1)^{\frac{r}{2}+1}}, \tag{6.11}$$

where $r = 1, 2, \ldots$ and

$$\omega_{ijkml}^{*} = \frac{(-1)^{i+j+k+m+l}\Gamma(i+2)\Gamma(\lambda(i+1))\Gamma(c(j+1))\Gamma(d(k+1))\Gamma(\beta(m+1))}{i!j!k!m!l!\Gamma(\lambda(i+1)-j)\Gamma(c(j+1)-k)\Gamma(d(k+1)-m)\Gamma(\beta(m+1)-l)}.$$

Proof. By definition

$$\begin{aligned}
\mu_r^{'} &= \int_0^{\infty} x^r \mathrm{g}(x)dx \\
&= \int_0^{\infty} x^r 2\lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\omega_{ijkm} f_{BX}(x;\,\alpha,\,\beta_{m+1})dx \\
&= 2\lambda cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\omega_{ijkm}\int_0^{\infty} x^r f_{BX}(x;\,\alpha,\,\beta_{m+1})dx,
\end{aligned}$$

where $f_{BX}(x;\,\alpha,\,\beta_{m+1}) = 2\alpha^2\beta_{m+1}xe^{-(\alpha x)^2}(1-e^{-(\alpha x)^2})^{\beta_{m+1}-1}$. But

$$(1-e^{-(\alpha x)^2})^{\beta_{m+1}-1} = \sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(\beta(m+1))}{l!\Gamma(\beta(m+1)-l)}e^{-l(\alpha x)^2}.$$

Hence,

$$\mu_r^{'} = 4\lambda\alpha^2\beta cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\omega_{ijkml}^{*}\int_0^{\infty} x^{r+1}e^{-(l+1)(\alpha x)^2}dx.$$

Letting $y = (l+1)(\alpha x)^2$ implies that if $x \to 0$, $y \to 0$ and if $x \to \infty$, $y \to \infty$. Also,

$x = \frac{y^{\frac{1}{2}}}{\alpha(l+1)^{\frac{1}{2}}}$ and $dx = \frac{y^{-\frac{1}{2}}}{2\alpha(l+1)^{\frac{1}{2}}}dy$. Hence,

$$\begin{aligned}
\mu_r^{'} &= 4\lambda\alpha^2\beta cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\omega_{ijkml}^{*}\int_0^{\infty}\left(\frac{y^{\frac{1}{2}}}{\alpha(l+1)^{\frac{1}{2}}}\right)^{r+1}e^{-y}\frac{dy}{2\alpha(l+1)^{\frac{1}{2}}y^{\frac{1}{2}}}\\
&= 4\lambda\alpha^2\beta cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\omega_{ijkml}^{*}\int_0^{\infty}\frac{y^{\frac{r}{2}}}{2\alpha^{r+2}(l+1)^{\frac{r}{2}+1}}e^{-y}dy\\
&= 2\lambda\beta cd\alpha^{-r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\omega_{ijkml}^{*}\,\Gamma(\frac{r}{2}+1)}{(l+1)^{\frac{r}{2}+1}}.
\end{aligned}$$

The values for the first six moments of the EGHLBX distribution for selected values of the parameters are displayed in Table 6.2. The values for the first six moments were obtained using numerical integration. The following parameter values were used for the computation. I : $\lambda = 1.3$, $\alpha = 0.2$, $\beta = 1.2$, $c = 8.9$, $d = 23.5$; II : $\lambda = 4.7$, $\alpha = 0.6$, $\beta = 3.5$, $c = 4.5$, $d = 10.5$, and III : $\lambda = 7.8$, $\alpha = 2.5$, $\beta = 3.5$, $c = 2.5$, $d = 10.3$.

Table 6.2: **First six moments of EGHLBX distribution**

r	I	II	III
$\mu_1^{'}$	2.106170	1.445147	0.300764
$\mu_2^{'}$	4.556683	2.101419	0.091317
$\mu_3^{'}$	10.113806	3.073818	0.027968
$\mu_4^{'}$	23.005018	4.521645	0.008636
$\mu_5^{'}$	53.577152	6.687550	0.002687
$\mu_6^{'}$	127.661348	9.942582	0.000842

6.4.3 Moment Generating Function

Proposition 6.2. The MGF of the EGHLBX distribution is

$$M_X(z) = 2\lambda\beta cd\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^{-r}\,z^r\,\omega_{ijkml}^{*}\,\Gamma(\frac{r}{2}+1)}{r!\,(l+1)^{\frac{r}{2}+1}}. \tag{6.12}$$

Proof. By definition

$$
\begin{aligned}
M_X(z) &= \int_0^\infty e^{zx}\mathrm{g}(x)dx \\
&= \sum_{r=0}^{\infty}\frac{z^r}{r!}\int_0^\infty x^r\mathrm{g}(x)dx \\
&= 2\lambda\beta cd\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\alpha^{-r}\,z^r\,\omega^*_{ijkml}\,\Gamma(\frac{r}{2}+1)}{r!\,(l+1)^{\frac{r}{2}+1}}.
\end{aligned}
$$

Note that the following series expansion $e^{zx}=\sum_{r=0}^{\infty}\frac{z^rx^r}{r!}$ was employed in the proof.

6.4.4 Incomplete Moment

Proposition 6.3. The r^{th} incomplete moment of the EGHLBX distribution is

$$
M_r(x) = 2\lambda\beta cd\alpha^{-r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\omega^*_{ijkml}\,\gamma(\frac{r}{2}+1,\,(l+1)(\alpha x)^2)}{(l+1)^{\frac{r}{2}+1}}, \tag{6.13}
$$

where $r=1,2,\ldots$ and $\gamma(s,\,x)=\int_0^x u^{s-1}e^{-u}du$ is the lower incomplete gamma function.

Proof. Using the definition of incomplete moment of a random variable and the approach for proving the moment of the EGHLBX distribution,

$$
\begin{aligned}
M_r(x) &= E(X^r|X\le x) \\
&= \int_0^x u^r\mathrm{g}(u)du \\
&= 4\lambda\alpha^2\beta cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\omega^*_{ijkml}\int_0^x u^{r+1}e^{-(l+1)(\alpha u)^2}du.
\end{aligned}
$$

Letting $y=(l+1)(\alpha u)^2$ implies that if $u\to 0$, $y\to 0$ and if $u\to x$, $y\to(l+1)(\alpha x)^2$. Also, $u=\frac{y^{\frac{1}{2}}}{\alpha(l+1)^{\frac{1}{2}}}$ and $du=\frac{y^{-\frac{1}{2}}}{2\alpha(l+1)^{\frac{1}{2}}}dy$. Thus,

$$
\begin{aligned}
M_r(x) &= 4\lambda\alpha^2\beta cd\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\omega^*_{ijkml}\int_0^{(l+1)(\alpha x)^2}\frac{y^{\frac{r}{2}}}{2\alpha^{r+2}(l+1)^{\frac{r}{2}+1}}e^{-y}dy\\
&= 2\lambda\beta cd\alpha^{-r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\omega^*_{ijkml}\,\gamma(\frac{r}{2}+1,\,(l+1)(\alpha x)^2)}{(l+1)^{\frac{r}{2}+1}}.
\end{aligned}
$$

6.4.5 Mean and Median Deviations

Proposition 6.4. The mean deviation of a random variable X having the EGHLBX distribution is

$$
\delta_1(x) = 2\mu G(\mu) - 4\lambda\beta cd\alpha^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha\mu)^2)}{(l+1)^{\frac{3}{2}}}, \qquad (6.14)
$$

where $\mu = \mu_1^{'}$ is the mean of X.

Proof. By definition

$$
\begin{aligned}
\delta_1(x) &= \int_0^{\infty}|x-\mu|\,\mathrm{g}(x)dx\\
&= \int_0^{\mu}(\mu-x)\mathrm{g}(x)dx + \int_{\mu}^{\infty}(x-\mu)\mathrm{g}(x)dx\\
&= 2\mu G(\mu) - 2\int_0^{\mu}x\mathrm{g}(x)dx\\
&= 2\mu G(\mu) - 4\lambda\beta cd\alpha^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha\mu)^2)}{(l+1)^{\frac{3}{2}}}
\end{aligned}
$$

where $\int_0^{\mu}x\mathrm{g}(x)dx$ is simplified using the first incomplete moment.

Proposition 6.5. The median deviation of a random variable X having the EGHLBX distribution is

$$\delta_2(x) = \mu - 4\lambda\beta cd\alpha^{-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha M)^2)}{(l+1)^{\frac{3}{2}}}, \tag{6.15}$$

where M is the median of X.

Proof. By definition

$$\begin{aligned}
\delta_2(x) &= \int_0^{\infty} |x-M|\,\mathrm{g}(x)dx \\
&= \int_0^{M} (M-x)\mathrm{g}(x)dx + \int_M^{\infty} (x-M)\mathrm{g}(x)dx \\
&= \mu - 2\int_0^{M} x\mathrm{g}(x)dx \\
&= \mu - 4\lambda\beta cd\alpha^{-1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha M)^2)}{(l+1)^{\frac{3}{2}}},
\end{aligned}$$

where $\int_0^M x\mathrm{g}(x)dx$ is simplified using the first incomplete moment.

6.4.6 Inequality Measures

Proposition 6.6. The Lorenz curve, $L_G(x)$ is given by

$$L_G(x) = \frac{2\lambda\beta cd\alpha^{-1}}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha x)^2)}{(l+1)^{\frac{3}{2}}}. \tag{6.16}$$

Proof. By definition

$$\begin{aligned}
L_G(x) &= \frac{1}{\mu}\int_0^{x} u\mathrm{g}(u)du \\
&= \frac{2\lambda\beta cd\alpha^{-1}}{\mu} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha x)^2)}{(l+1)^{\frac{3}{2}}}.
\end{aligned}$$

Proposition 6.7. The Bonferroni curve, $B_G(x)$ is given by

$$B_G(x) = \frac{2\lambda\beta cd\alpha^{-1}}{\mu G(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha x)^2)}{(l+1)^{\frac{3}{2}}}. \tag{6.17}$$

Proof. By definition

$$\begin{aligned} B_G(x) &= \frac{L_G(x)}{G(x)} \\ &= \frac{1}{\mu G(x)} \int_0^x u\mathrm{g}(u)du \\ &= \frac{2\lambda\beta cd\alpha^{-1}}{\mu G(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\omega^*_{ijkml}\,\gamma(\frac{3}{2},\,(l+1)(\alpha x)^2)}{(l+1)^{\frac{3}{2}}}. \end{aligned}$$

6.4.7 Entropy

In this subsection, the Rényi entropy of the random variable X was derived (Rényi, 1961).

Proposition 6.8. The Rényi entropy of a random variable X having the EGHLBX distribution is

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\frac{(4\lambda\beta cd)^{\delta}\alpha^{\delta-1}}{2}\sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varphi_{ijklm}\Gamma(2\delta+i)\Gamma(\delta(\lambda-1)+\lambda i+1)\Gamma(\frac{\delta+1}{2})}{(\delta+m)^{\frac{\delta+1}{2}}\Gamma(2\delta)\Gamma(\delta(\lambda-1)+\lambda i-j+1)}\right], \tag{6.18}$$

where $\delta \neq 1$, $\delta > 0$ and

$$\varphi_{ijklm} = \frac{(-1)^{i+j+k+l+m}\Gamma(\delta(c-1)+cj+1)\Gamma(\delta(d-1)+dk+1)\Gamma(\delta(\beta-1)+\beta l+1)}{i!j!k!l!m!\Gamma(\delta(c-1)+cj-k+1)\Gamma(\delta(d-1)+dk-l+1)\Gamma(\delta(\beta-1)+\beta l-m+1)}.$$

Proof. The Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^{\infty}\mathrm{g}^{\delta}(x)dx\right],\ \delta \neq 1,\ \delta > 0.$$

Using the same method for expanding the density,

$$g^{\delta}(x) = (4\lambda\beta\alpha^2 cd)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi_{ijklm}\Gamma(2\delta+i)\Gamma(\delta(\lambda-1)+\lambda i+1)x^{\delta}}{\Gamma(2\delta)\Gamma(\delta(\lambda-1)+\lambda i-j+1)} e^{-(\delta+m)(\alpha x)^2}.$$

Hence,

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[(4\lambda\beta\alpha^2 cd)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi_{ijklm}\Gamma(2\delta+i)\Gamma(\delta(\lambda-1)+\lambda i+1)}{\Gamma(2\delta)\Gamma(\delta(\lambda-1)+\lambda i-j+1)}\right] + \frac{1}{1-\delta}\log\left[\int_0^{\infty} x^{\delta} e^{-(\delta+m)(\alpha x)^2} dx\right].$$

Letting $y = (\delta+m)(\alpha x)^2$, when $x \to 0$, $y \to 0$ and when $x \to \infty$, $y \to \infty$. In addition, $x = \frac{y^{\frac{1}{2}}}{\alpha(\delta+m)^{\frac{1}{2}}}$ and $dx = \frac{y^{-\frac{1}{2}}}{2\alpha(\delta+m)^{\frac{1}{2}}}$. Thus,

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta}\log\left[(4\lambda\beta\alpha^2 cd)^{\delta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi_{ijklm}\Gamma(2\delta+i)\Gamma(\delta(\lambda-1)+\lambda i+1)}{\Gamma(2\delta)\Gamma(\delta(\lambda-1)+\lambda i-j+1)}\right] + \\ &\quad \frac{1}{1-\delta}\log\left[\int_0^{\infty} \frac{y^{\frac{\delta}{2}-\frac{1}{2}}}{2\alpha^{\delta+1}(\delta+m)^{\frac{\delta}{2}+\frac{1}{2}}} e^{-y} dy\right] \\ &= \frac{1}{1-\delta}\log\left[\frac{(4\lambda\beta cd)^{\delta}\alpha^{\delta-1}}{2} \sum_{i=}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{\varphi_{ijklm}\Gamma(2\delta+i)\Gamma(\delta(\lambda-1)+\lambda i+1)\Gamma(\frac{\delta+1}{2})}{(\delta+m)^{\frac{\delta+1}{2}}\Gamma(2\delta)\Gamma(\delta(\lambda-1)+\lambda i-j+1)}\right]. \end{aligned}$$

The Rényi entropy tends to Shannon entropy as $\delta \to 1$.

6.4.8 Stress-Strength Reliability

Proposition 6.9. If X_1 is the strength of a component and X_2 is the stress, such that both follow the EGHLBX distribution with the same parameters, then the stress-strength reliability is given by

$$R = 1 - 2\lambda\beta cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijklmn} \frac{\Gamma(n+1)}{m^{n+1}}, \tag{6.19}$$

where

$$\zeta_{ijklmn} = \frac{(-1)^{i+j+k+l+m+n}\Gamma(i+3)\Gamma(\lambda(i+2))\Gamma(c(j+1))\Gamma(d(k+1))\Gamma(\beta(l+1))}{i!j!k!l!m!n!\Gamma(\lambda(i+2)-j)\Gamma(c(j+1)-k)\Gamma(d(k+1)-l)\Gamma(\beta(l+1)-m)}$$

Proof. By definition

$$\begin{aligned} R &= \mathbb{P}(X_2 < X_1) \\ &= \int_0^{\infty} \mathrm{g}(x)G(x)dx \\ &= 1 - \int_0^{\infty} \mathrm{g}(x)S(x)dx \\ &= 1 - 4\lambda\beta cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijklmn}\alpha^{2(n+1)} \int_0^{\infty} x^{2n+1} e^{-m(\alpha x)^2} dx. \end{aligned}$$

Letting $y = m(\alpha x)^2$, when $x \to 0$, $y \to 0$ and when $x \to \infty$, $y \to \infty$. In addition, $x = \frac{y^{\frac{1}{2}}}{\alpha m^{\frac{1}{2}}}$ and $dx = \frac{y^{-\frac{1}{2}}}{2\alpha m^{\frac{1}{2}}} dy$. Hence,

$$\begin{aligned} R &= 1 - 4\lambda\beta cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijklmn}\alpha^{2(n+1)} \int_0^{\infty} \frac{y^n}{2\alpha^{2(n+1)}m^{n+1}} e^{-y} dy \\ &= 1 - 2\lambda\beta cd \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \zeta_{ijklmn} \frac{\Gamma(n+1)}{m^{n+1}}. \end{aligned}$$

6.4.9 Order Statistics

In this subsection, the order statistics of EGHLBX distribution was derived. Suppose $X_1, X_2, \ldots, X_n$ is random sample from EGHLBX and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are the corresponding order statistics. The PDF, $g_{p:\,n}(x)$, of the p^{th} order statistic $X_{p:\,n}$ is

$$g_{p:n}(x) = \frac{1}{B(p,\, n-p+1)} \sum_{s=0}^{n-p} (-1)^s \binom{n-p}{s} [G(x)]^{p+s-1} \, g(x). \tag{6.20}$$

Substituting the CDF and PDF of the EGHLBX distribution into equation (6.20) and using similar concept for expanding the density gives

$$g_{p:n}(x) = 2\lambda c d \sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{n-p}\sum_{u=0}^{p+s-1} \frac{(-1)^{q+v+w+t+s+u} n! \varpi_{qvwtsu}}{s!q!u!v!w!(t+1)!(p-1)!(n-p-s)!} f_{BX}(x;\, \alpha,\, \beta_{t+1}), \tag{6.21}$$

where $f_{BX}(x;\, \alpha,\, \beta_{t+1}) = 2\alpha^2 \beta_{t+1} x e^{-(\alpha x)^2}(1 - e^{-(\alpha x)^2})^{\beta_{t+1}-1}$ is the PDF of the Burr X distribution with parameters α and $\beta_{t+1} = \beta(t+1)$ and

$$\varpi_{qvwtsu} = \frac{\Gamma(p+s+q+1)\Gamma(p+s)\Gamma(\lambda(q+u+1))\Gamma(c(v+1))\Gamma(d(w+1))}{\Gamma(p+s+1)\Gamma(p+s-u)\Gamma(\lambda(q+u+1)-v)\Gamma(c(v+1)-w)\Gamma(d(w+1)-t)}.$$

It can be seen that the density of the p^{th} order statistic is a weighted function of the density of the Burr X distribution with different shape parameters. Equation (6.21) can be used to obtain several structural properties of $X_{p:n}$. For instance, the r^{th} moment of the p^{th} order statistic.

Proposition 6.10. The r^{th} non-central moment of the p^{th} order statistic is given by

$$\mu_r^{'(p:n)} = 2\lambda\beta c d \alpha^{-r} \sum_{z=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{n-p}\sum_{u=0}^{p+s-1} \frac{(-1)^{s+q+v+w+t+u+z} n!(p+s+q)! \varpi^*_{qvwtsuv} \Gamma(\frac{r}{2}+1)}{s!q!u!v!w!t!(z+1)!(p-1)!(n-p-s)!(p+s)!(z+1)^{\frac{1}{2}}}, \tag{6.22}$$

where $r = 1, 2, \ldots$ and

$$\varpi^*_{qvwtsuv} = \frac{\Gamma(p+s)\Gamma(\lambda(q+u+1))\Gamma(c(v+1))\Gamma(d(w+1))\Gamma(\beta(t+1))}{\Gamma(p+s-u)\Gamma(\lambda(q+u+1)-v)\Gamma(c(v+1)-w)\Gamma(d(w+1)-t)\Gamma(\beta(t+1)-z)}.$$

Proof. By definition

$$\begin{aligned}
\mu_r'^{(p:n)} &= \int_0^\infty x^r \mathrm{g}_{p:n}(x)dx \\
&= \int_0^\infty x^r 2\lambda cd \sum_{q=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \sum_{t=0}^\infty \sum_{s=0}^{n-p} \sum_{u=0}^{p+s-1} \frac{(-1)^{q+v+w+t+s+u} n! \varpi_{qvwtsu}}{s!q!u!v!w!(t+1)!(p-1)!(n-p-s)!} f_{BX}(x;\, \alpha,\, \beta_{t+1})dx \\
&= 2\lambda cd \sum_{q=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \sum_{t=0}^\infty \sum_{s=0}^{n-p} \sum_{u=0}^{p+s-1} \frac{(-1)^{q+v+w+t+s+u} n! \varpi_{qvwtsu}}{s!q!u!v!w!(t+1)!(p-1)!(n-p-s)!} \int_0^\infty x^r f_{BX}(x;\, \alpha,\, \beta_{t+1})dx.
\end{aligned}$$

Using the same approach for deriving the non-central moment,

$$\mu_r'^{(p:n)} = 2\lambda\beta cd\alpha^{-r} \sum_{z=0}^\infty \sum_{q=0}^\infty \sum_{v=0}^\infty \sum_{w=0}^\infty \sum_{t=0}^\infty \sum_{s=0}^{n-p} \sum_{u=0}^{p+s-1} \frac{(-1)^{s+q+v+w+t+u+z} n!(p+s+q)! \varpi^*_{qvwtsuv} \Gamma(\frac{r}{2}+1)}{s!q!u!v!w!t!(z+1)!(p-1)!(n-p-s)!(p+s)!(z+1)^{\frac{1}{2}}}.$$

6.5 Parameter Estimation

In this section, the unknown parameters of the EGHLBX distribution were estimated using the method of maximum likelihood estimation. Let $X_1,\, X_2,\, \ldots,\, X_n$ be a random sample of size n from the EGHLBX distribution. Let $z_i = e^{-(\alpha x_i)^2}$ and $\bar{z}_i = 1 - e^{-(\alpha x_i)^2}$, then the log-likelihood function is given by

$$\begin{aligned}
\ell = n\log(4cd\alpha^2\beta\lambda) + (\beta-1)\sum_{i=1}^n \log(\bar{z}_i) + (d-1)\sum_{i=1}^n \log(1-\bar{z}_i^\beta) + (c-1)\sum_{i=1}^n \log[1-(1-\bar{z}_i^\beta)^d] + \\
(\lambda-1)\sum_{i=1}^n \log[1-(1-(1-\bar{z}_i^d)^d)^c] - 2\sum_{i=1}^n \log[1+(1-(1-(1-\bar{z}_i^\beta)^d)^c)^\lambda] + \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \alpha^2 x_i^2.
\end{aligned} \tag{6.23}$$

Differentiating the log-likelihood function with respect to the parameters λ, c, d, β and α, the score functions are obtained as:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}] - \sum_{i=1}^{n} \frac{2[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda} \log[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}}, \tag{6.24}$$

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \log[1-(1-\bar{z}_i^{\beta})^{d}] - (\lambda-1)\sum_{i=1}^{n} \frac{(1-(1-\bar{z}_i^{\beta})^{d})^{c} \log[1-(1-\bar{z}_i^{\beta})^{d}]}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} - \sum_{i=1}^{n} \frac{2\lambda(1-(1-\bar{z}_i^{\beta})^{d})^{c}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1} \log[1-(1-\bar{z}_i^{\beta})^{d}]}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}}, \tag{6.25}$$

$$\frac{\partial \ell}{\partial d} = \frac{n}{d} + \sum_{i=0}^{n} \log(1-\bar{z}_i^{\beta}) - (c-1)\sum_{i=1}^{n} \frac{(1-\bar{z}_i^{\beta})^{d} \log(1-\bar{z}_i^{\beta})}{1-(1-\bar{z}_i^{\beta})^{d}} + (\lambda-1)\sum_{i=1}^{n} \frac{c(1-\bar{z}_i^{\beta})^{d}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1} \log(1-\bar{z}_i^{\beta})}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} - \sum_{i=1}^{n} \frac{2\lambda c(1-\bar{z}_i^{\beta})^{d}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1} \log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}}, \tag{6.26}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(\bar{z}_i) - (d-1)\sum_{i=1}^{n} \frac{\bar{z}_i^{\beta} \log(\bar{z}_i)}{1-\bar{z}_i^{\beta}} + (c-1)\sum_{i=1}^{n} \frac{d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1} \log(\bar{z}_i)}{1-(1-\bar{z}_i^{\beta})^{d}} - (\lambda-1)\sum_{i=1}^{n} \frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1} \log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} + \sum_{i=1}^{n} \frac{2\lambda cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1} \log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}}, \tag{6.27}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{2n}{\alpha} - \sum_{i=1}^{n} 2\alpha x_i^2 + (\beta - 1)\sum_{i=1}^{n} \frac{2\alpha x_i^2 z_i}{\bar{z}_i} - (d-1)\sum_{i=1}^{n} \frac{2\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}}{1-\bar{z}_i^{\beta}} +$$
$$(c-1)\sum_{i=1}^{n} \frac{2\alpha\beta d x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^{d}} - (\lambda-1)\sum_{i=1}^{n} \frac{2\alpha\beta c d x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$
$$\sum_{i=1}^{n} \frac{4\alpha\beta\lambda c d x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}}. \tag{6.28}$$

The maximum likelihood estimates for the parameters are obtained by equating the score functions to zero and solving the system of nonlinear equations numerically. To be able to construct confidence intervals for the parameters, the observed information matrix $J(\boldsymbol{\vartheta})$ was used because of the complex nature of the expected information matrix. The observed information matrix for the parameters is given by

$$J(\boldsymbol{\vartheta}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial c} & \frac{\partial^2 \ell}{\partial \lambda \partial d} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} & \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} \\ & \frac{\partial^2 \ell}{\partial c^2} & \frac{\partial^2 \ell}{\partial c \partial d} & \frac{\partial^2 \ell}{\partial c \partial \beta} & \frac{\partial^2 \ell}{\partial c \partial \alpha} \\ & & \frac{\partial^2 \ell}{\partial d^2} & \frac{\partial^2 \ell}{\partial d \partial \beta} & \frac{\partial^2 \ell}{\partial d \partial \alpha} \\ & & & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \alpha} \\ & & & & \frac{\partial^2 \ell}{\partial \alpha^2} \end{bmatrix}.$$

The elements of the observed information matrix are given in Appendix A3. When the usual regularity condition holds and the parameters are within the interior of the parameter space, that is not on the boundary, the distribution of $\sqrt{n}(\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta})$ converges to the multivariate normal distribution $N_5(\mathbf{0}, I^{-1}(\boldsymbol{\vartheta}))$, where $I(\boldsymbol{\vartheta})$ is the expected information matrix. The asymptotic behavior is still valid when $I(\boldsymbol{\vartheta})$ is replaced by the observed information matrix estimated at $J(\hat{\boldsymbol{\vartheta}})$. The asymptotic multivariate normal distribution $N_5(\mathbf{0}, J^{-1}(\hat{\boldsymbol{\vartheta}}))$ is an important distribution for constructing an approximate $100(1-\eta)\%$ two-sided confidence intervals for the model parameters.

6.6 Monte Carlo Simulation

This section presents the results of simulation experiment used to examine properties of the maximum likelihood estimator for the parameters of the EGHLBX distribution. Random samples for the simulation were generated using the quantile function in equation (6.9). The properties of the estimators were investigated by computing AB and RMSE for each of the parameters. The simulation experiment was replicated for $N = 1,000$ times each with sample sizes $n = 25, 50, 75, 100, 200, 300, 600$ and parameter values $(\lambda, \alpha, \beta, c, d) = (0.8, 0.6, 0.2, 1.5, 3.5)$ and $(2.5, 4.6, 1.2, 3.5, 0.5)$. Figure 6.3 and 6.4 respectively displays the AB and RMSE for the maximum likelihood estimators of $(\lambda, \alpha, \theta, c, d) = (0.8, 0.6, 0.2, 1.5, 3.5)$ for $n = 25, 50, 75, 100, 200, 300, 600$. The AB for the estimators fluctuate upward and downward while the RMSE for the estimators generally decreases as the sample size increases.

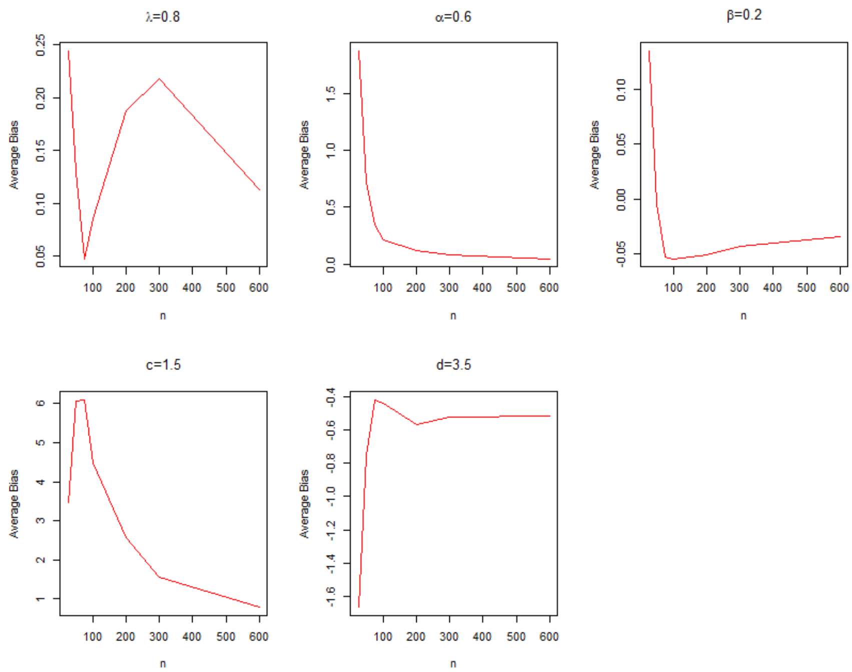

Figure 6.3: **AB for Estimators**

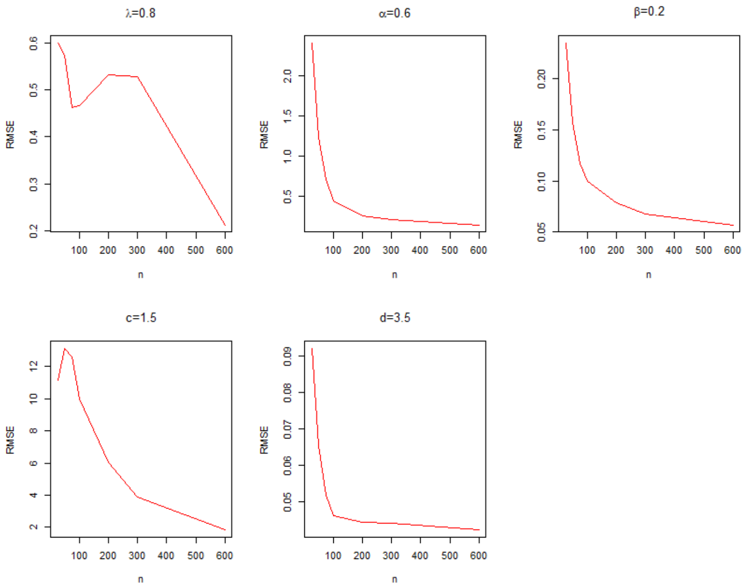

Figure 6.4: **RMSE for Estimators**

Figure 6.5 and 6.6 respectively displays the AB and RMSE for the maximum likelihood estimators of $(\lambda, \alpha, \beta, c, d) = (2.5, 4.6, 1.2, 3.5, 0.5)$ for $n = 25, 50, 75, 100, 200, 300, 600$. Both the AB and RMSE for the estimators of the parameters fluctuates upward and downward as the sample size increases.

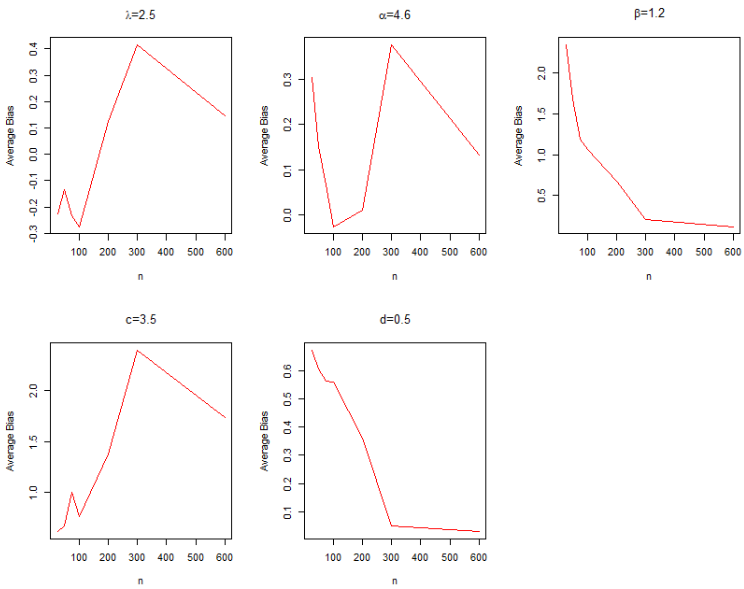

Figure 6.5: **AB for Estimators**

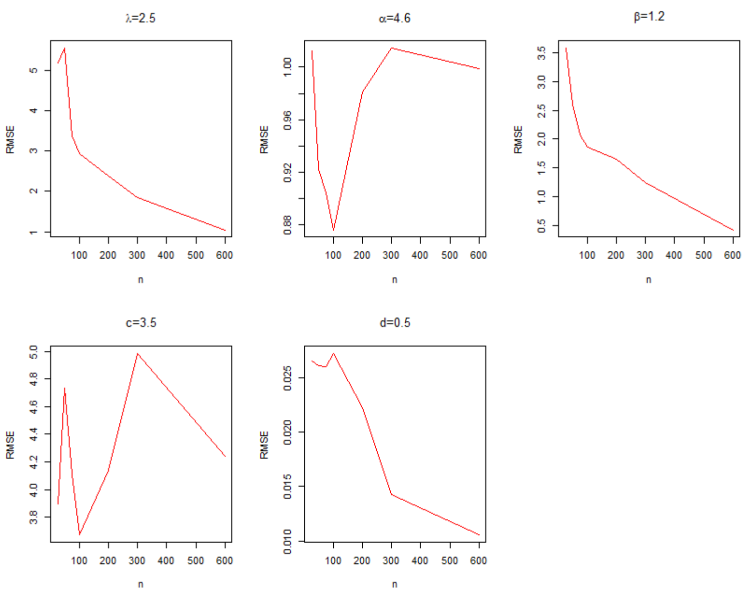

Figure 6.6: **RMSE for Estimators**

6.7 Application

In this section, the application of the EGHLBX distribution was demonstrated using real data set. The goodness-of-fit of the EGHLBX distribution was compared with that of its sub-models and the Weibull-Burr XII (WBXII) distribution. The data were obtained from Birnbaum and Saunders (1969) and consists of the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. Table 6.3 displays the data set.

Table 6.3: **Fatigue time of 101 6061-T6 aluminum coupons**

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

The PDF of the WBXII distribution is given by

$$g(x) = \frac{\alpha\beta cks^{-c}x^{c-1}[(1+(\frac{x}{s})^c)^k - 1]^{\beta-1}}{1+(\frac{x}{s})^c} e^{-\alpha[(1+(\frac{x}{s})^c)^k-1]^\beta}, \; x > 0, \tag{6.29}$$

where α, $s > 0$ are scale parameters and β, c, $k > 0$ are shape parameters. The data set has an increasing failure rate since the TTT transform plot is concave above the 45 degrees line as displayed in Figure 6.7.

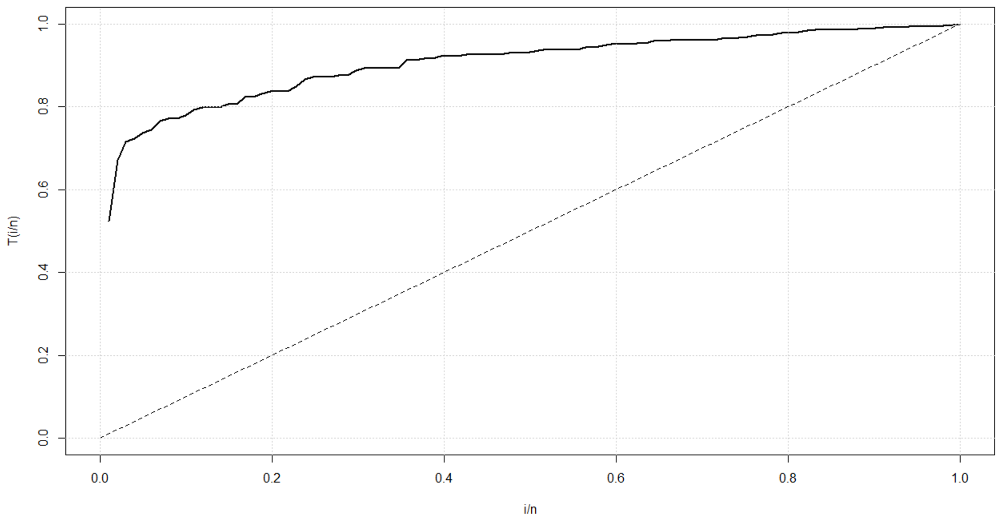

Figure 6.7: **TTT-transform plot for aluminum coupons data**

Table 6.4 shows the maximum likelihood estimates for the parameters of the fitted distribution with their corresponding standard errors in bracket. The parameters of the EGHLBX distribution were all significant at the 5% significance level.

Table 6.4: **Maximum likelihood estimates of parameters and standard errors for aluminum data**

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{d}$
EGHLBX	0.882	0.012	9.448	1.168	1.994
	(0.332)	(0.001)	(0.042)	(0.467)	(0.119)
EGSHLBX		0.003	2.198	1.767	122.752
		(2.206×10^{-4})	(1.538×10^{-1})	(9.060×10^{-2})	(1.063×10^{-4})
EHLBX	24.734	0.006	2.173	1.673	
	(1.461×10^{-3})	(3.523×10^{-4})	(9.794×10^{-2})	(1.275×10^{-1})	
HLBX	15.583	0.005	2.356		
	(1.091×10^{-3})	(3.328×10^{-4})	(2.561×10^{-1})		
SHLBX		0.023	1200.502		
		(1.773×10^{-4})	(3.178×10^{-10})		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{s}$	$\hat{k}$	$\hat{c}$
WBXII	34.698	0.672	145.356	0.011	13.453
	(2.078)	(0.225)	(9.697)	(0.014)	(3.878)

The EGHLBX distribution provides a better fit to the aluminum coupons data than its sub-models and the WBXII distribution. Table 6.5 revealed that the EGHLBX distribution has the highest log-likelihood and the smallest K-S, W*, AIC, AICc, and BIC values compared to the other fitted models.

Table 6.5: **Log-likelihood, goodness-of-fit statistics and information criteria for aluminum data**

Model	ℓ	AIC	AICc	BIC	K-S	W*
EGHLBX	**-455.790**	**921.575**	**922.206**	**934.650**	**0.050**	**0.038**
EGSHLBX	-458.540	925.087	925.504	935.547	0.078	0.083
EHLBX	-459.400	926.795	927.212	937.255	0.085	0.111
HLBX	-467.580	941.153	941.401	948.999	0.150	0.075
SHLBX	-627.260	1258.517	1258.639	1263.747	0.381	0.625
WBXII	-455.960	921.919	922.551	934.995	0.060	0.046

The LRT was performed to compare the EGHLBX distribution with its sub-models. Table 6.6 showed that the EGHLBX distribution provides a good fit to the data than its sub-models.

Table 6.6: **Likelihood ratio test statistic for aluminum data**

Model	Hypotheses	LRT	P-values
EGSHLBX	$H_0 : \lambda = 1$ vs $H_1 : H_0$ is false	5.512	0.019
EHLBX	$H_0 : d = 1$ vs $H_1 : H_0$ is false	7.220	0.007
HLBX	$H_0 : c = d = 1$ vs $H_1 : H_0$ is false	23.579	< 0.001
SHLBX	$H_0 : \lambda = c = d = 1$ vs $H_1 : H_0$ is false	342.940	< 0.001

The asymptotic variance-covariance matrix for the estimated parameters of the EGHLBX distribution is given by

$$J^{-1} = \begin{bmatrix} 0.110 & -4.787 \times 10^{-4} & -1.056 \times 10^{-2} & -0.128 & 0.039 \\ -4.787 \times 10^{-4} & 2.203 \times 10^{-6} & 5.387 \times 10^{-5} & 6.338 \times 10^{-4} & -1.680 \times 10^{-4} \\ -1.056 \times 10^{-2} & 5.387 \times 10^{-5} & 1.757 \times 10^{-3} & 1.944 \times 10^{-2} & -3.512 \times 10^{-3} \\ -0.128 & 6.338 \times 10^{-4} & 1.944 \times 10^{-2} & 0.218 & -0.043 \\ 0.039 & -1.680 \times 10^{-4} & -3.512 \times 10^{-3} & -0.043 & 0.014 \end{bmatrix}.$$

Thus, the approximate 95% confidence interval for the parameters λ, α, β, c and d are [0.231, 1.533], [0.009, 0.015], [9.366, 9.530], [0.253, 2.082] and [1.761, 2.227] respectively. The estimated confidence intervals for all the parameters do not contain zero. This implies that the estimated parameters of the EGHLBX distribution were all significant at the 5% significance level. The plots of the empirical density and fitted densities of the fitted distributions are shown in Figure 6.8.

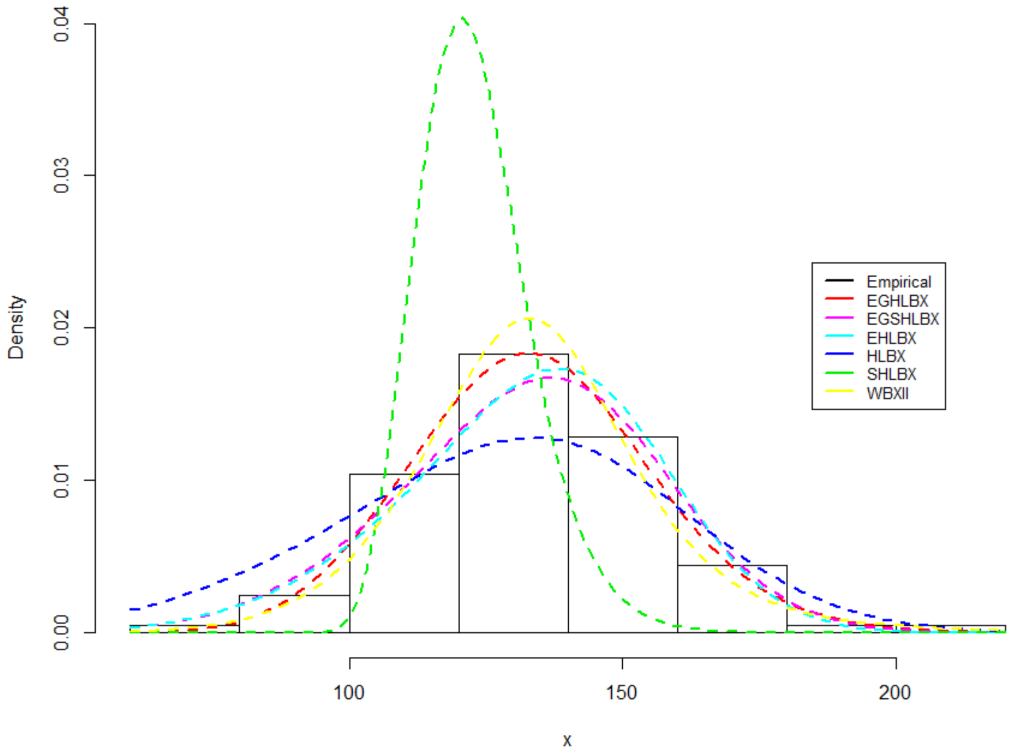

Figure 6.8: **Empirical and fitted densities plot of aluminum data**

6.8 Summary

In this chapter, the EGHLBX distribution was developed and its statistical properties were derived. The method of maximum likelihood was employed to estimate the parameters of the model and simulation studies were performed to examine the behavior of the estimators for the parameters. Application of the proposed distribution was demonstrated to show its usefulness using real data set. The results revealed that the new model provides a good fit to the data.

CHAPTER 7

EXPONENTIATED GENERALIZED POWER SERIES FAMILY OF DISTRIBUTIONS

7.1 Introduction

This chapter presents a new class of distributions called the EG power series (PS) (EGPS) family of distributions by compounding the EG class of distributions and the PS family of distributions.

7.2 Generalized Power Series Family

Let N represent the number of independent subsystems of a system functioning at a given time. Suppose that N has zero truncated power series distribution with probability mass function given by

$$\mathbb{P}(N = n) = \frac{a_n \lambda^n}{C(\lambda)},\ n = 1,\, 2,\, \ldots, \tag{7.1}$$

where $a_n > 0$, $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$, and $\lambda \in (0, s)$ (s can be ∞) is chosen such that $C(\lambda)$ is finite and its first, second and third derivatives are defined and denoted by $C^{'}(\cdot), C^{''}(\cdot)$ and $C^{'''}(\cdot)$. The PS family includes: binomial, Poisson, geometric and logarithmic distributions. Detailed information on the PS family can be found in Noack (1950). Suppose the failure time of each subsystem follows the EG class of distributions with CDF given by

$$H_{c,d}(x) = \left(1 - (1 - G(x;\boldsymbol{\psi}))^d\right)^c, x \in \mathbb{R}, \tag{7.2}$$

where c, $d > 0$ are extra shape parameters, $G(x;\boldsymbol{\psi})$ is the baseline CDF depending on parameter $\boldsymbol{\psi}$ and $\mathrm{g}(x;\boldsymbol{\psi})$ is its corresponding density function. For simplicity, $G(x;\boldsymbol{\psi})$ is written as $G(x)$. If T_j is the failure time of the j^{th} subsystem and X represents the time to failure of the first out of the N operating subsystems such that $X = \min(T_1, T_2, \ldots, T_N)$. Then the conditional CDF of X given N is

$$\begin{aligned} F(x|N=n) &= 1 - \mathbb{P}(X > x|N) \\ &= 1 - \mathbb{P}(T_1 > x, T_2 > x, \ldots, T_N > x) \\ &= 1 - (1 - \mathbb{P}(T_1 \leq x))^n \\ &= 1 - \left[1 - \left(1 - (1 - G(x))^d\right)^c\right]^n. \end{aligned} \tag{7.3}$$

Hence, the marginal CDF of X is given by

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \left\{1 - \left[1 - \left(1 - (1 - G(x))^d\right)^c\right]^n\right\} \\ &= 1 - \frac{C\left(\lambda\left[1 - \left(1 - (1 - G(x))^d\right)^c\right]\right)}{C(\lambda)}, \ x \in \mathbb{R}. \end{aligned} \tag{7.4}$$

The PDF is given by

$$f(x) = \lambda c d \mathrm{g}(x)(1 - G(x))^{d-1}(1 - (1 - G(x))^d)^{c-1} \frac{C'\left(\lambda\left[1 - \left(1 - (1 - G(x))^d\right)^c\right]\right)}{C(\lambda)}. \tag{7.5}$$

Table 7.1 summarizes some particular cases of zero truncated PS distributions.

Table 7.1: **Useful quantities for some power series distributions**

Distribution	a_n	$C(\lambda)$	$C^{'}(\lambda)$	$C^{''}(\lambda)$	$C^{'''}(\lambda)$	s
Geometric	1	$\lambda(1-\lambda)^{-1}$	$(1-\lambda)^{2}$	$2(1-\lambda)^{-3}$	$6(1-\lambda)^{-4}$	1
Poisson	$\frac{1}{n!}$	$e^{\lambda}-1$	e^{λ}	e^{λ}	e^{λ}	∞
Logarithmic	n^{-1}	$-\log(1-\lambda)$	$(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	1
Binomial	$\binom{m}{n}$	$(1+\lambda)^{m}-1$	$\frac{m}{(1+\lambda)^{1-m}}$	$\frac{m(m-1)}{(1+\lambda)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1+\lambda)^{3-m}}$	∞

The survival function and the hazard rate function of the EGPS class of distributions are respectively given by

$$S(x) = \frac{C\left(\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]\right)}{C(\lambda)}, \tag{7.6}$$

and

$$\tau(x) = \lambda c d g(x)(1-G(x))^{d-1}(1-(1-G(x))^{d})^{c-1}\frac{C^{'}\left(\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]\right)}{C\left(\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]\right)}. \tag{7.7}$$

Remark 1. If $X = \max(T_1, T_2, \ldots, T_N)$, then the CDF of the EGPS class is given by

$$F(x) = \frac{C\left(\lambda(1-(1-G(x))^{d})^{c}\right)}{C(\lambda)}. \tag{7.8}$$

Remark 2. If $C(\lambda) = \lambda$, then the EG class is a special case of the EGPS class.

Proposition 7.1. The EG class is a limiting case of the EGPS class when $\lambda \longrightarrow 0$.

Proof.

$$\lim_{\lambda \longrightarrow 0} F(x) = 1 - \lim_{\lambda \longrightarrow 0} \frac{C\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}$$
$$= 1 - \lim_{\lambda \longrightarrow 0} \frac{\sum_{n=1}^{\infty} a_n \lambda^n \left[1-\left(1-(1-G(x))^d\right)^c\right]^n}{\sum_{n=1}^{\infty} a_n \lambda^n}.$$

Applying L'Hôpital's rule,

$$\lim_{\lambda \longrightarrow 0} F(x) = 1 - \lim_{\lambda \longrightarrow 0} \frac{\sum_{n=1}^{\infty} n a_n \lambda^{n-1} \left[1-\left(1-(1-G(x))^d\right)^c\right]^n}{\sum_{n=1}^{\infty} n a_n \lambda^{n-1}}$$
$$= 1 - \lim_{\lambda \longrightarrow 0} \frac{a_1\left[\left(1-(1-G(x))^d\right)^c\right] + \sum_{n=2}^{\infty} n a_n \lambda^{n-1}\left[1-\left(1-(1-G(x))^d\right)^c\right]^n}{a_1 + \sum_{n=2}^{\infty} n a_n \lambda^{n-1}}$$
$$= \left(1-(1-G(x))^d\right)^c.$$

Proposition 7.2. The exponentiated PS class is a limiting special case of the EGPS class when $d \longrightarrow 1$.

Proof.

$$\lim_{d \longrightarrow 1} F(x) = 1 - \lim_{d \longrightarrow 1} \frac{C\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}$$
$$= 1 - \frac{C\left(\lambda\left(1-G(x)^c\right)\right)}{C(\lambda)}.$$

Lemma 7.1. The EGPS class density has a linear representation of the form

$$f(x) = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} E_N\left[\mathrm{g}(x) G(x)^k\right], \tag{7.9}$$

where $E_N(\cdot)$ is the expectation with respect to the random variable N and

$$\omega_{ijk} = \frac{(-1)^{i+j+k}\Gamma(n+1)\Gamma(c(i+1))\Gamma(d(j+1))}{i!\,j!\,k!\Gamma(n-i)\Gamma(c(i+1)-j)\Gamma(d(j+1)-k)}.$$

Proof. The EGPS PDF can be written as

$$f(x) = cd\sum_{n=1}^{\infty}\mathbb{P}(N=n)n\mathrm{g}(x)(1-G(x))^{d-1}\left(1-(1-G(x))^{d}\right)^{c-1}\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]^{n-1}.$$

For a real non-integer $\eta > 0$, a series representation for $(1-z)^{\eta-1}$, for $|z| < 1$ is

$$(1-z)^{\eta-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(\eta)}{i!\,\Gamma(\eta-i)}z^i. \tag{7.10}$$

Using the series expansion in equation (7.10) thrice and the fact that $0 \leq 1-G(x) \leq 1$, yields

$$f(x) = cd\sum_{j,\,k=0}^{\infty}\sum_{i=0}^{n-1}\omega_{ijk}E_N\left[\mathrm{g}(x)G(x)^k\right].$$

The linear representation of the density function makes it easy to study the statistical properties of the EGPS class. Alternatively, it can be written in terms of the exp-G density function as

$$f(x) = cd\sum_{j,\,k=0}^{\infty}\sum_{i=0}^{n-1}\omega^{*}_{ijk}E_N\left[\varphi_{k+1}\right], \tag{7.11}$$

where $\omega^{*}_{ijk} = \frac{\omega_{ijk}}{(k+1)}$ and $\varphi_{k+1} = (k+1)\mathrm{g}(x)G(x)^k$ is the exp-G density with power parameter $k+1$.

7.3 Sub-Families

In this section, a number of sub-families of the EGPS family were discussed. These include: EG Poisson (EGP), EG binomial (EGB), EG geometric (EGG) and EG logarithmic (EGL) classes of distributions.

7.3.1 Exponentiated Generalized Poisson Class

The zero truncated Poisson distribution is a special case of PS distributions with $a_n = \frac{1}{n!}$ and $C(\lambda) = e^{\lambda} - 1$, $(\lambda > 0)$. Using the CDF in equation (7.4), the CDF and PDF of the EGP class of distributions are respectively given by

$$F(x) = \frac{e^{\lambda} - e^{\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]}}{e^{\lambda} - 1}, \tag{7.12}$$

and

$$f(x) = \lambda c d g(x)(1 - G(x))^{d-1}(1 - (1 - G(x))^{d})^{c-1} \frac{e^{\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]}}{e^{\lambda} - 1}, \quad x \in \mathbb{R}. \tag{7.13}$$

7.3.2 Exponentiated Generalized Binomial Class

The zero truncated binomial distribution is a special case of PS distributions with $a_n = \binom{m}{n}$ and $C(\lambda) = (1 + \lambda)^{m} - 1$,$(\lambda > 0)$, where $m\,(n \leq m)$ is the number of replicas and is a positive integer. The CDF and PDF of the EGB class of distributions are respectively given by

$$F(x) = 1 - \frac{\left[1 + \lambda\left[1 - \left(1 - (1 - G(x))^{d}\right)^{c}\right]\right]^{m} - 1}{(1 + \lambda)^{m} - 1}, \tag{7.14}$$

and

$$f(x) = m\lambda c d g(x)(1-G(x))^{d-1}(1-(1-G(x))^{d})^{c-1}\frac{\left[1+\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]\right]^{m-1}}{((1+\lambda)^{m}-1)}, \ x \in \mathbb{R}. \tag{7.15}$$

The EGP class is a limiting case of the EGB class if $m\lambda \longrightarrow \theta > 0$, when $m \longrightarrow \infty$.

7.3.3 Exponentiated Generalized Geometric Class

The zero truncated geometric distribution is a special case of PS distributions with $a_n = 1$ and $C(\lambda) = \frac{\lambda}{1-\lambda}$, $(0 < \lambda < 1)$. The CDF and PDF of the EGG class of distributions are respectively given by

$$F(x) = 1 - \frac{(1-\lambda)\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]}{1-\lambda\left[1-(1-(1-G(x))^{d})^{c}\right]}, \tag{7.16}$$

and

$$f(x) = \frac{(1-\lambda)c d g(x)(1-G(x))^{d-1}(1-(1-G(x))^{d})^{c-1}}{\left[1-\lambda\left[1-(1-(1-G(x))^{d})^{c}\right]\right]^{2}}, \ x \in \mathbb{R}. \tag{7.17}$$

7.3.4 Exponentiated Generalized Logarithmic Class

The zero truncated logarithmic distribution is another special case of the PS distributions with $a_n = \frac{1}{n}$ and $C(\lambda) = -\log(1-\lambda)$, $(0 < \lambda < 1)$. The CDF and PDF of the EGL class are respectively given by

$$F(x) = 1 - \frac{\log\left[1-\lambda\left[1-\left(1-(1-G(x))^{d}\right)^{c}\right]\right]}{\log(1-\lambda)}, \tag{7.18}$$

and

$$f(x) = \frac{\lambda c d g(x)(1-G(x))^{d-1}(1-(1-G(x))^{d})^{c-1}}{\log(1-\lambda)\left[\lambda\left[1-(1-(1-G(x))^{d})^{c}\right]-1\right]}, \quad x \in \mathbb{R}. \tag{7.19}$$

7.4 Statistical Properties

In this section, the statistical properties of the EGPS class of distributions were discussed. These include: the quantile function, moments, MGF, incomplete moment, mean residual life, stochastic ordering property, reliability, Shannon entropy and order statistics.

7.4.1 Quantile function

The quantile function is another way of describing the distribution of a random variable. It plays a key role when simulating random samples from a distribution and it provides an alternative means for describing the shapes of a distribution.

Proposition 7.3. The quantile function of the EGPS class is given by

$$Q_F(u) = G^{-1}\left\{1-\left[1-\left(1-\frac{C^{-1}\left((1-u)C(\lambda)\right)}{\lambda}\right)^{\frac{1}{c}}\right]^{\frac{1}{d}}\right\}, \tag{7.20}$$

where $u \in [0,\, 1]$ and $C^{-1}(\cdot)$ is the inverse of $C(\cdot)$.

Proof. By definition, the quantile function is given by $F(x_u) = \mathbb{P}(X \leq x_u) = u$. Thus, setting $Q_F(u) = x_u$ in equation (7.4) and solving for $Q_F(u)$ yields the quantile function of the EGPS class.

The median of the EGPS class is obtained by substituting $u = 0.5$ into equation (7.20).

7.4.2 Moments

The moments of a random variable plays an important role in statistical analysis. For instance, it is used in the computation of the variance, skewness and kurtosis of the distribution of the random variable. This subsection, presents the moments of the EGPS family.

Proposition 7.4. The r^{th} non-central moment of the EGPS class is given by

$$\mu_r^{'} = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\int_{-\infty}^{\infty} x^r \mathrm{g}(x) G(x)^k dx \right],\; r = 1,\, 2,\, \ldots. \tag{7.21}$$

Proof. By definition, the r^{th} non-central moment is given by

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x) dx.$$

Substituting the linear representation of the density function into the definition and simplifying yields

$$\begin{aligned}
\mu_r^{'} &= \int_{-\infty}^{\infty} x^r cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} E_N \left[\mathrm{g}(x) G(x)^k \right] dx \\
&= cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} \int_{-\infty}^{\infty} x^r E_N \left[\mathrm{g}(x) G(x)^k \right] dx \\
&= cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\int_{-\infty}^{\infty} x^r \mathrm{g}(x) G(x)^k dx \right].
\end{aligned}$$

Alternatively, the moments can be expressed in terms of the quantile function of the baseline. Let $G(x) = u$, then

$$\mu_r^{'} = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} \, E_N \left[\int_0^1 Q_G(u)^r u^k du \right]. \tag{7.22}$$

7.4.3 Moment Generating Function

The MGF are special functions that can be used to compute the moments of a random variable. In this subsection, the MGF of the EGPS class was derived.

Proposition 7.5. The MGF of EGPS class of distribution is given by

$$M_X(t) = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} \, E_N \left[\int_{-\infty}^{\infty} e^{tx} \mathrm{g}(x) G(x)^k dx \right]. \tag{7.23}$$

Proof. By definition

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} \, E_N \left[\int_{-\infty}^{\infty} e^{tx} \mathrm{g}(x) G(x)^k dx \right]. \end{aligned}$$

In terms of the quantile function of the baseline, the MGF is given by

$$M_X(t) = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} \, E_N \left[\int_0^1 e^{tQ_G(u)} u^k du \right]. \tag{7.24}$$

7.4.4 Incomplete Moment

The incomplete moment plays a useful role in estimating the mean deviation, median deviation, inequality measures and mean residual life of the distribution of a random variable.

Proposition 7.6. The r^{th} incomplete moment of the EGPS class of distributions is given by

$$M_r(t) = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\int_{-\infty}^{t} x^r \mathrm{g}(x) G(x)^k dx \right],\ r = 1,\, 2,\, \ldots. \tag{7.25}$$

Proof. By definition

$$\begin{aligned} M_r(t) &= \int_{-\infty}^{t} x^r f(x) dx \\ &= cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\int_{-\infty}^{t} x^r \mathrm{g}(x) G(x)^k dx \right]. \end{aligned}$$

Letting $u = G(x)$, the incomplete moment can be expressed in terms of the baseline quantile function as

$$M_r(t) = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\int_{0}^{G(t)} Q_G(u)^r u^k du \right]. \tag{7.26}$$

Using the power series expansion of the quantile function of the baseline as $Q_G(u) = \sum_{h=0}^{\infty} e_h u^h$, where $e_h (h = 0,\, 1,\, \ldots)$ are suitably chosen real numbers that depend on the parameters of the $G(x)$ distribution,

$$(Q_G(u))^r = \left(\sum_{h=0}^{\infty} e_h u^h \right)^r = \sum_{h=0}^{\infty} e'_{r,h} u^h,$$

where $e'_{r,\,h} = (h\, e_0)^{-1} \sum_{z=1}^{h} [z(r+1) - h] e_z e'_{r,h-z}$, $e'_{r,0} = (e_0)^h$ and $r(r \geq 1)$ is a positive integer. The incomplete moment can now be expressed as

$$M_r(t) = cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\sum_{h=0}^{\infty} e^{'}_{r,h} \int_0^{G(t)} u^{k+h} du \right]$$
$$= cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\sum_{h=0}^{\infty} \frac{e^{'}_{r,h} G(t)^{k+h+1}}{k+h+1} \right].$$

7.4.5 Residual and Mean Residual Life

A system's residual lifetime when it is still operating at time t, is $X_t = X - t|X > t$ which has the PDF

$$f(x;t) = \frac{f(x)}{1-F(t)}$$
$$= \lambda c d g(x)(1-G(x))^{d-1}(1-(1-G(x))^d)^{c-1} \frac{C^{'}\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C\left(\lambda\left[1-\left(1-(1-G(t))^d\right)^c\right]\right)}.$$

Proposition 7.7. The mean residual life of X_t is given by

$$m(t) = \frac{1}{1-F(t)} \left[\mu - cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk}\, E_N \left[\sum_{h=0}^{\infty} \frac{e_h G(t)^{k+h+1}}{k+h+1} \right] \right] - t, \qquad (7.27)$$

where $\mu = \mu^{'}_1$ is the mean and $e_h (h = 0, 1, \ldots)$ are suitably chosen real numbers that depend on the parameters of the $G(x)$ distribution.

Proof. The mean residual life is defined as

$$m(t) = E[X - t|X > t] = \frac{\int_t^{\infty} (x-t) f(x) dx}{1-F(t)} = \frac{\mu^{'}_1 - \int_{-\infty}^{t} x f(x) dx}{1-F(t)} - t. \qquad (7.28)$$

The integral $\int_{-\infty}^{t} x f(x) dx$ is the first incomplete moment. Thus, substituting the first incomplete moment into equation (7.28) yields the mean residual life.

7.4.6 Stochastic Ordering Property

Stochastic ordering is the common way of showing ordering mechanism in lifetime distribution. A random variable X_1 is said to be greater than a random variable X_2 in likelihood ratio order if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ is an increasing function of x.

Proposition 7.8. Let $X_1 \sim \text{EGPS}(x; c, d, \lambda, \boldsymbol{\psi})$ and $X_2 \sim \text{EG}(x; c, d, \boldsymbol{\psi})$, then X_2 is greater than X_1 in likelihood ratio order ($X_1 \leq_{lr} X_2$) provided $\lambda > 0$.

Proof.

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\lambda C'\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}.$$

Thus,

$$\frac{d}{dx}\frac{f_{X_1}(x)}{f_{X_2}(x)} = -\lambda^2 cdg(x)(1-G(x))^{d-1}(1-(1-G(x))^d)^{c-1}\frac{C''\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}.$$

Since $\frac{d}{dx}\frac{f_{X_1}(x)}{f_{X_2}(x)} < 0$ for all $x > 0$, $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ is a decreasing function provided $\lambda > 0$.

From proposition 7.8, it is considered that the hazard rate order, the usual stochastic order and the mean residual life order between X_1 and X_2 hold.

7.4.7 Stress-Strength Reliability

Reliability plays a useful role in the analysis of stress-strength of models. If X_1 is the strength of a component and X_2 is the stress, then the component fails when $X_1 \leq X_2$. The estimate of the stress-strength reliability of the component R is $\mathbb{P}(X_2 < X_1)$.

Proposition 7.9. If $X_1 \sim \text{EGPS}(x; c, d, \lambda, \boldsymbol{\psi})$ and $X_2 \sim \text{EGPS}(x; c, d, \lambda, \boldsymbol{\psi})$, then the stress-strength reliability is given by

$$R = 1 - \sum_{j,k=0}^{\infty}\sum_{i=0}^{n-1}\omega_{ijk}E_N\left[\int_0^1 u^k \frac{C\left(\lambda\left[1-\left(1-(1-u)^d\right)^c\right]\right)}{C(\lambda)}du\right]. \tag{7.29}$$

Proof. The stress-strength reliability is defined as

$$\begin{aligned} R &= \int_{-\infty}^{\infty} f(x)F(x)dx \\ &= 1 - \int_{-\infty}^{\infty} f(x)S(x)dx \\ &= 1 - \sum_{j,k=0}^{\infty}\sum_{i=0}^{n-1}\omega_{ijk}E_N\left[\int_{-\infty}^{\infty} \mathrm{g}(x)G(x)^k S(x)dx\right]. \end{aligned}$$

Letting $G(x) = u$ yields

$$R = 1 - \sum_{j,k=0}^{\infty}\sum_{i=0}^{n-1}\omega_{ijk}E_N\left[\int_0^1 u^k \frac{C\left(\lambda\left[1-\left(1-(1-u)^d\right)^c\right]\right)}{C(\lambda)}du\right].$$

7.4.8 Shannon Entropy

The entropy of a random variable is a measure of variation or uncertainty of the random variable. The Shannon entropy of a random variable X with PDF $f(x)$ is given by $\eta_X = E\{-\log f(x)\}$ (Shannon, 1948).

Proposition 7.10. The Shannon entropy of the EGPS class random variable is given by

$$\eta_X = -\log\left(\frac{cd\lambda}{C(\lambda)}\right) - E\left[\log \mathrm{g}(X)\right] + (1-d)\delta_1 + (1-c)\delta_2 - E\left[\log C'\left(\lambda\left[1-H_{c,d}(X)\right]\right)\right], \tag{7.30}$$

where $H_{c,\,d}(x)$ is the CDF of the EG class,

$$\delta_1 = -cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} E_N \left[\sum_{q=1}^{\infty} \frac{1}{q(k+q+1)} \right],$$

and

$$\delta_2 = -cd \sum_{j,\,k=0}^{\infty} \sum_{i=0}^{n-1} \omega_{ijk} E_N \left[\sum_{s=0}^{\infty} \sum_{q=1}^{\infty} \frac{(-1)^s \binom{dq}{s}}{q(k+q+1)} \right].$$

Proof. By definition

$$\eta_X = -\log\left(\frac{cd\lambda}{C(\lambda)}\right) - E\left[\log \mathrm{g}(X)\right] + (1-d)E\left[\log(1-G(X))\right]$$
$$+ (1-c)E\left[\log(1-(1-G(X))^d)\right] - E\left[\log C'\left(\lambda\left[1-H_{c,\,d}(X)\right]\right)\right]. \qquad (7.31)$$

Let $\delta_1 = E\left[\log(1-G(X))\right]$ and $\delta_2 = E\left[\log(1-(1-G(X))^d)\right]$. Using the identity $\log(1-z) = -\sum_{q=1}^{\infty} \frac{z^q}{q}$, $|z|<1$, yields

$$\log(1-G(x)) = -\sum_{q=1}^{\infty} \frac{G(x)^q}{q},$$
$$\log(1-(1-G(x))^d) = -\sum_{s=0}^{\infty} \sum_{q=1}^{\infty} \frac{(-1)^s \binom{dq}{s} G(x)^s}{q}.$$

Putting $G(x) = u$ and taking the expectation with respect to the random variable X, give the values of δ_1 and δ_2 after some algebraic manipulation.

7.4.9 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from EGPS, then the PDF of the p^{th} order statistic, say $X_{p:n}$, is given by

$$f_{p:n}(x) = \frac{1}{B(p,\, n-p+1)} \left[F(x)\right]^{p-1} \left[1 - F(x)\right]^{n-p} f(x),$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the EGPS class of distributions respectively, and $B(\cdot,\cdot)$ is the beta function. Thus,

$$f_{p:n}(x) = \frac{n!}{(p-1)!(n-p)!} f(x) \left[1 - \frac{C\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}\right]^{p-1} \times \left[\frac{C\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}\right]^{n-p}. \tag{7.32}$$

The largest and smallest order statistics play an important role in statistical analysis. For instance, the difference between the largest order statistic and the smallest order statistic is the range. Hence, the PDF of the largest order statistic, $f_{X_{(n)}}(x)$ is given by

$$f_{X_{(n)}}(x) = n\, f(x) \left[1 - \frac{C\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}\right]^{n-1},$$

and that of the smallest order statistic, $f_{X_{(1)}}(x)$ is

$$f_{X_{(1)}}(x) = n\, f(x) \left[\frac{C\left(\lambda\left[1-\left(1-(1-G(x))^d\right)^c\right]\right)}{C(\lambda)}\right]^{n-1}.$$

7.5 Parameter Estimation

Different methods for parameter estimation exist in literature but the maximum likelihood approach is the most commonly used. The maximum likelihood estimators have several desirable properties and can be used for constructing confidence intervals. Thus, the maximum likelihood method was employed for the estimation of the parameters of the EGPS distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the EGPS distribution. Let $z_i = 1 - G(x_i; \boldsymbol{\psi})$, then the log-likelihood is given by

$$\begin{aligned}\ell =& n\log(\lambda cd) + \sum_{i=1}^{n}\log \mathrm{g}(x_i; \boldsymbol{\psi}) + (d-1)\sum_{i=1}^{n}\log(z_i) + \sum_{i=1}^{n}\log C'\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right) + \\ & (c-1)\sum_{i=1}^{n}\log(1-z_i^d) - n\log C(\lambda). \end{aligned} \tag{7.33}$$

Taking the partial derivative of the log-likelihood function with respect to the parameters yields the following score functions:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \frac{nC'(\lambda)}{C(\lambda)} + \sum_{i=1}^{n}\frac{[1-(1-z_i^d)^c]C''\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}{C'\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}, \tag{7.34}$$

$$\frac{\partial \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\log(1-z_i^d) - \sum_{i=1}^{n}\frac{\lambda(1-z_i^d)^c\log(1-z_i^d)C''\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}{C'\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}, \tag{7.35}$$

$$\frac{\partial \ell}{\partial d} = \frac{n}{d} + \sum_{i=1}^{n}\log(z_i) - (c-1)\sum_{i=1}^{n}\frac{z_i^d\log(z_i)}{1-z_i^d} + \sum_{i=1}^{n}\frac{\lambda c(1-z_i^d)^{c-1}z_i^d\log(z_i)C''\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}{C'\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}, \tag{7.36}$$

$$\frac{\partial \ell}{\partial \boldsymbol{\psi}} = \sum_{i=}^{n} \frac{g'(x_i;\, \boldsymbol{\psi})}{g(x_i;\, \boldsymbol{\psi})} - (d-1)\sum_{i=1}^{n} \frac{G'(x_i;\, \boldsymbol{\psi})}{z_i} + (c-1)\sum_{i=1}^{n} \frac{d z_i^{d-1} G'(x_i;\, \boldsymbol{\psi})}{1 - z_i^d} - \sum_{i=1}^{n} \frac{\lambda c d z_i^{d-1}(1-z_i^d)^{c-1} G'(x_i;\, \boldsymbol{\psi}) C''\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}{C'\left(\lambda\left[1-\left(1-z_i^d\right)^c\right]\right)}, \tag{7.37}$$

where $g'(x_i;\, \boldsymbol{\psi}) = \frac{\partial g'(x_i;\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}$ and $G'(x_i;\, \boldsymbol{\psi}) = \frac{\partial G(x_i;\boldsymbol{\psi})}{\partial \boldsymbol{\psi}}$. The score functions do not have closed form, thus it is more convenient to solve them using numerical techniques. For the purpose of interval estimation of the parameters, a $p \times p$ observed information matrix can be obtained as $J(\boldsymbol{\vartheta}) = -\left\{\frac{\partial^2 \ell}{\partial q \partial r}\right\}$ (for $q,\, r = \lambda,\, c,\, d,\, \boldsymbol{\psi}$), whose elements can be computed numerically. Under the usual regularity conditions as $n \longrightarrow \infty$, the distribution of $\hat{\boldsymbol{\vartheta}} = (\hat{\lambda},\, \hat{c},\, \hat{d},\, \hat{\boldsymbol{\psi}}^T)^T$ approximately converges to a multivariate normal $N_p(\mathbf{0},\, J(\hat{\boldsymbol{\vartheta}})^{-1})$ distribution. $J(\hat{\boldsymbol{\vartheta}})$ is the observed information matrix evaluated at $\hat{\boldsymbol{\vartheta}}$. The asymptotic normal distribution is useful for constructing approximate $100(1-\eta)\%$ confidence intervals.

7.6 Extensions via Copula

In this section, bivariate and multivariate extensions of the EGPS class of distributions were proposed using Clayton copula. Consider a random pair $(X_1,\, X_2)$, a copula C^* associated with the pair is simply a joint distribution of the random vector $(F_{X_1}(x_1),\, F_{X_2}(x_2))$. Suppose that $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ are marginal CDFs of the random variables X_1 and X_2 respectively and C^* is the copula associated to $(X_1,\, X_2)$. Sklar (1959) established that the joint CDF $F_{X_1X_2}(x_1,\, x_2)$ of the pair $(X_1,\, X_2)$ is given by

$$F_{X_1X_2}(x_1,\, x_2) = C^*(F_{X_1}(x_1),\, F_{X_2}(x_2)).$$

Suppose $(X_1,\, X_2)$ follows bivariate EGPS random variables with marginal distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$. Let the copula associated with $(X_1,\, X_2)$ belong to Clayton copula family given by

$$C^*(z_1,\, z_2) = \left[z_1^{-\theta} + z_2^{-\theta} - 1\right]^{\frac{-1}{\theta}},\, \theta \geq 0.$$

The joint CDF, $F_{X_1X_2}(x_1,\, x_2)$, for the bivariate EGPS class is given by

$$F_{X_1X2}(x_1,\, x_2) = \left\{\sum_{i=1}^{2}\left[1 - \frac{C\left(\lambda_i\left[1-\left(1-(1-G(x_i;\, \boldsymbol{\psi_i}))^{d_i}\right)^{c_i}\right]\right)}{C(\lambda_i)}\right]^{-\theta} - 1\right\}^{\frac{-1}{\theta}}, \tag{7.38}$$

where λ_i, c_i, d_i and $\boldsymbol{\psi_i}$ describe the marginal parameters while θ is the Clayton copula parameter. A p-dimensional multivariate extension from the above is given by

$$F_{X_1X_2,\ldots,X_p}(x_1,\, x_2,\, \ldots,\, x_p) = \left\{\sum_{i=1}^{p}\left[1 - \frac{C\left(\lambda_i\left[1-\left(1-(1-G(x_i;\, \boldsymbol{\psi_i}))^{d_i}\right)^{c_i}\right]\right)}{C(\lambda_i)}\right]^{-\theta} - p + 1\right\}^{\frac{-1}{\theta}}. \tag{7.39}$$

7.7 Special Distributions

In this section, four special distributions were presented. These include: EGP inverse exponential (EGPIE) distribution, EGB inverse exponential (EGBIE) distribution, EGG inverse exponential (EGGIE) distribution and EGL inverse exponential (EGLIE) distribution. Suppose the baseline distribution follows an inverse exponential distribution with

CDF $G(x) = e^{-\gamma x^{-1}}$, $\gamma > 0$, $x > 0$. The densities, hazard rate functions and quantiles of the EGPIE, EGBIE, EGGIE and EGLIE distributions are defined as follows.

7.7.1 EGPIE Distribution

The density function of the EGPIE distribution is obtained by substituting the baseline CDF and its corresponding PDF into equation (7.13). Thus, the PDF of EGPIE distribution is given by

$$f(x) = \lambda\gamma c d x^{-2} e^{-\gamma x^{-1}} (1 - e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1} \frac{e^{\lambda\left[1-\left(1-\left(1-e^{-\gamma x^{-1}}\right)^{d}\right)^{c}\right]}}{e^{\lambda} - 1},$$

$$\lambda,\ \gamma,\ c,\ d > 0,\ x > 0. \tag{7.40}$$

The corresponding hazard rate function is given by

$$\tau(x) = \lambda\gamma c d x^{-2} e^{-\gamma x^{-1}} (1 - e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1} \frac{e^{\lambda\left[1-\left(1-\left(1-e^{-\gamma x^{-1}}\right)^{d}\right)^{c}\right]}}{e^{\lambda\left[1-\left(1-\left(1-e^{-\gamma x^{-1}}\right)^{d}\right)^{c}\right]} - 1},$$

$$\lambda,\ \gamma,\ c,\ d > 0,\ x > 0. \tag{7.41}$$

Figure 7.1 displays the plots of the density and hazard rate function of the EGPIE distribution. From the figure, the density exhibit right skewed shape with varied degrees of kurtosis and an approximately symmetric shape. The hazard rate function shows an upside down bathtub shapes.

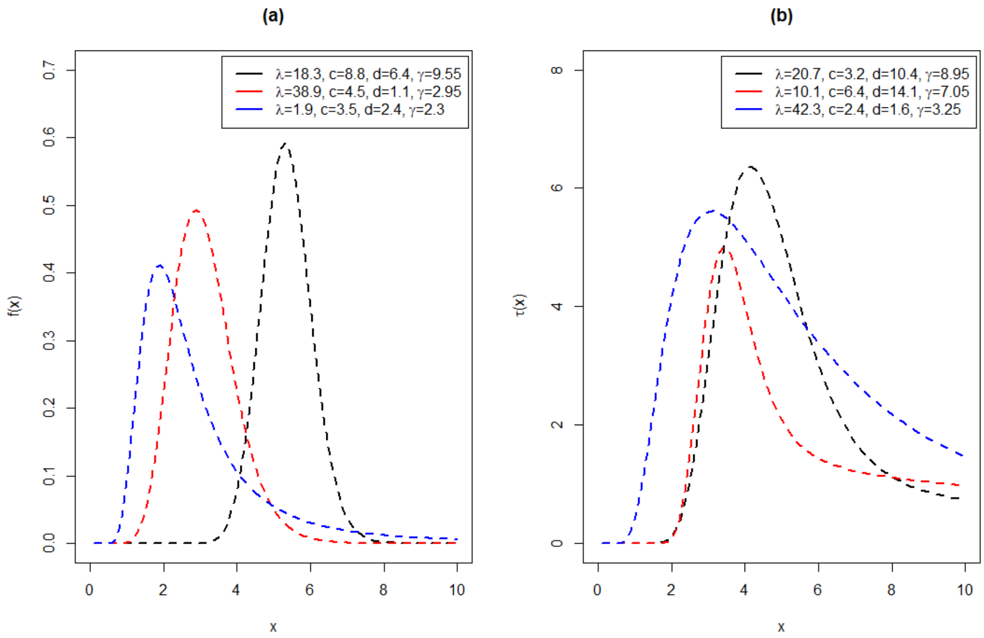

Figure 7.1: **Plots of EGPIE (a) PDF and (b) hazard rate function for some parameter values**

The quantile function of the EGPIE distribution is given by

$$Q(u) = \left\{ \frac{-1}{\gamma} \log \left[1 - \left[1 - \left(1 - \frac{\log\left(e^{\lambda} - u(e^{\lambda} - 1)\right)}{\lambda} \right)^{\frac{1}{c}} \right]^{\frac{1}{d}} \right] \right\}^{-1}. \tag{7.42}$$

7.7.2 EGBIE Distribution

Using equation (7.15), the PDF of EGBIE distribution is given by

$$f(x) =$$

$$m\lambda\gamma c d x^{-2} e^{-\gamma x^{-1}} (1 - e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1} \frac{\left[1 + \lambda \left[1 - \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c}\right]\right]^{m-1}}{(1+\lambda)^m - 1},$$

$$\lambda,\ \gamma,\ c,\ d > 0,\ x > 0. \tag{7.43}$$

The corresponding hazard rate function is given by

$\tau(x) =$

$$m\lambda\gamma c d x^{-2} e^{-\gamma x^{-1}} (1 - e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1} \frac{\left[1 + \lambda\left[1 - \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c}\right]\right]^{m-1}}{\left[1 + \lambda\left[1 - \left(1 - (1 - e^{-\gamma x^{-1}})^{d}\right)^{c}\right]\right]^{m} - 1},$$

$\lambda,\ \gamma,\ c,\ d\ > 0,\ x > 0.$ (7.44)

The plots of the density and hazard rate function of the EGBIE distribution for $m = 5$ are shown in Figure 7.2. The density function exhibit right skewed and approximately symmetric shapes. The hazard rate function exhibit an upside down bathtub shapes and an upside down bathtub shape followed by a bathtub and then upside down bathtub shape.

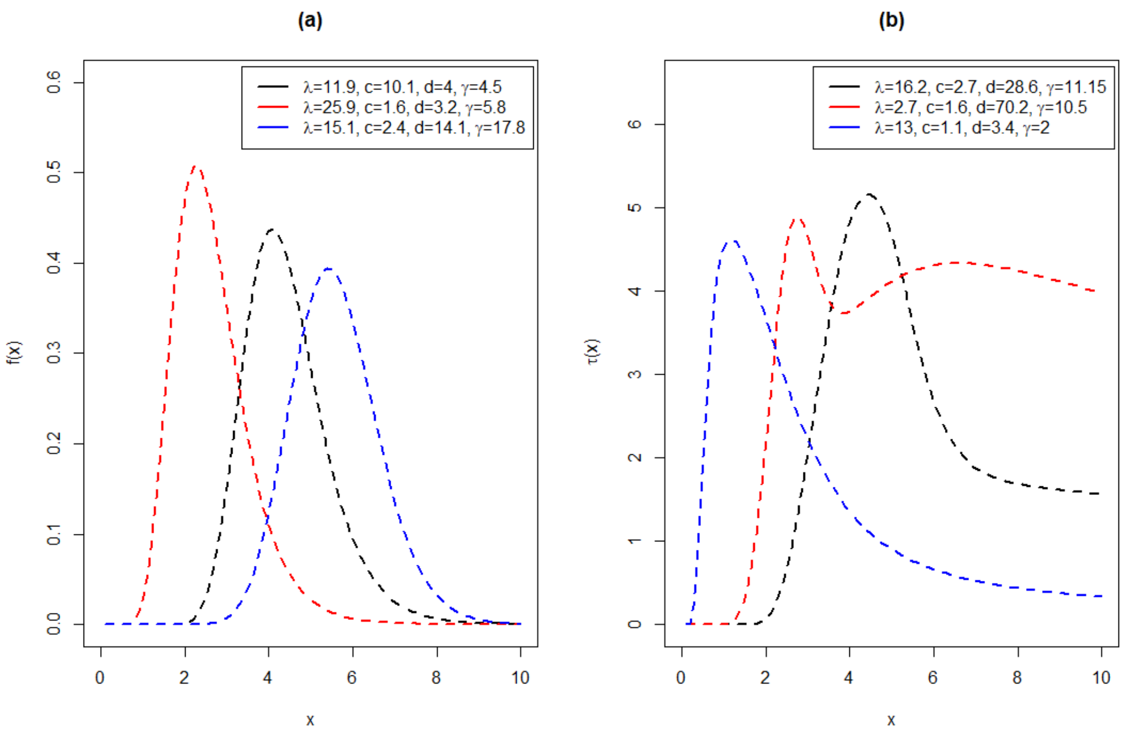

Figure 7.2: **Plots of EGBIE (a) PDF and (b) hazard rate function for some parameter values**

The quantile function of the EGBIE distribution is given by

$$Q(u) = \left\{ \frac{-1}{\gamma} \log \left[1 - \left[1 - \left(1 - \frac{(((1+\lambda)^m - 1)(1-u) + 1)^{\frac{1}{m}} - 1}{\lambda} \right)^{\frac{1}{c}} \right]^{\frac{1}{d}} \right] \right\}^{-1}. \tag{7.45}$$

7.7.3 EGGIE Distribution

From equation (7.17), the PDF of EGGIE distribution is given by

$$f(x) = \frac{(1-\lambda)\gamma c d x^{-2} e^{-\gamma x^{-1}} (1 - e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1}}{\left[1 - \lambda \left[1 - \left(1 - (1 - e^{-\gamma x^{-1}})^{d}\right)^{c}\right]\right]^{2}}, \; 0 < \lambda < 1, \; \gamma, \; c, \; d > 0, \; x > 0. \tag{7.46}$$

The associated hazard rate function is given by

$$\tau(x) = \frac{\gamma c d x^{-2} e^{-\gamma x^{-1}} (1 - e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1}}{\left[1 - \left(1 - (1 - e^{-\gamma x^{-1}})^{d}\right)^{c}\right] \left[1 - \lambda \left[1 - \left(1 - (1 - e^{-\gamma x^{-1}})^{d}\right)^{c}\right]\right]}, \; 0 < \lambda < 1, \; \gamma, \; c, \; d > 0, \; x > 0. \tag{7.47}$$

The density and hazard rate function plots of the EGGIE distribution for some parameter values are displayed in Figure 7.3. The density and the hazard rate function exhibits similar shapes like that of the EGBIE distribution.

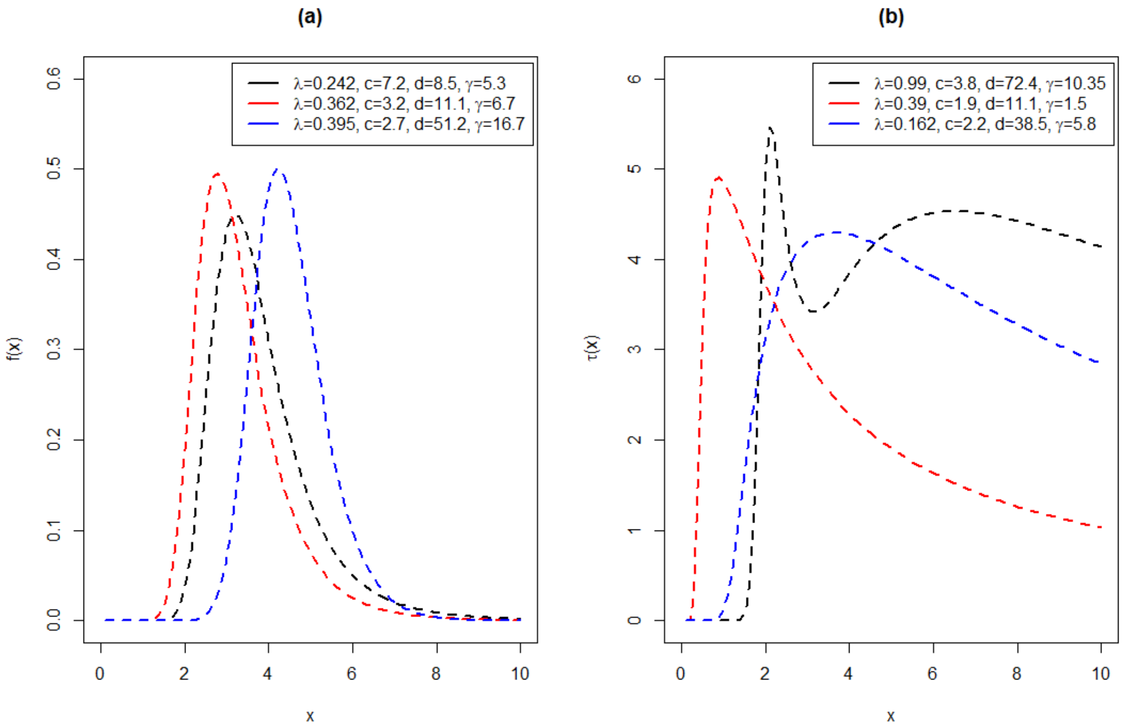

Figure 7.3: **Plots of EGGIE (a) PDF and (b) hazard rate function for some parameter values**

The quantile function of the EGGIE distribution is given by

$$Q(u) = \left\{ \frac{-1}{\gamma} \log \left[1 - \left(1 - \left(\frac{u(1-\lambda)}{1-u\lambda} \right)^{\frac{1}{c}} \right)^{\frac{1}{d}} \right] \right\}^{-1}. \tag{7.48}$$

7.7.4 EGLIE Distribution

From equation (7.19), the PDF of EGLIE distribution is given by

$$f(x) = \frac{\lambda\gamma c d x^{-2} e^{-\gamma x^{-1}} (1-e^{-\gamma x^{-1}})^{d-1} \left(1 - \left(1 - e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1}}{\log(1-\lambda) \left[\lambda \left[1 - \left(1 - (1-e^{-\gamma x^{-1}})^{d}\right)^{c}\right] - 1\right]}, \; 0 < \lambda < 1, \; \gamma, \; c, \; d > 0, \; x > 0. \tag{7.49}$$

The corresponding hazard rate function is given by

$$\tau(x)=\frac{\lambda\gamma c d x^{-2}e^{-\gamma x^{-1}}(1-e^{-\gamma x^{-1}})^{d-1}\left(1-\left(1-e^{-\gamma x^{-1}}\right)^{d}\right)^{c-1}}{\log\left[1-\lambda\left[1-\left(1-(1-e^{-\gamma x^{-1}})^{d}\right)^{c}\right]\right]\left[\lambda\left[1-\left(1-(1-e^{-\gamma x^{-1}})^{d}\right)^{c}\right]-1\right]},$$

$$0<\lambda<1,\ \gamma,\ c,\ d>0,\ x>0. \tag{7.50}$$

Figure 7.4 shows the density and hazard rate function of the EGLIE distribution for some parameter values. The density exhibit approximately symmetric shapes with different degrees of kurtosis. The hazard rate function shows upside down bathtub shapes.

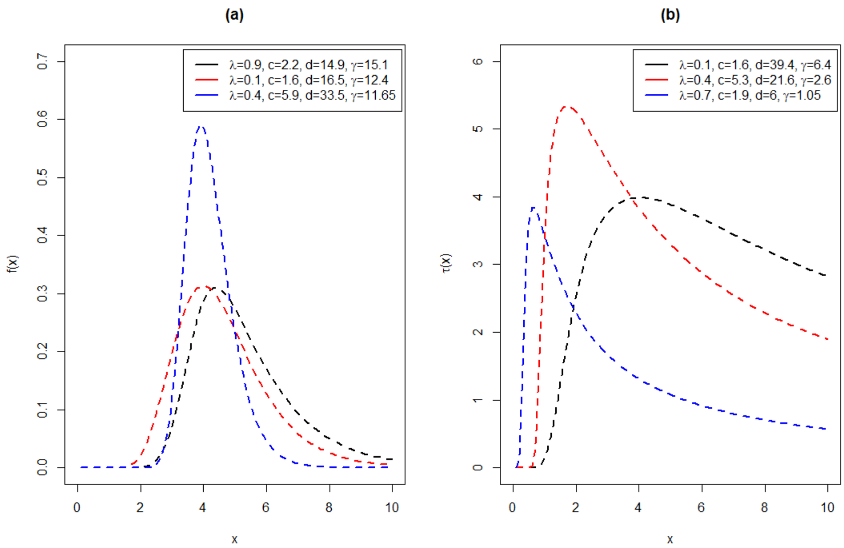

Figure 7.4: **Plots of EGLIE (a) PDF and (b) hazard rate function for some parameter values**

The quantile function of the EGLIE distribution is given by

$$Q(u)=\left\{\frac{-1}{\gamma}\log\left[1-\left(1-\left(1-\frac{1-(1-\lambda)^{1-u}}{\lambda}\right)^{\frac{1}{c}}\right)^{\frac{1}{d}}\right]\right\}^{-1}. \tag{7.51}$$

7.8 Monte Carlo Simulation

In this section, Monte Carlo simulations were performed to examine the finite sample properties of the maximum likelihood estimators for the parameters of the EGPIE, EGBIE, EGGIE and EGLIE distributions. For the case of the EGBIE distribution, $m = 5$ was used during the simulation. The simulation steps are as follows:

1. Specify the values of the parameters λ, c, d, γ and the sample size n.
2. Generate random samples of size $n = 25$, 50, 75, 100 from EGPIE, EGBIE, EGGIE and EGLIE distributions using their respective quantiles.
3. Find the maximum likelihood estimates for the parameters.
4. Repeat steps $2 - 3$ for $N = 1500$ times.
5. Calculate the average estimate (AE) and RMSE for the parameters of the distributions.

Table 7.2 shows the simulation results for the EGPIE and EGBIE distributions whereas Table 7.3 displays that of the EGGIE and EGLIE distributions. From both tables it can be seen that the AE for the estimators were quite close to the actual values. The RMSE for the estimators of the parameters decreases as the sample size increases.

Table 7.2: **Monte Carlo simulation results: AE and RMSE for EGPIE and EGBIE distributions**

	Parameter				EGPIE								EGBIE							
					AE				RMSE				AE				RMSE			
n	λ	c	d	γ	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$
25	0.8	0.5	0.2	0.3	0.692	0.450	0.219	0.370	0.313	0.097	0.065	0.091	0.638	0.438	0.236	0.368	0.325	0.109	0.077	0.091
50	0.8	0.5	0.2	0.3	0.654	0.448	0.214	0.364	0.337	0.088	0.054	0.087	0.635	0.434	0.229	0.367	0.311	0.103	0.067	0.089
75	0.8	0.5	0.2	0.3	0.628	0.445	0.213	0.360	0.354	0.085	0.049	0.084	0.638	0.440	0.228	0.360	0.300	0.102	0.064	0.087
100	0.8	0.5	0.2	0.3	0.609	0.444	0.211	0.357	0.363	0.308	0.043	0.081	0.638	0.438	0.227	0.360	0.295	0.101	0.306	0.086
25	0.1	0.9	0.1	0.5	0.156	0.873	0.121	0.587	0.094	0.129	0.032	0.095	0.131	0.885	0.120	0.585	0.083	0.121	0.033	0.095
50	0.1	0.9	0.1	0.5	0.153	0.902	0.119	0.582	0.092	0.100	0.027	0.089	0.113	0.898	0.118	0.580	0.077	0.105	0.026	0.089
75	0.1	0.9	0.1	0.5	0.143	0.909	0.117	0.579	0.091	0.089	0.022	0.086	0.102	0.904	0.117	0.576	0.074	0.089	0.023	0.084
100	0.1	0.9	0.1	0.5	0.140	0.910	0.116	0.574	0.091	0.082	0.020	0.082	0.0914	0.901	0.117	0.572	0.073	0.085	0.021	0.081
25	0.5	2.5	6.2	3.5	0.582	2.749	5.612	3.215	0.233	0.331	1.116	0.476	0.537	2.744	5.855	3.263	0.223	0.320	0.983	0.439
50	0.5	2.5	6.2	3.5	0.578	2.723	5.785	3.298	0.230	0.327	0.891	0.364	0.540	2.718	5.892	3.314	0.210	0.313	0.848	3.314
75	0.5	2.5	6.2	3.5	0.582	2.725	5.840	3.309	0.228	0.315	0.797	0.335	0.538	2.698	5.904	3.323	0.208	0.310	0.800	0.334
100	0.5	2.5	6.2	3.5	0.571	2.712	5.886	3.324	0.229	0.314	0.705	0.305	0.531	2.700	5.901	3.316	0.202	0.301	0.776	0.326

Table 7.3: **Monte Carlo simulation results: AE and RMSE for EGGIE and EGLIE distributions**

	Parameter				EGGIE								EGLIE							
					AE				RMSE				AE				RMSE			
n	λ	c	d	γ	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$
25	0.8	0.5	0.2	0.3	0.817	0.556	0.174	0.282	0.099	0.083	0.047	0.039	0.771	0.508	0.218	0.318	0.144	0.080	0.080	0.078
50	0.8	0.5	0.2	0.3	0.834	0.558	0.175	0.285	0.072	0.078	0.043	0.031	0.784	0.505	0.215	0.316	0.101	0.073	0.071	0.068
75	0.8	0.5	0.2	0.3	0.834	0.554	0.175	0.285	0.068	0.075	0.042	0.029	0.780	0.502	0.219	0.311	0.092	0.073	0.069	0.062
100	0.8	0.5	0.2	0.3	0.835	0.552	0.175	0.285	0.065	0.072	0.040	0.030	0.779	0.503	0.220	0.308	0.081	0.073	0.065	0.060
25	0.1	0.9	0.1	0.5	0.087	0.939	0.121	0.561	0.161	0.112	0.035	0.100	0.073	0.936	0.122	0.560	0.138	0.113	0.035	0.100
50	0.1	0.9	0.1	0.5	0.066	0.933	0.119	0.559	0.139	0.100	0.028	0.090	0.066	0.933	0.119	0.562	0.133	0.097	0.027	0.088
75	0.1	0.9	0.1	0.5	0.053	0.926	0.118	0.555	0.123	0.090	0.024	0.081	0.055	0.926	0.118	0.555	0.121	0.090	0.024	0.082
100	0.1	0.9	0.1	0.5	0.043	0.923	0.117	0.553	0.118	0.085	0.022	0.076	0.045	0.924	0.117	0.554	0.114	0.085	0.022	0.076
25	0.5	2.5	6.2	3.5	0.360	2.601	5.825	3.286	0.319	0.415	1.014	0.436	0.360	2.601	5.825	3.286	0.319	0.415	1.014	0.436
50	0.5	2.5	6.2	3.5	0.409	2.640	5.937	3.329	0.286	0.355	0.842	0.370	0.409	2.641	5.937	3.329	0.286	0.355	0.842	0.370
75	0.5	2.5	6.2	3.5	0.422	2.644	5.983	3.346	0.270	0.332	0.764	0.336	0.422	2.644	5.983	3.346	0.270	0.332	0.764	0.336
100	0.5	2.5	6.2	3.5	0.428	2.660	6.004	3.342	0.263	0.314	0.727	0.336	0.427	2.660	6.004	3.342	0.263	0.314	0.727	0.336

7.9 Application

The application of the EGPIE, EGBIE (with $m = 5$), EGGIE and EGLIE distributions were demonstrated in this section using real data set. The performance of the distributions with regards to providing reasonable parametric fit to data set were compared using the K-S statistic, W*, AIC, AICc, and BIC. The data set comprises 101 observations corresponding to the failure time in hours of Kevlar 49/epoxy strands with pressure at 90%. The data set displayed in Table 7.4 can be found in Barlow et al. (1984) and Andrews and Herzberg (2012).

Table 7.4: **Failure times of Kevlar 49/epoxy strands with pressure at** 90%

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07
0.08	0.09	0.09	0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20
0.23	0.24	0.24	0.29	0.34	0.35	0.36	0.38	0.40	0.42	0.43	0.52
0.54	0.56	0.60	0.60	0.63	0.65	0.67	0.68	0.72	0.72	0.72	0.73
0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33
1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53	1.54	1.54	1.55	1.58
1.60	1.63	1.64	1.80	1.80	1.81	2.02	2.05	2.14	2.17	2.33	3.03
3.03	3.34	4.20	4.69	7.89							

An exploratory analysis of the failure rate function of the data set using the TTT transform plot revealed that the data exhibit a modified bathtub shape. The TTT curve shows a convex shape and then concave shape followed by a convex shape as shown in Figure 7.5.

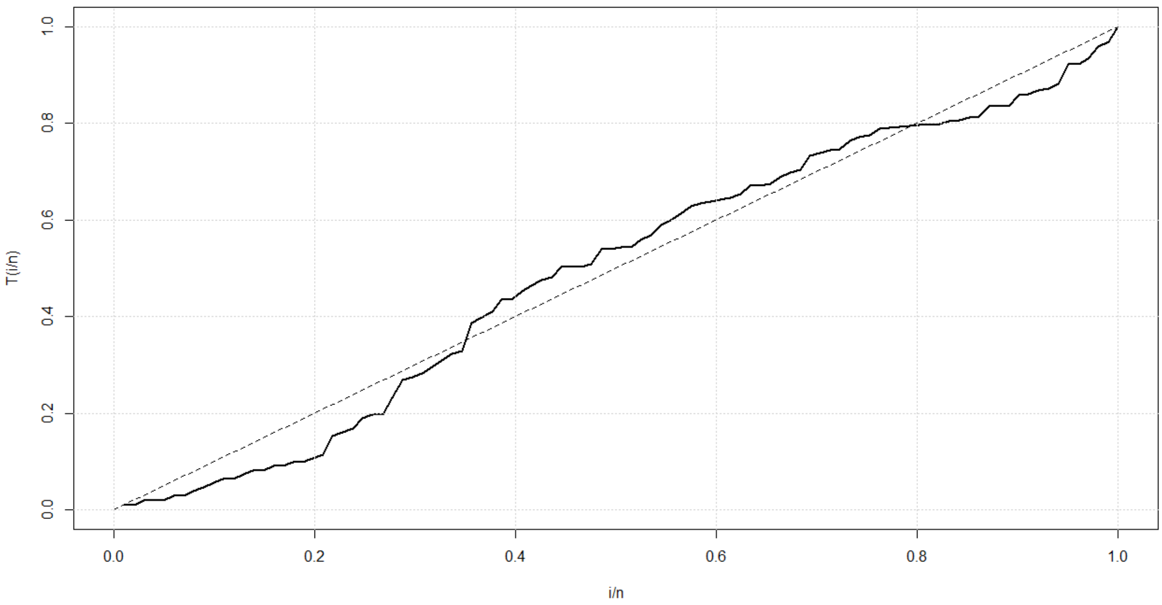

Figure 7.5: **TTT-transform plot for Kevlar data set**

Table 7.5 displays the maximum likelihood estimates for the parameters of the fitted distributions with their corresponding standard errors in brackets. To test for the significance of the parameters of the fitted models, the standard error test was employed. The parameters of the EGPIE and EGGIE distributions were all significant at the 5% level with the exception of the parameter γ for the two distributions. The parameters of the EGBIE distribution were all significant at the 5% level. The EGLIE distribution parameters were all significant at the 5% level with the exception of the parameter λ.

Table 7.5: **Maximum likelihood estimates of parameters and standard errors of Kevlar data**

Model	$\hat{\lambda}$	$\hat{c}$	$\hat{d}$	$\hat{\gamma}$
EGPIE	26.062	7.320	0.175	0.002
	(0.009)	(1.770)	(0.019)	(0.002)
EGBIE	11.644	8.862	0.313	0.003
	(1.925×10^{-5})	(1.136×10^{-4})	(2.141×10^{-2})	(8.755×10^{-5})
EGGIE	0.664	20.525	0.498	0.002
	(2.412×10^{-1})	(4.798×10^{-3})	(1.360×10^{-1})	(9.954×10^{-4})
EGLIE	0.018	19.277	0.616	0.002
	(5.004×10^{-1})	(3.845×10^{-3})	(6.431×10^{-2})	(6.301×10^{-4})

The EGPIE distribution provides a better fit to the data set compared to the other models. From Table 7.6, the EGPIE distribution has the highest log-likelihood and the smallest K-S, W*, AIC, AICc and BIC values compared to the other fitted models.

Table 7.6: **Log-likelihood, goodness-of-fit statistics and information criteria of Kevlar data**

Model	ℓ	AIC	AICc	BIC	K-S	W*
EGPIE	**-116.660**	**241.314**	**241.946**	**251.774**	**0.182**	**0.738**
EGBIE	-122.930	253.868	254.500	264.328	0.195	0.926
EGGIE	-140.090	288.170	288.802	298.631	0.237	1.386
EGLIE	-134.010	276.025	276.657	286.486	0.203	1.211

The plots of the empirical density, the fitted densities, the empirical CDF and the CDF of the fitted distributions are shown in Figure 7.6.

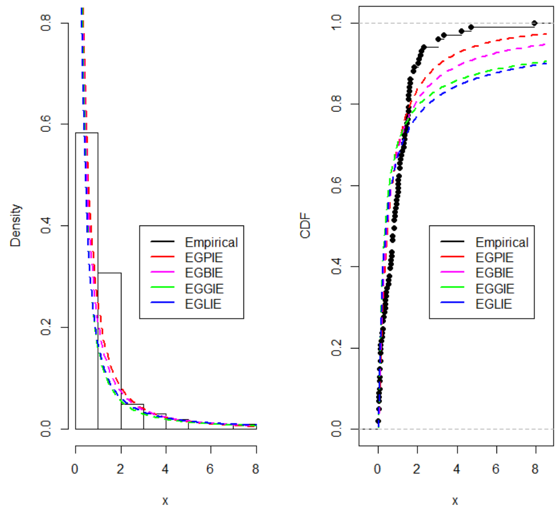

Figure 7.6: **Empirical and fitted density and CDF plots of Kevlar data**

In addition, the P-P plots in Figure 7.7 shows that the EGPIE and EGBIE distributions provide a more reasonable fit to the data compared to the EGGIE and EGLIE distributions.

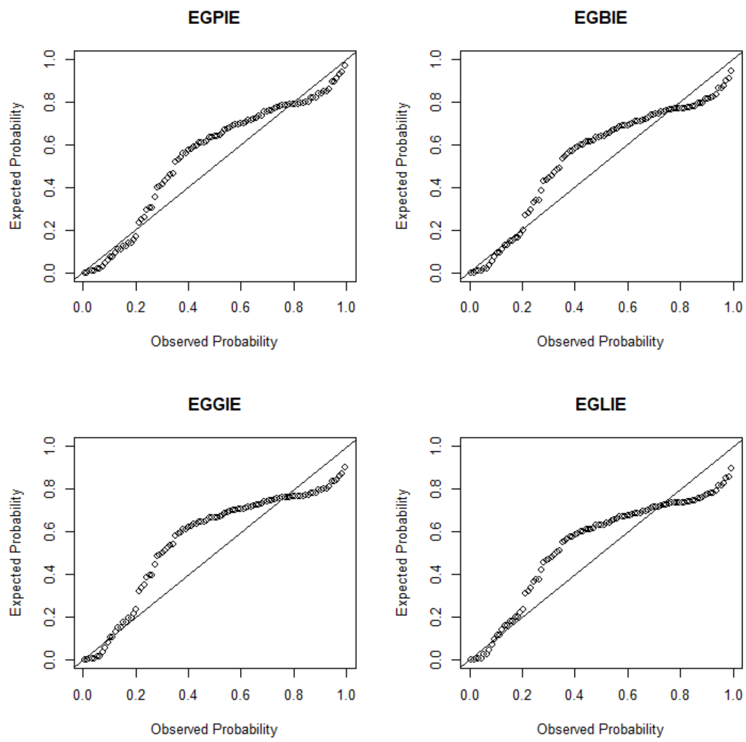

Figure 7.7: **P-P plots of the fitted distributions**

7.10 Summary

In this chapter, the EGPS family of distributions was developed and studied. Various statistical properties such as the quantile function, moments, moment generating function, incomplete moment, reliability, residual life, mean residual life, Shannon entropy and order statistics were derived. The method of maximum likelihood estimation was proposed for the estimation of the parameters of the family. Bivariate and multivariate extensions of the family was proposed using the Clayton copula. Some special distributions were defined and Monte Carlo simulations were performed to investigate the statistical properties of the estimators for the parameters of the special distributions. Finally, an application of the special distributions was illustrated using real data set.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 Introduction

This chapter presents the conclusions and recommendations for future works.

8.2 Conclusions

The knowledge of an appropriate statistical distribution in modeling lifetime data is imperative in different fields of study. Most parametric inferences in these areas heavily depend on some distributional assumptions. However, some of the data sets from these fields may not be well described by the existing standard distributions. Hence, researchers in the area of distribution theory are developing barrage of generators for modifying existing statistical distributions to make them more flexible in providing reasonable parametric fit to data sets.

In this study, a new statistical distribution generator called EG T-X family was proposed and studied. Some sub-families of the generator were developed. These include: EGE-X, EG beta-exponential-X, EG exponentiated exponential-X, EG gamma-X, EG Gompertz-X, EGHL-X, EG lomax-X, EG Burr XII-X and EG Weibull-X. The statistical properties of the generator, such as the quantile function, moments, moment generating function and Shannon entropy were derived.

Further, the EGE-X family was used to modify the Dagum and modified inverse Rayleigh distributions to obtain the EGED and NEGMIR distributions respectively. Sub-models of these modified distributions were defined and their statistical properties derived. The maximum likelihood estimation technique was employed to develop estimators for the parameters of the distributions and simulation studies were performed to assess the properties of the estimators. The applications of the EGED and NEGMIR distributions were demonstrated using real data sets and their performance compared to that of their sub-models and other existing candidate models. The goodness-of-fit statistics and the information criteria used all revealed that the new models were better than their sub-models and the other competing models. Also, the EGHL-X generator was used to modify the Burr X distribution to obtain the EGHLBX distribution. Sub-models of the EGHLBX were defined and the statistical properties derived. The parameters of the distribution were estimated using the maximum likelihood method and the properties of the estimators for the parameters were investigated using simulation.

Finally, an extension of the EG class was proposed by compounding it with the PS class to form a new family of distribution called the EGPS family. Sub-families of the EGPS such as the EGP, EGB, EGG and EGL were defined. The statistical properties were derived and the parameters of the family were estimated using the maximum likelihood technique. Bivariate and multivariate extensions of the family using the Clayton copula were proposed. Some special distributions such as EGPIE, EGBIE, EGGIE and EGLE were defined and simulation studies were conducted to examine the properties of the estimators for the parameters of these distributions. The usefulness of the special distributions was demonstrated using real data set.

8.3 Recommendations

The data sets used in this study were complete samples. However, incomplete samples may arise in different fields of studies. For instance, in a follow-up study in medical research, cancer or tuberculosis patients may die before the study end or survive beyond the duration of the study. Hence, further studies should consider the use of censored data in demonstrating the applications of the developed models.

Several sub-family generators for modifying distributions were proposed in this study. Therefore, subsequent further research should consider using these generators to modify existing distributions and investigate their performance in terms of providing reasonable parametric fit to both complete and incomplete data sets.

Again, a phenomenon may be influenced by a number of independent variables. For example, the trap efficiency in a dam may be influenced by the age of the dam, annual rainfall and inflow among others. It is important to investigate how each of the factors affects the output variable. Thus, parametric regression models for studying the relationship between an output variable and input variables may be developed in subsequent studies using the proposed distributions.

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APPENDICES

Appendix A1

Elements of the observed information matrix of the EGED distribution.

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial c} = -\sum_{i=1}^{n} \frac{[1-(1-z_i^{-\beta})^d]^c \log[1-(1-z_i^{-\beta})^d]}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial d} = \sum_{i=1}^{n} \frac{c(1-z_i^{-\beta})^d[1-(1-z_i^{-\beta})^d]^{c-1} \log(1-z_i^{-\beta})}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \beta} = \sum_{i=1}^{n} \frac{cdz_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1} \log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \theta} = -\sum_{i=1}^{n} \frac{\alpha\beta cdx_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1} \log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \frac{\beta cdx_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial c^2} = -\frac{n}{c^2} - (\lambda-1)\sum_{i=1}^{n} \frac{[1-(1-z_i^{-\beta})^d]^{2c} \log[1-(1-z_i^{-\beta})^d]^2}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{[1-(1-z_i^{-\beta})^d]^c \log[1-(1-z_i^{-\beta})^d]^2}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial c \partial d} = -\sum_{i=1}^{n} \frac{(1-z_i^{-\beta})^d \log(1-z_i^{-\beta})}{1-(1-z_i^{-\beta})^d} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(1-z_i^{-\beta})^d[1-(1-z_i^{-\beta})^d]^{2c-1}\log(1-z_i^{-\beta})\log[1-(1-z_i^{-\beta})^d]}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{(1-z_i^{-\beta})^d[1-(1-z_i^{-\beta})^d]^{c-1}\log(1-z_i^{-\beta})}{1-[1-(1-z_i^{-\beta})^d]^c} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(1-z_i^{-\beta})^d[1-(1-z_i^{-\beta})^d]^{c-1}\log(1-z_i^{-\beta})\log[1-(1-z_i^{-\beta})^d]}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial c \partial \beta} = -\sum_{i=1}^{n} \frac{dz_i^{-\beta}(1-z_i^{-\beta})^{d-1}\log(z_i)}{1-(1-z_i^{-\beta})^d} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{cdz_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{2c-1}\log(z_i)\log[1-(1-z_i^{-\beta})^d]}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{dz_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{cdz_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)\log[1-(1-z_i^{-\beta})^d]}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial c \partial \theta} = \sum_{i=1}^{n} \frac{\alpha\beta dx_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(x_i)}{1-(1-z_i^{-\beta})^d} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{\alpha\beta cdx_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{2c-1}\log(x_i)\log[1-(1-z_i^{-\beta})^d]}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{\alpha\beta dx_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{\alpha\beta cdx_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)\log[1-(1-z_i^{-\beta})^d]}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2\ell}{\partial c\partial\alpha} = -\sum_{i=1}^{n}\frac{\beta d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}}{1-(1-z_i^{-\beta})^d}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta c d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{2c-1}\log[1-(1-z_i^{-\beta})^d]}{[1-(1-(1-z_i^{-\beta})^d)^c]^2}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta c d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log[1-(1-z_i^{-\beta})^d]}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2\ell}{\partial d^2} = -\frac{n}{d^2}-(c-1)\sum_{i=1}^{n}\frac{(1-z_i^{-\beta})^{2d}\log(1-z_i^{-\beta})^2}{[1-(1-z_i^{-\beta})^d]^2}-(c-1)\sum_{i=1}^{n}\frac{(1-z_i^{-\beta})^{d}\log(1-z_i^{-\beta})^2}{[1-(1-z_i^{-\beta})^d]^2}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c^2(1-z_i^{-\beta})^{2d}[1-(1-z_i^{-\beta})^d]^{2(c-1)}\log(1-z_i^{-\beta})^2}{[1-(1-(1-z_i^{-\beta})^d)^c]^2}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)(1-z_i^{-\beta})^{2d}[1-(1-z_i^{-\ beta})^d]^{c-2}\log(1-z_i^{-\beta})^2}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c(1-z_i^{-\beta})^{d}[1-(1-z_i^{-\beta})^d]^{c-1}\log(1-z_i^{-\beta})^2}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2\ell}{\partial d\partial\beta} = \sum_{i=1}^{n}\frac{z_i^{-\beta}\log(z_i)}{1-z_i^{-\beta}}-(c-1)\sum_{i=1}^{n}\frac{d z_i^{-\beta}(1-z_i^{-\beta})^{2d-1}\log(z_i)\log(1-z_i^{-\beta})}{[1-(1-z_i^{-\beta})^d]^2}-$$
$$(c-1)\sum_{i=1}^{n}\frac{z_i^{-\beta}(1-z_i^{-\beta})^{d-1}\log(z_i)}{1-(1-z_i^{-\beta})^d}-(c-1)\sum_{i=1}^{n}\frac{d z_i^{-\beta}(1-z_i^{-\beta})^{d-1}\log(z_i)\log(1-z_i^{-\beta})}{1-(1-z_i^{-\beta})^d}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c^2 d z_i^{-\beta}(1-z_i^{-\beta})^{2d-1}[1-(1-z_i^{-\beta})^d]^{2(c-1)}\log(z_i)\log(1-z_i^{-\beta})}{[1-(1-(1-z_i^{-\beta})^d)^c]^2}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1) d z_i^{-\beta}(1-z_i^{-\beta})^{2d-1}[1-(1-z_i^{-\beta})^d]^{c-2}\log(z_i)\log(1-z_i^{-\beta})}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c z_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c d z_i^{-\beta}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)\log(1-z_i^{-\beta})}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial d \partial \alpha} = \sum_{i=1}^{n} \frac{\beta x_i^{-\theta} z_i^{-\beta-1}}{1 - z_i^{-\beta}} - (c-1) \sum_{i=1}^{n} \frac{\beta d x_i^{-\theta} (1 - z_i^{-\beta})^{2d-1} \log(1 - z_i^{-\beta})}{[1 - (1 - z_i^{-\beta})^d]^2} -$$
$$(c-1) \sum_{i=1}^{n} \frac{\beta x_i^{-\theta} z_i^{-\beta-1} (1 - z_i^{-\beta})^{d-1}}{1 - (1 - z_i^{-\beta})^d} - (c-1) \sum_{i=1}^{n} \frac{\beta d x_i^{-\theta} z_i^{-\beta-1} (1 - z_i^{-\beta})^{d-1} \log(1 - z_i^{-\beta})}{1 - (1 - z_i^{-\beta})^d} -$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{\beta c^2 d x_i^{-\theta} z_i^{-\beta-1} (1 - z_i^{-\beta})^{2d-1} [1 - (1 - z_i^{-\beta})^d]^{2(c-1)} \log(1 - z_i^{-\beta})}{[1 - (1 - (1 - z_i^{-\beta})^d)^c]^2} -$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{\beta c(c-1) d x_i^{-\theta} z_i^{-\beta-1} (1 - z_i^{-\beta})^{2d-1} [1 - (1 - z_i^{-\beta})^d]^{c-2} \log(1 - z_i^{-\beta})}{1 - [1 - (1 - z_i^{-\beta})^d]^c} +$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{\beta c x_i^{-\theta} z_i^{-\beta-1} (1 - z_i^{-\beta})^{d-1} [1 - (1 - z_i^{-\beta})^d]^{c-1}}{1 - [1 - (1 - z_i^{-\beta})^d]^c} +$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{\beta c d x_i^{-\theta} z_i^{-\beta-1} (1 - z_i^{-\beta})^{d-1} [1 - (1 - z_i^{-\beta})^d]^{c-1} \log(1 - z_i^{-\beta})}{1 - [1 - (1 - z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} - (d-1) \sum_{i=1}^{n} \frac{z_i^{-2\beta} \log(z_i)^2}{(1 - z_i^{-\beta})^2} - (d-1) \sum_{i=1}^{n} \frac{z_i^{-\beta} \log(z_i)^2}{1 - z_i^{-\beta}} +$$
$$(c-1) \sum_{i=1}^{n} \frac{d z_i^{-\beta} (1 - z_i^{-\beta})^{d-1} \log(z_i)^2}{1 - (1 - z_i^{-\beta})^d} - (c-1) \sum_{i=1}^{n} \frac{d^2 z_i^{-2\beta} (1 - z_i^{-\beta})^{2(d-1)} \log(z_i)^2}{[1 - (1 - z_i^{-\beta})^d]^2} -$$
$$(c-1) \sum_{i=1}^{n} \frac{d(d-1) z_i^{-2\beta} (1 - z_i^{-\beta})^{d-2} \log(z_i)^2}{1 - (1 - z_i^{-\beta})^d} -$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{c^2 d^2 z_i^{-2\beta} (1 - z_i^{-\beta})^{2(d-1)} [1 - (1 - z_i^{-\beta})^d]^{2(c-1)} \log(z_i)^2}{[1 - (1 - (1 - z_i^{-\beta})^d)^c]^2} -$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^{-2\beta} (1 - z_i^{-\beta})^{2(d-1)} [1 - (1 - z_i^{-\beta})^d]^{c-2} \log(z_i)_2}{1 - [1 - (1 - z^{-\beta})^d]^c} +$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{c d(d-1) z_i^{-2\beta} (1 - z_i^{-\beta})^{d-2} [1 - (1 - z_i^{-\beta})^d]^{c-1} \log(z_i)^2}{1 - [1 - (1 - z_i^{-\beta})^d]^c} -$$
$$(\lambda-1) \sum_{i=1}^{n} \frac{c d z_i^{-\beta} (1 - z_i^{-\beta})^{d-1} [1 - (1 - z_i^{-\beta})^d]^{c-1} \log(z_i)^2}{1 - [1 - (1 - z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2\ell}{\partial\beta\partial\theta} = \sum_{i=1}^{n}\frac{\alpha x_i^{-\theta}\log(x_i)}{z_i} + (d-1)\sum_{i=1}^{n}\frac{\alpha\beta x_i^{-\theta}z_i^{-2\beta-1}\log(x_i)\log(z_i)}{(1-z_i^{-\beta})^2} - (d-1)\sum_{i=1}^{n}\frac{\alpha x_i^{-\theta}z_i^{-\beta-1}\log(x_i)}{1-z_i^{-\beta}}$$
$$+ (d-1)\sum_{i=1}^{n}\frac{\alpha\beta x_i^{-\theta}z_i^{-\beta-1}\log(x_i)\log(z_i)}{1-z_i^{-\beta}} - (c-1)\sum_{i=1}^{n}\frac{\alpha\beta d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(x_i)\log(z_i)}{1-(1-z_i^{-\beta})^d}+$$
$$(c-1)\sum_{i=1}^{n}\frac{\alpha d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(x_i)}{1-(1-z_i^{-\beta})^d} + (c-1)\sum_{i=1}^{n}\frac{\alpha\beta d^2 x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{2(d-1)}\log(x_i)\log(z_i)}{[1-(1-z_i^{-\beta})^d]^2}+$$
$$(c-1)\sum_{i=1}^{n}\frac{\alpha\beta d(d-1)x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{d-2}\log(x_i)\log(z_i)}{1-(1-z_i^{-\beta})^d}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha c d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta c^2 d^2 x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{2(c-1)}\log(x_i)\log(z_i)}{[1-(1-(1-z_i^{-\beta})^d)^c]^2}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta c(c-1)d^2 x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{c-2}\log(x_i)\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta c d(d-1)x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{d-2}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta c d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2\ell}{\partial\beta\partial\alpha} = -\sum_{i=1}^{n}\frac{x_i^{-\theta}}{z_i} - (d-1)\sum_{i=1}^{n}\frac{\beta x_i^{-\theta}z_i^{-2\beta-1}\log(z_i)}{(1-z_i^{-\beta})^2} + (d-1)\sum_{i=1}^{n}\frac{x_i^{-\theta}z_i^{-\beta-1}}{1-z_i^{-\beta}}-$$
$$(d-1)\sum_{i=1}^{n}\frac{\beta x_i^{-\theta}z_i^{-\beta-1}\log(z_i)}{1-z_i^{-\beta}} - (c-1)\sum_{i=1}^{n}\frac{\beta d^2 x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{2(d-1)}\log(z_i)}{[1-(1-z_i^{-\beta})^d]^2}-$$
$$(c-1)\sum_{i=1}^{n}\frac{\beta d(d-1)x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{d-2}\log(z_i)}{1-(1-z_i^{-\beta})^d} - (c-1)\sum_{i=1}^{n}\frac{d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}}{1-(1-z_i^{-\beta})^d}+$$
$$(c-1)\sum_{i=1}^{n}\frac{\beta d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(z_i)}{1-(1-z_i^{-\beta})^d} + (\lambda-1)\sum_{i=1}^{n}\frac{c d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}}{1-[1-(1-z_i^{-\beta})^d]^c}$$
$$-(\lambda-1)\sum_{i=1}^{n}\frac{\beta c^2 d^2 x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})_d]^{2(c-1)}\log(z_i)}{[1-(1-(1-z_i^{-\beta})^d)^c]^2}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta c(c-1)d^2 x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{c-2}\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta c d(d-1)x_i^{-\theta}z_i^{-2\beta-1}(1-z_i^{-\beta})^{d-2}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta c d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(z_i)}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$
\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2} + (\beta+1)\sum_{i=1}^{n} \frac{\alpha^2 x_i^{-2\theta}\log(x_i)^2}{z_i^2} - (\beta+1)\sum_{i=1}^{n} \frac{\alpha x_i^{-\theta}\log(x_i)^2}{z_i} -
$$

$$
(d-1)\sum_{i=1}^{n} \frac{\alpha^2\beta^2 x_i^{-2\theta} z_i^{-2(\beta+1)}\log(x_i)^2}{(1-z_i^{-\beta})^2} + (d-1)\sum_{i=1}^{n} \frac{\alpha^2\beta(-\beta-1) x_i^{-2\theta} z_i^{-\beta-2}\log(x_i)^2}{1-z_i^{-\beta}} +
$$

$$
(d-1)\sum_{i=1}^{n} \frac{\alpha\beta x_i^{-\theta} z_i^{-\beta-1}\log(x_i)^2}{1-z_i^{-\beta}} - (c-1)\sum_{i=1}^{n} \frac{\alpha^2\beta^2 d^2 x_i^{-2\theta} z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}\log(x_i)^2}{[1-(1-z_i^{-\beta})^d]^2} -
$$

$$
(c-1)\sum_{i=1}^{n} \frac{\alpha^2\beta^2 d(d-1) x_i^{-2\theta} z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{d-2}\log(x_i)^2}{1-(1-z_i^{-\beta})^d} -
$$

$$
(c-1)\sum_{i=1}^{n} \frac{\alpha^2\beta(-\beta-1) d x_i^{-2\theta} z_i^{-\beta-2}(1-z_i^{-\beta})^{d-1}\log(x_i)^2}{1-(1-z_i^{-\beta})^d} - (c-1)\sum_{i=1}^{n} \frac{\alpha\beta d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(x_i)^2}{1-(1-z_i^{-\beta})^d}
$$

$$
-(\lambda-1)\sum_{i=1}^{n} \frac{\alpha^2\beta^2 c^2 d^2 x_i^{-2\theta} z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{2(c-1)}\log(x_i)^2}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} -
$$

$$
(\lambda-1)\sum_{i=1}^{n} \frac{\alpha^2\beta^2 c(c-1) d^2 x_i^{-2\theta} z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{c-2}\log(x_i)^2}{1-[1-(1-z_i^{-\beta})^d]^c} +
$$

$$
(\lambda-1)\sum_{i=1}^{n} \frac{\alpha^2\beta^2 c d(d-1) x_i^{-2\theta} z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{d-2}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)^2}{1-[1-(1-z_i^{-\beta})^d]^c} +
$$

$$
(\lambda-1)\sum_{i=1}^{n} \frac{\alpha^2\beta(-\beta-1) c d x_i^{-2\theta} z_i^{-\beta-2}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)^2}{1-[1-(1-z_i^{-\beta})^d]^c} +
$$

$$
(\lambda-1)\sum_{i=1}^{n} \frac{\alpha\beta c d x_i^{-\theta} z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)^2}{1-[1-(1-z_i^{-\beta})^d]^c},
$$

$$\frac{\partial^2 \ell}{\partial\theta\partial\alpha} = -(\beta+1)\sum_{i=1}^{n}\frac{\alpha x_i^{-2\theta}\log(x_i)}{z_i^2} + (\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\theta}\log(x_i)}{z_i} + (d-1)\sum_{i=1}^{n}\frac{\alpha\beta^2 x_i^{-2\theta} z_i^{-2(\beta+1)}\log(x_i)}{(1-z_i^{-\beta})^2}$$

$$-(d-1)\sum_{i=1}^{n}\frac{\alpha\beta(-\beta-1)x_i^{-2\theta}z_i^{-\beta-2}\log(x_i)}{1-z_i^{-\beta}} - (d-1)\sum_{i=1}^{n}\frac{\beta x_i^{-\theta}z_i^{-\beta-1}\log(x_i)}{1-z_i^{-\beta}} +$$

$$(c-1)\sum_{i=1}^{n}\frac{\alpha\beta^2 d^2 x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}\log(x_i)}{[1-(1-z_i^{-\beta})^d]^2} +$$

$$(c-1)\sum_{i=1}^{n}\frac{\alpha\beta^2 d(d-1)x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{d-2}\log(x_i)}{1-(1-z_i^{-\beta})^d} +$$

$$(c-1)\sum_{i=1}^{n}\frac{\alpha\beta(-\beta-1)dx_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{d-1}\log(x_i)}{1-(1-z_i^{-\beta})^d} + (c-1)\sum_{i=1}^{n}\frac{\beta d x_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}\log(x_i)}{1-(1-z_i^{-\beta})^d} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta^2 c^2 d^2 x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{2(c-1)}\log(x_i)}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta^2 c(c-1)d^2 x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{c-2}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta^2 cd(d-1)x_i^{-2\theta}z_i^{-2(\beta-1)}(1-z_i^{-\beta})^{d-2}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\beta(-\beta-1)cdx_i^{-2\theta}z_i^{-\beta-2}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta cdx_i^{-\theta}z_i^{-\beta-1}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}\log(x_i)}{1-[1-(1-z_i^{-\beta})^d]^c},$$

$$\frac{\partial^2 \ell}{\partial\alpha^2} = -\frac{n}{\alpha^2} + (\beta+1)\sum_{i=1}^{n}\frac{x_i^{-2\theta}}{z_i^2} - (d-1)\sum_{i=1}^{n}\frac{\beta^2 x_i^{-2\theta}z_i^{-2(\beta+1)}}{(1-z_i^{-\beta})^2} + (d-1)\sum_{i=1}^{n}\frac{\beta(-\beta-1)x_i^{-2\theta}z_i^{-\beta-2}}{1-z_i^{-\beta}}$$

$$-(c-1)\sum_{i=1}^{n}\frac{\beta^2 d^2 x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}}{[1-(1-z_i^{-\beta})^d]^2} - (c-1)\sum_{i=1}^{n}\frac{\beta^2 d(d-1)x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{d-2}}{1-(1-z_i^{-\beta})^d}$$

$$-(c-1)\sum_{i=1}^{n}\frac{\beta(-\beta-1)dx_i^{-2\theta}z_i^{-\beta-2}(1-z_i^{-\beta})^{d-1}}{1-(1-z_i^{-\beta})^d} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta^2 c^2 d^2 x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{2(c-1)}}{[1-(1-(1-z_i^{-\beta})^d)^c]^2} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta^2 c(c-1)d^2 x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{2(d-1)}[1-(1-z_i^{-\beta})^d]^{c-2}}{1-[1-(1-z_i^{-\beta})^d]^c} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta^2 cd(d-1)x_i^{-2\theta}z_i^{-2(\beta+1)}(1-z_i^{-\beta})^{d-2}[1-(1-z_i^{-\beta})^d]^{c-1}}{1-[1-(1-z_i^{-\beta})^d]^c} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{\beta(-\beta-1)cdx_i^{-2\theta}z_i^{-\beta-2}(1-z_i^{-\beta})^{d-1}[1-(1-z_i^{-\beta})^d]^{c-1}}{1-[1-(1-z_i^{-\beta})^d]^c}.$$

Appendix A2

Elements of the observed information matrix for the NEGMIR distribution.

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial c} = -\sum_{i=1}^{n} \frac{(1-\bar{z}_i^d)^c \log(1-\bar{z}_i^d)}{1-(1-\bar{z}_i^d)^c},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial d} = \sum_{i=1}^{n} \frac{c\bar{z}_i^d (1-\bar{z}_i^d)^{c-1} \log(\bar{z}_i)}{1-(1-\bar{z}_i^d)^c},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \frac{cdz_i \bar{z}_i^{d-1} (1-\bar{z}^d)^{c-1}}{x_i \left[1-(1-\bar{z}_i^d)^c\right]},$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \theta} = \sum_{i=1}^{n} \frac{cdz_i \bar{z}_i^{d-1} (1-\bar{z}^d)^{c-1}}{x_i^2 \left[1-(1-\bar{z}_i^d)^c\right]},$$

$$\frac{\partial^2 \ell}{\partial c^2} = -\frac{n}{c^2} - (\lambda-1)\sum_{i=1}^{n} \frac{(1-\bar{z}_i^d)^{2c} \log(1-\bar{z}_i^d)^2}{\left[1-(1-\bar{z}_i^d)^c\right]^2} - (\lambda-1)\sum_{i=1}^{n} \frac{(1-\bar{z}_i^d)^c \log(1-\bar{z}_i^d)^2}{1-(1-\bar{z}_i^d)^c},$$

$$\frac{\partial^2 \ell}{\partial c \partial d} = -\sum_{i=1}^{n} \frac{\bar{z}_i^d \log(\bar{z}_i)}{1-\bar{z}_i^d} + (\lambda-1)\sum_{i=1}^{n} \frac{\bar{z}_i^d (1-\bar{z}_i^d)^{c-1} \log(\bar{z}_i)}{1-(1-\bar{z}_i^d)^c} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c\bar{z}_i^d (1-\bar{z}_i^d)^{2c-1} \log(\bar{z}_i) \log(1-\bar{z}_i^d)}{\left[1-(1-\bar{z}_i^d)^c\right]^2} + (\lambda-1)\sum_{i=1}^{n} \frac{c\bar{z}_i^d (1-\bar{z}_i^d)^{c-1} \log(\bar{z}_i) \log(1-\bar{z}_i^d)}{1-(1-\bar{z}_i^d)^c},$$

$$\frac{\partial^2 \ell}{\partial c \partial \alpha} = (\lambda-1)\sum_{i=1}^{n} \frac{dz_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i\left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{cdz_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{2c-1}\log(1-\bar{z}_i^d)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]^2} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{cdz_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(1-\bar{z}_i^d)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{dz_i \bar{z}_i^{d-1}}{x_i(1-\bar{z}_i^d)},$$

$$\frac{\partial^2 \ell}{\partial c \partial \theta} = (\lambda-1)\sum_{i=1}^{n} \frac{dz_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{cdz_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{2c-1}\log(1-\bar{z}_i^d)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{cdz_i \bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(1-\bar{z}_i^d)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{dz_i \bar{z}_i^{d-1}}{x_i^2(1-\bar{z}_i^d)},$$

$$\frac{\partial^2 \ell}{\partial d^2} = -\frac{n}{d^2} - (c-1)\sum_{i=1}^{n} \frac{\bar{z}_i^{2d}\log(\bar{z}_i)^2}{(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{\bar{z}_i^{d}\log(\bar{z}_i)^2}{1-\bar{z}_i^d} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c^2\bar{z}_i^{2d}(1-\bar{z}_i^d)^{2(c-1)}\log(\bar{z}_i)^2}{\left[1-(1-\bar{z}_i^d)^c\right]^2} - (\lambda-1)\sum_{i=1}^{n} \frac{c(c-1)\bar{z}_i^{2d}(1-\bar{z}_i^d)^{c-2}\log(\bar{z}_i)^2}{1-(1-\bar{z}_i^d)^c} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c\bar{z}_i^{d}(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)^2}{1-(1-\bar{z}_i^d)^c}$$

$$\frac{\partial^2 \ell}{\partial d \partial \alpha} = \sum_{i=1}^{n} \frac{z_i}{x_i\bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{z_i\bar{z}_i^{d-1}}{x_i(1-\bar{z}_i^d)} - (c-1)\sum_{i=1}^{n} \frac{dz_i\bar{z}_i^{2d-1}\log(\bar{z}_i)}{x_i(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{dz_i\bar{z}_i^{d-1}\log(\bar{z}_i)}{x_i(1-\bar{z}_i^d)} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{cz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i\left[1-(1-\bar{z}_i^d)^c\right]^2} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2dz_i\bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{2(c-1)}\log(\bar{z}_i)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1)dz_i\bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{c-2}\log(\bar{z}_i)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{x_i\left[1-(1-\bar{z}_i^d)^c\right]},$$

$$\frac{\partial^2 \ell}{\partial d \partial \theta} = \sum_{i=1}^{n} \frac{z_i}{x_i^2\bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{z_i\bar{z}_i^{d-1}}{x_i^2(1-\bar{z}_i^d)} - (c-1)\sum_{i=1}^{n} \frac{dz_i\bar{z}_i^{2d-1}\log(\bar{z}_i)}{x_i^2(1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{dz_i\bar{z}_i^{d-1}\log(\bar{z}_i)}{x_i^2(1-\bar{z}_i^d)} +$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{cz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2dz_i\bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{2(c-1)}\log(\bar{z}_i)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1)dz_i\bar{z}_i^{2d-1}(1-\bar{z}_i^d)^{c-2}\log(\bar{z}_i)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{cdz_i\bar{z}_i^{d-1}(1-\bar{z}_i^d)^{c-1}\log(\bar{z}_i)}{x_i^2\left[1-(1-\bar{z}_i^d)^c\right]},$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -(d-1)\sum_{i=1}^{n} \frac{z_i^2}{x_i^2 \bar{z}_i^2} - (d-1)\sum_{i=1}^{n} \frac{z_i}{x_i^2 \bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{d^2 z_i^2 \bar{z}_i^{2(d-1)}}{x_i^2 (1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d(d-1) z_i^2 \bar{z}_i^{d-2}}{x_i^2 (1-\bar{z}_i^d)} +$$
$$(c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^2 (1-\bar{z}_i^d)} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d^2 z_i^2 \bar{z}_i^{2(d-1)} (1-\bar{z}_i^d)^{2(c-1)}}{x_i^2 \left[1-(1-\bar{z}_i^d)^c\right]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^2 \bar{z}_i^{2(d-1)} (1-\bar{z}_i^d)^{c-2}}{x_i^2 \left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{c d(d-1) z_i^2 \bar{z}_i^{d-2} (1-\bar{z}_i^d)^{c-1}}{x_i^2 \left[1-(1-\bar{z}_i^d)^c\right]} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1} (1-\bar{z}_i^d)^{c-1}}{x_i^2 \left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{1}{x_i^4 \left(\frac{\alpha}{x_i^2} + \frac{\theta}{x_i^3}\right)^2},$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = -(d-1)\sum_{i=1}^{n} \frac{z_i^2}{x_i^3 \bar{z}_i^2} - (d-1)\sum_{i=1}^{n} \frac{z_i}{x_i^3 \bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{d^2 z_i^2 \bar{z}_i^{2(d-1)}}{x_i^3 (1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d(d-1) z_i^2 \bar{z}_i^{d-2}}{x_i^3 (1-\bar{z}_i^d)} +$$
$$(c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^3 (1-\bar{z}_i^d)} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d^2 z_i^2 \bar{z}_i^{2(d-1)} (1-\bar{z}_i^d)^{2(c-1)}}{x_i^3 \left[1-(1-\bar{z}_i^d)^c\right]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^2 \bar{z}_i^{2(d-1)} (1-\bar{z}_i^d)^{c-2}}{x_i^3 \left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{c d(d-1) z_i^2 \bar{z}_i^{d-2} (1-\bar{z}_i^d)^{c-1}}{x_i^3 \left[1-(1-\bar{z}_i^d)^c\right]} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1} (1-\bar{z}_i^d)^{c-1}}{x_i^3 \left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{1}{x_i^5 \left(\frac{\alpha}{x_i^2} + \frac{\theta}{x_i^3}\right)^2},$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -(d-1)\sum_{i=1}^{n} \frac{z_i^2}{x_i^4 \bar{z}_i^2} - (d-1)\sum_{i=1}^{n} \frac{z_i}{x_i^4 \bar{z}_i} - (c-1)\sum_{i=1}^{n} \frac{d^2 z_i^2 \bar{z}_i^{2(d-1)}}{x_i^4 (1-\bar{z}_i^d)^2} - (c-1)\sum_{i=1}^{n} \frac{d(d-1) z_i^2 \bar{z}_i^{d-2}}{x_i^4 (1-\bar{z}_i^d)} +$$
$$(c-1)\sum_{i=1}^{n} \frac{d z_i \bar{z}_i^{d-1}}{x_i^4 (1-\bar{z}_i^d)} - (\lambda-1)\sum_{i=1}^{n} \frac{c^2 d^2 z_i^2 \bar{z}_i^{2(d-1)} (1-\bar{z}_i^d)^{2(c-1)}}{x_i^4 \left[1-(1-\bar{z}_i^d)^c\right]^2} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1) d^2 z_i^2 \bar{z}_i^{2(d-1)} (1-\bar{z}_i^d)^{c-2}}{x_i^4 \left[1-(1-\bar{z}_i^d)^c\right]} + (\lambda-1)\sum_{i=1}^{n} \frac{c d(d-1) z_i^2 \bar{z}_i^{d-2} (1-\bar{z}_i^d)^{c-1}}{x_i^4 \left[1-(1-\bar{z}_i^d)^c\right]} -$$
$$(\lambda-1)\sum_{i=1}^{n} \frac{c d z_i \bar{z}_i^{d-1} (1-\bar{z}_i^d)^{c-1}}{x_i^4 \left[1-(1-\bar{z}_i^d)^c\right]} - \sum_{i=1}^{n} \frac{1}{x_i^6 \left(\frac{\alpha}{x_i^2} + \frac{\theta}{x_i^3}\right)^2}.$$

Appendix A3

Elements of the observed information matrix for the EGHLBX distribution

$$\frac{\partial^2\ell}{\partial\lambda^2} = -\frac{n}{\lambda^2} + 2\sum_{i=1}^{n}\frac{[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2\lambda}\log[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^2}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2} -$$
$$2\sum_{i=1}^{n}\frac{[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\log[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^2}{1+(1-(1-(1-\bar{z}_i^{\beta})^d)^c)^{\lambda}},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial c} = -\sum_{i=1}^{n}\frac{(1-(1-\bar{z}_i^{\beta})^d)^c\log(1-(1-\bar{z}_i^{\beta})^d)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} +$$
$$2\sum_{i=1}^{n}\frac{(1-(1-\bar{z}_i^{\beta})^d)^c[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(1-(1-\bar{z}_i^{\beta})^d)}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}} -$$
$$2\sum_{i=1}^{n}\frac{\lambda(1-(1-\bar{z}_i^{\beta})^d)^c[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2\lambda-1}\log(1-(1-\bar{z}_i^{\beta})^d)\log[1-(1-(1-\bar{z}_i^{\beta})^d)^c]}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2} +$$
$$2\sum_{i=1}^{n}\frac{\lambda(1-(1-\bar{z}_i^{\beta})^d)^c[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(1-(1-\bar{z}_i^{\beta})^d)\log[1-(1-(1-\bar{z}_i^{\beta})^d)^c]}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}},$$

$$\frac{\partial^2\ell}{\partial\lambda\partial d} = \sum_{i=1}^{n}\frac{c(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(1-\bar{z}_i^{\beta})}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} -$$
$$2\sum_{i=1}^{n}\frac{c(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} +$$
$$2\sum_{i=1}^{n}\frac{\lambda c(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2\lambda-1}\log(1-\bar{z}_i^{\beta})\log[1-(1-(1-\bar{z}_i^{\beta})^d)^c]}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2} -$$
$$2\sum_{i=1}^{n}\frac{\lambda c(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(1-\bar{z}_i^{\beta})\log[1-(1-(1-\bar{z}_i^{\beta})^d)^c]}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}},$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\beta} = -\sum_{i=1}^{n} \frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}\log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$2\sum_{i=1}^{n} \frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$2\sum_{i=1}^{n} \frac{\lambda cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2\lambda-1}\log(\bar{z}_i)\log[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2} +$$

$$2\sum_{i=1}^{n} \frac{\lambda cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(\bar{z}_i)\log[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}},$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\alpha} = -2\sum_{i=1}^{n} \frac{cd\alpha\beta x_i^2 z_i \bar{z}^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$4\sum_{i=1}^{n} \frac{cd\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$4\sum_{i=1}^{n} \frac{cd\alpha\beta\lambda z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2\lambda-1}\log[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2} +$$

$$4\sum_{i=1}^{n} \frac{cd\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}},$$

$$\frac{\partial^2 \ell}{\partial c^2} = -\frac{n}{c^2} - (\lambda-1)\sum_{i=1}^{n} \frac{(1-(1-\bar{z}_i^{\beta})^{d})^{2c}\log(1-(1-\bar{z}_i^{\beta})^{d})^2}{[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^2} -$$

$$(\lambda-1)\sum_{i=1}^{n} \frac{(1-(1-\bar{z}_i^{\beta})^{d})^{c}\log(1-(1-\bar{z}_i^{\beta})^{d})^2}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$2\sum_{i=1}^{n} \frac{\lambda(1-(1-\bar{z}_i^{\beta})^{d})^{c}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(1-(1-\bar{z}_i^{\beta})^{d})^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$2\sum_{i=1}^{n} \frac{\lambda(\lambda-1)(1-(1-\bar{z}_i^{\beta})^{d})^{2c}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-2}\log(1-(1-\bar{z}_i^{\beta})^{d})^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} +$$

$$2\sum_{i=1}^{n} \frac{\lambda^2(1-(1-\bar{z}_i^{\beta})^{d})^{2c}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2(\lambda-1)}\log(1-(1-\bar{z}_i^{\beta})^{d})^2}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2},$$

$$\frac{\partial^2\ell}{\partial c\partial d} = -\sum_{i=1}^{n}\frac{(1-\bar{z}_i^{\beta})^d\log(1-\bar{z}_i^{\beta})}{1-(1-\bar{z}_i^{\beta})^d} + (\lambda-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(1-\bar{z}_i^{\beta})}{1-(1-(1-\bar{z}_i^{\beta})^d)^c}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{2c-1}\log(1-\bar{z}_i^{\beta})\log(1-(1-\bar{z}_i^{\beta})^d)}{[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^2}+$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{c(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(1-\bar{z}_i^{\beta})\log(1-(1-\bar{z}_i^{\beta})^d)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c}-$$
$$2\sum_{i=1}^{n}\frac{\lambda(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}}-$$
$$2\sum_{i=1}^{n}\frac{c\lambda(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(1-\bar{z}_i^{\beta})\log(1-(1-\bar{z}_i^{\beta})^d)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}}+$$
$$2\sum_{i=1}^{n}\frac{c\lambda(\lambda-1)(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{2c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-2}\log(1-\bar{z}_i^{\beta})\log(1-(1-\bar{z}_i^{\beta})^d)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}}-$$
$$2\sum_{i=1}^{n}\frac{c\lambda^2(1-\bar{z}_i^{\beta})^d(1-(1-\bar{z}_i^{\beta})^d)^{2c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2(\lambda-1)}\log(1-\bar{z}_i^{\beta})\log(1-(1-\bar{z}_i^{\beta})^d)}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2},$$

$$\frac{\partial^2\ell}{\partial c\partial \beta} = \sum_{i=1}^{n}\frac{d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^{\beta})^d} - (\lambda-1)\sum_{i=1}^{n}\frac{d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{2c-1}\log(\bar{z}_i)\log(1-(1-\bar{z}_i^{\beta})^d)}{[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^2}-$$
$$(\lambda-1)\sum_{i=1}^{n}\frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(\bar{z}^i)\log(1-(1-\bar{z}_i^{\beta})^d)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c}+$$
$$2\sum_{i=1}^{n}\frac{d\lambda\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}}+$$
$$2\sum_{i=1}^{n}\frac{cd\lambda\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)\log(1-(1-\bar{z}_i^{\beta})^d)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}}-$$
$$2\sum_{i=1}^{n}\frac{cd\lambda(\lambda-1)\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{2c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-2}\log(\bar{z}_i)\log(1-(1-\bar{z}_i^{\beta})^d)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}}+$$
$$2\sum_{i=1}^{n}\frac{cd\lambda^2\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{2c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2(\lambda-1)}\log(\bar{z}_i)\log(1-(1-\bar{z}_i^{\beta})^d)}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2},$$

$$\frac{\partial^2\ell}{\partial c\partial\alpha} = 2\sum_{i=1}^{n}\frac{d\alpha\beta x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^{d}} - 2(\lambda-1)\sum_{i=1}^{n}\frac{d\alpha\beta x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} -$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha\beta x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2c-1}\log(1-(1-\bar{z}_i^{\beta})^{d})}{[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^2} -$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha\beta x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}\log(1-(1-\bar{z}_i^{\beta})^{d})}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$4\sum_{i=1}^{n}\frac{d\alpha\beta\lambda x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} +$$

$$4\sum_{i=1}^{n}\frac{cd\alpha\beta\lambda z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(1-(1-\bar{z}_i^{\beta})^{d})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$4\sum_{i=1}^{n}\frac{cd\alpha\beta\lambda(\lambda-1)x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-2}\log(1-(1-\bar{z}_i^{\beta})^{d})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}}$$

$$-4\sum_{i=1}^{n}\frac{cd\alpha\beta\lambda^2 x_i^2 z_i\bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2(\lambda-1)}\log(1-(1-\bar{z}_i^{\beta})^{d})}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2},$$

$$\frac{\partial^2\ell}{\partial d^2} = -\frac{n}{d^2} - (c-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^{\beta})^{2d}\log(1-\bar{z}_i^{\beta})^2}{(1-(1-\bar{z}_i^{\beta})^{d})^2} - (c-1)\sum_{i=1}^{n}\frac{(1-\bar{z}_i^{\beta})^{d}\log(1-\bar{z}_i^{\beta})^2}{1-(1-\bar{z}_i^{\beta})^{d}} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{c^2(1-\bar{z}_i^{\beta})^{2d}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}\log(1-\bar{z}_i^{\beta})^2}{[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^2} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)(1-\bar{z}_i^{\beta})^{2d}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}\log(1-\bar{z}_i^{\beta})^2}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{c(1-\bar{z}_i^{\beta})^{d}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}\log(1-\bar{z}_i^{\beta})^2}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$2\sum_{i=1}^{n}\frac{c(c-1)\lambda(1-\bar{z}_i^{\beta})^{2d}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(1-\bar{z}_i^{\beta})^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$2\sum_{i=1}^{n}\frac{c\lambda(1-\bar{z}_i^{\beta})^{d}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(1-\bar{z}_i^{\beta})^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$2\sum_{i=1}^{n}\frac{c^2\lambda(\lambda-1)(1-\bar{z}_i^{\beta})^{2d}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-2}}\log(1-\bar{z}_i^{\beta})^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} +$$

$$2\sum_{i=1}^{n}\frac{c^2\lambda^2(1-\bar{z}_i^{\beta})^{2d}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2(\lambda-1)}\log(1-\bar{z}_i^{\beta})^2}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2},$$

$$\frac{\partial^2\ell}{\partial d\partial\beta} = -\sum_{i=1}^{n}\frac{\bar{z}_i^{\beta}\log(\bar{z}_i)}{1-\bar{z}_i^{\beta}} + (c-1)\sum_{i=1}^{n}\frac{\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^{\beta})^{d}} +$$

$$(c-1)\sum_{i=1}^{n}\frac{d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{2d-1}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{(1-(1-\bar{z}_i^{\beta})^{d})^2} + (c-1)\sum_{i=1}^{n}\frac{d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{1-(1-\bar{z}_i^{\beta})^{d}} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{c\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}\log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{c^2 d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^2} +$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} -$$

$$(\lambda-1)\sum_{i=1}^{n}\frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} +$$

$$2\sum_{i=1}^{n}\frac{c\lambda\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$2\sum_{i=1}^{n}\frac{c(c-1)d\lambda\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} +$$

$$2\sum_{i=1}^{n}\frac{cd\lambda\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} +$$

$$2\sum_{i=1}^{n}\frac{c^2 d\lambda(\lambda-1)\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-2}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} -$$

$$2\sum_{i=1}^{n}\frac{c^2 d\lambda^2\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2(\lambda-1)}\log(\bar{z}_i)\log(1-\bar{z}_i^{\beta})}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2},$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial d \partial \alpha} &= -2\sum_{i=1}^{n}\frac{\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}}{1-\bar{z}_i^{\beta}} + 2(c-1)\sum_{i=1}^{n}\frac{\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^{d}} + \\
& 2(c-1)\sum_{i=1}^{n}\frac{d\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{2d-1}\log(1-\bar{z}_i^{\beta})}{(1-(1-\bar{z}_i^{\beta})^{d})^2} + 2(c-1)\sum_{i=1}^{n}\frac{d\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}\log(1-\bar{z}_i^{\beta})}{1-(1-\bar{z}_i^{\beta})^{d}} \\
& -2(\lambda-1)\sum_{i=1}^{n}\frac{c\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} + \\
& 2(\lambda-1)\sum_{i=1}^{n}\frac{c^2 d\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}\log(1-\bar{z}_i^{\beta})}{[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^2} + \\
& 2(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)d\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}\log(1-\bar{z}_i^{\beta})}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} - \\
& 2(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}\log(1-\bar{z}_i^{\beta})}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} + \\
& 4\sum_{i=1}^{n}\frac{c\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} - \\
& 4\sum_{i=1}^{n}\frac{c(c-1)d\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} + \\
& 4\sum_{i=1}^{n}\frac{cd\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} + \\
& 4\sum_{i=1}^{n}\frac{c^2 d\alpha\beta\lambda(\lambda-1) z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-2}\log(1-\bar{z}_i^{\beta})}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} - \\
& 4\sum_{i=1}^{n}\frac{c^2 d\alpha\beta\lambda^2 x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{2d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2(\lambda-1)}\log(1-\bar{z}_i^{\beta})}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2},
\end{aligned}
$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} - (d-1)\sum_{i=1}^{n} \frac{\bar{z}_i^{2\beta}\log(\bar{z}_i)^2}{(1-\bar{z}_i^{\beta})^2} - (d-1)\sum_{i=1}^{n} \frac{\bar{z}_i^{\beta}\log(\bar{z}_i)^2}{1-\bar{z}_i^{\beta}} - (c-1)\sum_{i=1}^{n} \frac{d^2\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{2(d-1)}\log(\bar{z}_i)^2}{(1-(1-\bar{z}_i^{\beta})^d)^2} -$$

$$(c-1)\sum_{i=1}^{n} \frac{d(d-1)\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{d-2}\log(\bar{z}_i)^2}{1-(1-\bar{z}_i^{\beta})^d} + (c-1)\sum_{i=1}^{n} \frac{d\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}\log(\bar{z}_i)^2}{1-(1-\bar{z}_i^{\beta})^d} -$$

$$(\lambda-1)\sum_{i=1}^{n} \frac{c^2d^2\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{2(c-1)}\log(\bar{z}_i)^2}{[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^2} -$$

$$(\lambda-1)\sum_{i=1}^{n} \frac{c(c-1)d^2\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^c)^{c-2}\log(\bar{z}_i)^2}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} +$$

$$(\lambda-1)\sum_{i=1}^{n} \frac{cd(d-1)\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{d-2}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(\bar{z}_i)^2}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} -$$

$$(\lambda-1)\sum_{i=1}^{n} \frac{cd\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(\bar{z}_i)^2}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} +$$

$$2\sum_{i=1}^{n} \frac{c(c-1)d^2\lambda\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{c-2}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} -$$

$$2\sum_{i=1}^{n} \frac{cd(d-1)\lambda\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{d-2}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} +$$

$$2\sum_{i=1}^{n} \frac{cd\lambda\bar{z}_i^{\beta}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} +$$

$$2\sum_{i=1}^{n} \frac{c^2d^2\lambda(\lambda-1)\bar{z}_i^{2\beta}()^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-2}\log(\bar{z}_i)^2}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} -$$

$$2\sum_{i=1}^{n} \frac{c^2d^2\lambda^2\bar{z}_i^{2\beta}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2(\lambda-1)}\log(\bar{z}_i)^2}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2},$$

$$\frac{\partial^2 \ell}{\partial\beta\partial\alpha} = 2\sum_{i=1}^{n}\frac{\alpha x_i^2 z_i}{\bar{z}_i} - 2(d-1)\sum_{i=1}^{n}\frac{\alpha z_i \bar{z}_i^{\beta-1}}{1-\bar{z}_i^{\beta}} - 2(d-1)\sum_{i=1}^{n}\frac{\alpha\beta x_i^2 z_i \bar{z}_i^{2\beta-1}\log(\bar{z}_i)}{(1-\bar{z}_i^{\beta})^2} -$$

$$2(d-1)\sum_{i=1}^{n}\frac{\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}\log(\bar{z}_i)}{1-\bar{z}_i^{\beta}} + 2(c-1)\sum_{i=1}^{n}\frac{d\alpha x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^d} -$$

$$2(c-1)\sum_{i=1}^{n}\frac{d^2\alpha\beta x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{2(d-1)}\log(\bar{z}_i)}{(1-(1-\bar{z}_i^{\beta})^d)^2} -$$

$$2(c-1)\sum_{i=1}^{n}\frac{d(d-1)\alpha\beta x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{d-2}\log(\bar{z}_i)}{1-(1-\bar{z}_i^{\beta})^d} + 2(c-1)\sum_{i=1}^{n}\frac{d\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}\log(\bar{z}_i)}{1-(1-\bar{z}_i^{\beta})^d} -$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} -$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{c^2 d^2\alpha\beta x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{2(d-1)}\log(\bar{z}_i)}{[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^2} -$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)d^2\alpha\beta x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{c-2}\log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} +$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{cd(d-1)\alpha\beta x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{d-2}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} -$$

$$2(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}\log(\bar{z}_i)}{1-(1-(1-\bar{z}_i^{\beta})^d)^c} +$$

$$4\sum_{i=1}^{n}\frac{cd\alpha\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} +$$

$$4\sum_{i=1}^{n}\frac{c(c-1)d^2\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{c-2}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} -$$

$$4\sum_{i=1}^{n}\frac{cd(d-1)\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{d-2}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} +$$

$$4\sum_{i=1}^{n}\frac{cd\alpha\beta\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^d)^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-1}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} -$$

$$4\sum_{i=1}^{n}\frac{c^2 d^2\alpha\beta\lambda(\lambda-1)x_i^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda-2}\log(\bar{z}_i)}{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}} +$$

$$4\sum_{i=1}^{n}\frac{c^2 d^2\alpha\beta\lambda^2 z_i \bar{z}_i^{2\beta-1}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^d)^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{2(\lambda-1)}\log(\bar{z}_i)}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^d)^c]^{\lambda}\right\}^2},$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha^2} &= -\frac{2n}{\alpha^2} - 2\sum_{i=1}^{n} x_i^2 + 2(\beta-1)\sum_{i=1}^{n}\frac{x_i^2 z_i}{\bar{z}_i} - 4(\beta-1)\sum_{i=1}^{n}\frac{\alpha^2 x_i^4 z_i^2}{\bar{z}_i^2} - 4(\beta-1)\sum_{i=1}^{n}\frac{\alpha^2 x_i^4 z_i}{\bar{z}_i} - \\
&2(d-1)\sum_{i=1}^{n}\frac{\beta x_i^2 z_i \bar{z}_i^{\beta-1}}{1-\bar{z}_i^{\beta}} + 4(d-1)\sum_{i=1}^{n}\frac{\alpha^2\beta x_i^4 z_i \bar{z}_i^{\beta-1}}{1-\bar{z}_i^{\beta}} - 4(d-1)\sum_{i=1}^{n}\frac{\alpha^2\beta(\beta-1)x_i^4 x_i^4 z_i^2 \bar{z}_i^{\beta-2}}{1-\bar{z}_i^{\beta}} - \\
&4(d-1)\sum_{i=1}^{n}\frac{\alpha^2\beta^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}}{(1-\bar{z}_i^{\beta})^2} + 2(c-1)\sum_{i=1}^{n}\frac{d\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^{d}} - \\
&4(c-1)\sum_{i=1}^{n}\frac{d\alpha^2\beta x_i^4 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^{d}} + 4(c-1)\sum_{i=1}^{n}\frac{d\alpha^2\beta(\beta-1)x_i^4 z_i^2 \bar{z}_i^{\beta-2}(1-\bar{z}_i^{\beta})^{d-1}}{1-(1-\bar{z}_i^{\beta})^{d}} - \\
&4(c-1)\sum_{i=1}^{n}\frac{d^2\alpha^2\beta^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{2(d-1)}}{(1-(1-\bar{z}_i^{\beta})^{d})^2} - 4(c-1)\sum_{i=1}^{n}\frac{d(d-1)\alpha^2\beta^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{d-2}}{1-(1-\bar{z}_i^{\beta})^{d}} - \\
&2(\lambda-1)\sum_{i=1}^{n}\frac{cd\beta x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} + \\
&4(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha^2\beta x_i^4 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} - \\
&4(\lambda-1)\sum_{i=1}^{n}\frac{cd\alpha^2\beta(\beta-1)x_i^4 z_i^2 \bar{z}_i^{\beta-2}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} - \\
&4(\lambda-1)\sum_{i=1}^{n}\frac{c^2d^2\alpha^2\beta^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}}{[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^2} - \\
&4(\lambda-1)\sum_{i=1}^{n}\frac{c(c-1)d^2\alpha^2\beta^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} + \\
&4(\lambda-1)\sum_{i=1}^{n}\frac{cd(d-1)\alpha^2\beta^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{d-2}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}}{1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}} + \\
&4\sum_{i=1}^{n}\frac{cd\beta\lambda x_i^2 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} - \\
&8\sum_{i=1}^{n}\frac{cd\alpha^2\beta\lambda x_i^4 z_i \bar{z}_i^{\beta-1}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} + \\
&8\sum_{i=1}^{n}\frac{cd\alpha^2\beta(\beta-1)\lambda x_i^4 z_i^2 \bar{z}_i^{\beta-2}(1-\bar{z}_i^{\beta})^{d-1}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} + \\
&8\sum_{i=1}^{n}\frac{c(c-1)d^2\alpha^2\beta^2\lambda x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^{d})^{c-2}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} - \\
&8\sum_{i=1}^{n}\frac{cd(d-1)\alpha^2\beta^2\lambda x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{d-2}(1-(1-\bar{z}_i^{\beta})^{d})^{c-1}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-1}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} - \\
&8\sum_{i=1}^{n}\frac{c^2d^2\alpha^2\beta^2\lambda(\lambda-1)x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda-2}}{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}} + \\
&8\sum_{i=1}^{n}\frac{c^2d^2\alpha^2\beta^2\lambda^2 x_i^4 z_i^2 \bar{z}_i^{2(\beta-1)}(1-\bar{z}_i^{\beta})^{2(d-1)}(1-(1-\bar{z}_i^{\beta})^{d})^{2(c-1)}[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{2(\lambda-1)}}{\left\{1+[1-(1-(1-\bar{z}_i^{\beta})^{d})^{c}]^{\lambda}\right\}^2}.
\end{aligned}
$$

Appendix A4

R Codes

```
### EGED Distribution PDF ####
Dagum<-function(x,lambda,alpha,beta,theta,c,d){
A<-(1+alpha*(x^(-theta)))^(-beta-1)
B<-1-(1+alpha*(x^(-theta)))^(-beta)
fxn<-lambda*alpha*beta*theta*c*d*
(x^(-theta-1))*A*(B^(d-1))*((1-(B^d))^(c-1))*((1-(1-(B^d))^c)^(lambda-1))
return(fxn)
}


## EGED Distribution Hazard function ######
Hazard<-function(x,lambda,alpha,beta,theta,c,d){
A<-(1+alpha*(x^(-theta)))^(-beta-1)
B<-1-(1+alpha*(x^(-theta)))^(-beta)
fxn<-(lambda*alpha*beta*theta*c*d*(x^(-theta-1))*
A*(B^(d-1))*((1-(B^d))^(c-1)))/(1-(1-(B^d))^c)
return(fxn)
}
#### EGED Distribution Quantile function #######
quantile<-function(lambda,alpha,beta,theta,c,d,u){
A<-(1-u)^(1/lambda)
B<-(1-A)^(1/c)
C<-(1-B)^(1/d)
```

```
D<-(1-C)^(-1/beta)
 result<-((1/alpha)*(D-1))^(-1/theta)
return(result)}
#### EGED Distribution Moment #####
Moment<-function(alpha,lambda,beta,theta,c,d,r){
 f<-function(x,alpha,lambda,beta,theta,c,d,r){ (x^r)*
 (Dagum(x,alpha,lambda,beta,theta,c,d))}
 results<-integrate(f,lower=0,upper=Inf,subdivisions = 10000,alpha=alpha,
lambda=lambda,beta=beta,theta=theta,c=c,d=d,r=r)$value
return(results)
}
### EGED Distribution Negative Log-likelihood for Optimization ###
Dagum_LL<-function(lambda,alpha,beta,theta,c,d){
A<-(1+alpha*(x^(-theta)))^(-beta-1)
B<-1-(1+alpha*(x^(-theta)))^(-beta)
 fxn<- -sum(log(lambda*alpha*beta*theta*c*d*(x^(-theta-1))*
A*(B^(d-1))*((1-(B^d))^(c-1))*((1-(1-(B^d))^c)^(lambda-1))))
return(fxn)
}
#### EGED Distribution Optimization #####
library(bbmle) #### Calling R package bbmle ######
 fit<-mle2(Dagum_LL,start=list(lambda=lambda,
alpha=alpha,beta=beta,theta=theta,c=c,d=d),method="BFGS",data=list(x))
summary(fit) ###### Summary of Results ####
```

```
#### NEGMIR Distribution PDF ####
NEGMIR<-function(x,lambda,alpha,theta,c,d){
A<-((alpha/(x^2))+((2*theta)/(x^3)))
A1<-((alpha/(x))+(theta/(x^2)))
A2<-exp(-A1)
A3<-(1-A2)^(d-1)
A4<-(1-(1-A2)^d)^(c-1)
A5<-(1-(1-(1-A2)^d)^(c))^(lambda-1)
fxn<-lambda*c*d*A*A2*A3*A4*A5
return(fxn)
}
### NEGMIR Distribution Hazard function #####
Hazard<-function(x,lambda,alpha,theta,c,d){
A<-((alpha/(x^2))+((2*theta)/(x^3)))
A1<-((alpha/(x))+(theta/(x^2)))
A2<-exp(-A1)
A3<-(1-A2)^(d-1)
A4<-(1-(1-A2)^d)^(c-1)
A5<-(1-(1-(1-A2)^d)^(c))^(-1)
fxn<-lambda*c*d*A*A2*A3*A4*A5
return(fxn)
}
### NEGMIR Distribution Quantile function ###
quantile<-function(lambda,alpha,theta,c,d,u){
A<-(1-u)^(1/lambda)
```

```
A1<-(1-A)^(1/c)
A2<-(1-A1)^(1/d)
A3<-log(1-A2)
A4<-4*theta*A3
A5<-(alpha^2)-A4
result<-(2*theta)/(-alpha+sqrt(A5))
return(result)
}
### NEGMIR Distribution Moment ######
Moment<-function(lambda, alpha, theta, c, d, r){
f<-function(x, lambda, alpha, theta, c, d, r){(x^r)*
(NGMIR(x, lambda, alpha, theta, c, d))}
results<-integrate(f, lower=0, upper=Inf, subdivisions=10000,
lambda=lambda, alpha=alpha, theta=theta, c=c, d=d, r=r)$value
return(results)
}
#### NEGMIR Distribution Negative log-likelihood function for
  Optimization ####
NEGMIR_LL<-function(lambda, alpha, theta, c, d){
A<-lambda*c*d*((alpha/(x^2))+(2*theta)/(x^3))
A1<-exp(-((alpha/x)+(theta/x^2)))
A2<-(1-exp(-((alpha/x)+(theta/x^2))))^(d-1)
A3<-(1-(1-exp(-((alpha/x)+(theta/x^2))))^(d))^(c-1)
A4<-(1-(1-(1-exp(-((alpha/x)+(theta/x^2))))^(d))^(c))^(lambda-1)
fxn<- -sum(log(A*A1*A2*A3*A4))
```

```
return(fxn)
}
#### NEGMIR Distribution Optimization ###
library(bbmle) ### Calling R package bbmle ###
fit<-mle2(NEGMIR_LL,start=list(lambda=lambda,alpha=alpha,theta=theta,
c=c,d=d),method="BFGS",data=list(x))
summary(fit) #### Summary of Results ######

#### EGHLBX Distribution PDF ####
EGHLBX<-function(x,lambda,alpha,beta,c,d){
A<-4*lambda*((alpha)^2)*beta*c*d*x
A1<-exp(-((alpha*x)^(2)))
A2<-(1-A1)^(beta-1)
A3<-(1-((1-A1)^beta))^(d-1)
A4<-(1-((1-((1-A1)^beta))^(d)))^(c-1)
A5<-(1-((1-((1-((1-A1)^beta))^(d)))^(c)))^(lambda-1)
A6<-(1+(1-((1-((1-((1-A1)^beta))^(d)))^(c)))^(lambda))^(-2)
fxn<-A*A1*A2*A3*A4*A5*A6
return(fxn)
}

#### EGHLBX Hazard function ###
Hazard<-function(x,lambda,alpha,beta,c,d){
A<-2*lambda*((alpha)^2)*beta*c*d*x
A1<-exp(-((alpha*x)^(2)))
```

```
A2<-(1-A1)^(beta-1)

A3<-(1-((1-A1)^beta))^(d-1)

A4<-(1-((1-((1-A1)^beta))^(d)))^(c-1)

A5<-(1-((1-((1-((1-A1)^beta))^(d)))^(c)))^(-1)

A6<-(1+(1-((1-((1-((1-A1)^beta))^(d)))^(c)))^(lambda))^(-1)

fxn<-A*A1*A2*A3*A4*A5*A6

return(fxn)

}

#### EGHLBX Distribution Quantile function #####

quantile<-function(lambda,alpha,beta,c,d,u){

A<-((1-u)/(1+u))^(1/lambda)

B<-(1-A)^(1/c)

C<-(1-B)^(1/d)

D<-(1-C)^(1/beta)

E<--log(1-D)

result<-sqrt(E)/alpha

return(result)

}

#### EGHLBX Distribution Moment ####

Moment<-function(lambda,alpha,beta,c,d,r){

f<-function(x,lambda,alpha,beta,c,d,r){(x^r)*

(EGHLBX(x,lambda,alpha,beta,c,d))}

results<-integrate(f,lower=0,upper=Inf,subdivisions =10000,

lambda=lambda,alpha=alpha,beta=beta,c=c,d=d,r=r)$value

return(results)
```

```
}
### EGHLBX Distribution Negative log-likelihood function for
 Optimization ####
EGHLBX_LL<-function(lambda,alpha,beta,c,d){
A<-4*lambda*((alpha)^2)*beta*c*d*x
A1<-exp(-((alpha*x)^(2)))
A2<-(1-A1)^(beta-1)
A3<-(1-((1-A1)^beta))^(d-1)
A4<-(1-((1-((1-A1)^beta))^(d)))^(c-1)
A5<-(1-((1-((1-((1-A1)^beta))^(d)))^(c)))^(lambda-1)
A6<-(1+(1-((1-((1-((1-A1)^beta))^(d)))^(c)))^(lambda))^(-2)
fxn<- -sum(log(A*A1*A2*A3*A4*A5*A6))
return(fxn)
}
### EGHLBX Distribution Optimization #####
library(bbmle) ### Calling R package bbmle ###
fit<-mle2(EGHLBX_LL,start=list(lambda=lambda,alpha=alpha,
beta=beta,c=c,d=d),method="BFGS",data=list(x))
summary(fit) #### Summary of Results ####
### R Codes for EGPS Special Distributions ###
##### EGPIE Distribution PDF ####
EGPIE<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-(lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1))*
```

```
exp(lambda*(1-((1-A^d)^(c)))))/(exp(lambda)-1)
return(fxn)
}
### EGPIE Distribution Hazard function ####
Hazard<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-(lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1))*
exp(lambda*(1-((1-A^d)^(c)))))/(exp(lambda*(1-((1-A^d)^(c))))-1)
return(fxn)
}
#### EGPIE Distribution Quantile function ###
quantile<-function(lambda,c,d,gamma,u){
Z<-log(exp(lambda)-u*(exp(lambda)-1))
fxn<-((-1/gamma)*log(1-(1-(1-(Z/lambda))^(1/c))^(1/d)))^(-1)
return(fxn)
}
#### EGBIE Distribution PDF #####
EGBIE<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-(5*lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1))*
(1+lambda*(1-((1-A^d)^(c))))^(5-1))/(((1+lambda)^5)-1)
return(fxn)
}
```

```
### EGBIE Distribution Hazard function ####
Hazard<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-(5*lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1))*
(1+lambda*(1-((1-A^d)^(c))))^(5-1))/(((1+lambda*(1-((1-A^d)^(c))))^(5))-1)
return(fxn)
}
### EGBIE Distribution Quantile Function ####
quantile<-function(lambda,c,d,gamma,u){
Z<-(((((1+lambda)^(5))-1)*(1-u)+1)^(1/5))-1
fxn<-((-1/gamma)*log(1-(1-(1-(Z/lambda))^(1/c))^(1/d)))^(-1)
return(fxn)
}
### EGGIE Distribution PDF ###
EGGIE<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-((1-lambda)*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1)))/
((1-lambda*(1-((1-A^d)^(c))))^2)
return(fxn)
}
### EGGIE Distribution Hazard function ###
Hazard<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
```

```
B<-exp(-gamma*(x^(-1)))
fxn<-(c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1)))/((1-((1-A^d)^(c)))*
(1-lambda*(1-((1-A^d)^(c)))))
return(fxn)
}
#### EGGIE Distribution Quantile Function #####
quantile<-function(lambda,c,d,gamma,u){
Z<-(u*(1-lambda)/(1-u*lambda))^(1/c)
fxn<-((-1/gamma)*log(1-(1-Z)^(1/d)))^(-1)
return(fxn)
}
#### EGLIE Distribution PDF #####
EGLIE<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-(lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1)))/
((log(1-lambda))*(lambda*(1-((1-A^d)^(c)))-1))
return(fxn)
}
#### EGLIE Distribution Hazard function ####
EGLIEH<-function(x,lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fxn<-(lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1)))/
((log(1-lambda*(1-((1-A^d)^(c)))))*(lambda*(1-((1-A^d)^(c)))-1))
```

```
return(fxn)
}
### EGLIE Distribution Quantile Function ####
quantile(lambda,c,d,gamma,u){
Z<-(1-(((1-lambda)^(1-u))))/lambda
fxn<-((-1/gamma)*log(1-(1-(1-Z)^(1/c))^(1/d)))^(-1)
return(fxn)
}
#### EGPS Special Distributions Negative Log-likelihoods ####
#### EGPIE Negative Log-likelihood ####
EGPIE_LL<-function(lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fx<-(lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1))*
exp(lambda*(1-((1-A^d)^(c)))))/(exp(lambda)-1)
fxn<- -sum(log(fx))
return(fxn)
}
##### EGBIE Negative Log-likelihood ####
EGBIE_LL<-function(lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fx<-(5*lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1))*
(1+lambda*(1-((1-A^d)^(c))))^(5-1))/(((1+lambda)^5)-1)
fxn<- -sum(log(fx))
```

```
return(fxn)
}
#### EGGIE Negative Log-likelihood ####
EGGIE_LL<-function(lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fx<-((1-lambda)*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1)))/
((1-lambda*(1-((1-A^d)^(c))))^2)
fxn<- -sum(log(fx))
return(fxn)
}
#### EGLIE Negative Log-likelihood #####
EGLIE_LL<-function(lambda,c,d,gamma){
A<-1-exp(-gamma*(x^(-1)))
B<-exp(-gamma*(x^(-1)))
fx<-(lambda*c*d*gamma*(x^(-2))*B*(A^(d-1))*((1-A^d)^(c-1)))/
((log(1-lambda))*(lambda*(1-((1-A^d)^(c)))-1))
fxn<- -sum(log(fx))
return(fxn)
}
#### EGPS Special Distributions Optimization ###
library(bbmle) ### Calling R package bbmle ####
fit1<-mle2(EGPIE_LL,start=list(lambda=lambda,c=c,d=d,gamma=gamma),
method="BFGS",data=list(x))
summary(fit1) #### Summary of Results ###
```

```
fit2<-mle2(EGBIE_LL, start=list(lambda=lambda, c=c, d=d, gamma=gamma),
method="BFGS", data=list(x))
summary(fit2)
fit3<-mle2(EGGIE_LL, start=list(lambda=lambda, c=c, d=d, gamma=gamma),
method="BFGS", data=list(x))
summary(fit3)
fit4<-mle2(EGPIE_LL, start=list(lambda=lambda, c=c, d=d, gamma=gamma),
method="BFGS", data=list(x))
summary(fit4)


#### Simulation Code for EGED Distribution ###
### Specifying EGED Qunatile function ####
quantile<-function(lambda, alpha, beta, theta, c, d, u){
result<-((1/alpha)*((1-(1-(1-(1-u)^(1/lambda))^(1/c))^(1/d))^
(-1/beta)-1))^(-1/theta)
return(result)
}
####  Negative Log-likelihood function of EGED Dsitribution ####
EGED_LL<-function(par){-sum(log(par[1]*par[2]*par[3]*par[4]*par[5]*
par[6]*(x^(-par[4]-1))*((1+par[2]*x^(-par[4]))^( -par[3]-1))*
((1-(1+par[2]*x^(-par[4])))^( -par[3]))^(par[6]-1))*
(((1-(1+par[2]*x^(-par[4]))^( -par[3]))^(par[6]))^(par[5]-1))*
((((1-(1+par[2]*x^(-par[4]))^( -par[3]))^(par[6]))^(par[5]))^(par[1]-1))))
}
### Algorithm for Monte Carlo Simulation Study for EGED Distribution ###
```

```
library(numDeriv)
library(Matrix)
lambda=lambda
alpha=alpha
beta=beta
theta=theta
c=c
d=d
n1=c(25,50,75,100,200,300,600)
for(j in 1:length(n1)){
n=n1[j]
N=1000
mle_lambda<-c(rep(0,N))
mle_alpha<-c(rep(0,N))
mle_beta<-c(rep(0,N))
mle_theta<-c(rep(0,N))
mle_c<-c(rep(0,N))
mle_d<-c(rep(0,N))
LC_lambda<-c(rep(0,N))
UC_lambda<-c(rep(0,N))
LC_alpha<-c(rep(0,N))
UC_alpha<-c(rep(0,N))
LC_beta<-c(rep(0,N))
UC_beta<-c(rep(0,N))
LC_theta<-c(rep(0,N))
```

```
UC_theta<-c(rep(0,N))
LC_c<-c(rep(0,N))
UC_c<-c(rep(0,N))
LC_d<-c(rep(0,N))
UC_d<-c(rep(0,N))
count_lambda=0
count_alpha=0
count_beta=0
count_theta=0
count_c=0
count_d=0
temp=1
HH1<-matrix(c(rep(2,36)),nrow=6,ncol=6)
HH2<-matrix(c(rep(2,36)),nrow=6,ncol=6)
for(i in 1:N)
{
print(i)
flush.console()
repeat{
x<-c(rep(0,n))
# Generate a random variable from uniform distribution
u<-0
u<-runif(n,min=0,max=1)
for(k in 1:n){
x[k]<-quantile(lambda,alpha,beta,theta,c,d,u[k])
```

```r
}
#Maximum likelihood estimation
mle.result<-nlminb(c(lambda,alpha,beta,theta,c,d),EGED_LL,lower=0,
upper=Inf)
temp=mle.result$convergence
if(temp==0){
temp_lambda<-mle.result$par[1]
temp_alpha<-mle.result$par[2]
temp_beta<-mle.result$par[3]
temp_theta<-mle.result$par[4]
temp_c<-mle.result$par[5]
temp_d<-mle.result$par[6]
HH1<-hessian(EGED_LL, c(temp_lambda,temp_alpha,temp_beta,temp_theta,
temp_c,temp_d))
if(sum(is.nan(HH1))==0&(diag(HH1)[1]>0)&(diag(HH1)[2]>0)&(diag(HH1)[3]>0)
&(diag(HH1)[4]>0)&(diag(HH1)[5]>0)&(diag(HH1)[6]>0)){
HH2<-solve(HH1)
#print(det(HH1))
}
else{
temp=1}
}
if((temp==0)&(diag(HH2)[1]>0)&(diag(HH2)[2]>0)&(diag(HH2)[3]>0)&
(diag(HH2)[4]>0)&(diag(HH2)[5]>0)&(diag(HH2)[6]>0)&(sum(is.nan(HH2))==0)){
break
```

```
}
else{
temp=1}
}
temp=1
mle_lambda[i]<-mle.result$par[1]
mle_alpha[i]<-mle.result$par[2]
mle_beta[i]<-mle.result$par[3]
mle_theta[i]<-mle.result$par[4]
mle_c[i]<-mle.result$par[5]
mle_d[i]<-mle.result$par[6]
HH<-hessian(EGED_LL,c(mle_lambda[i],mle_alpha[i],mle_beta[i],mle_theta[i],
mle_c[i],mle_d[i]))
H<-solve(HH)
LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[1])
if((LC_lambda[i]<=lambda)&(lambda<=UC_lambda[i])){
count_lambda=count_lambda+1
}
LC_alpha[i]<-mle_alpha[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_alpha[i]<-mle_alpha[i]+qnorm(0.975)*sqrt(diag(H)[2])
if((LC_alpha[i]<=alpha)&(alpha<=UC_alpha[i])){
count_alpha=count_alpha+1
}
LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[3])
```

```
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[3])
if((LC_beta[i]<=beta)&(beta<=UC_beta[i])){
count_beta=count_beta+1
}
LC_theta[i]<-mle_theta[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_theta[i]<-mle_theta[i]+qnorm(0.975)*sqrt(diag(H)[4])
if((LC_theta[i]<=theta)&(theta<=UC_theta[i])){
count_theta=count_theta+1
}
LC_c[i]<-mle_c[i]-qnorm(0.975)*sqrt(diag(H)[5])
UC_c[i]<-mle_c[i]+qnorm(0.975)*sqrt(diag(H)[5])
if((LC_c[i]<=c)&(s<=UC_c[i])){
count_c=count_c+1
}
LC_d[i]<-mle_d[i]-qnorm(0.975)*sqrt(diag(H)[6])
UC_d[i]<-mle_d[i]+qnorm(0.975)*sqrt(diag(H)[6])
if((LC_d[i]<=d)&(d<=UC_d[i])){
count_d=count_d+1
}
}
# Calculate Average Bias
ABias_lambda<-sum(mle_lambda-lambda)/N
ABias_alpha<-sum(mle_alpha-alpha)/N
ABias_beta<-sum(mle_beta-beta)/N
ABias_theta<-sum(mle_theta-theta)/N
```

```
ABias_c<-sum(mle_c-c)/N
ABias_d<-sum(mle_d-d)/N
print(cbind(ABias_lambda,ABias_alpha,ABias_beta,ABias_theta,ABias_c,
ABias_d))
# Calculate RMSE
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/N)
RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)
RMSE_theta<-sqrt(sum((theta-mle_theta)^2)/N)
RMSE_c<-sqrt(sum((c-mle_c)^2)/N)
RMSE_d<-sqrt(sum((d-mle_d)^2))/N
print(cbind(RMSE_lambda,RMSE_alpha,RMSE_beta,RMSE_theta,RMSE_c,RMSE_d))
}
#### Simulation Code for EGHLBX Distribution #####
#### Specifying EGHLBX Qunatile function ####
quantile<-function(lambda,alpha,beta,c,d,u){
A<-((1-u)/(1+u))^(1/lambda)
B<-(1-A)^(1/c)
C<-(1-B)^(1/d)
D<-(1-C)^(1/beta)
E<--log(1-D)
result<-sqrt(E)/alpha
return(result)
}
#### Negative Log-likelihood function for EGHLBX Distribution ####
```

```
EGHLBX_LL<-function(par){-sum(log( 4*par[1]*((par[2])^2)*par[3]*par[4]*
par[5]*x*(exp(-((par[2]*x)^(2))))*((1-exp(-((par[2]*x)^(2))))^(par[3]-1))*
((1-((1-exp(-((par[2]*x)^(2))))^par[3]))^(par[5]-1))*
((1-((1-((1-exp(-((par[2]*x)^(2))))^par[3]))^(par[5])))^(par[4]-1))*
((1-((1-((1-((1-exp(-((par[2]*x)^(2))))^par[3]))^(par[5])))^(par[4])))^
(par[1]-1))*((1+(1-((1-((1-((1-exp(-((par[2]*x)^(2))))^par[3]))^(par[5]))
)^(par[4])))^(par[1]))^(-2))))
}
### Algorithm for Monte Carlo Simulation Study for EGHLBX Distribution ###
library(numDeriv)
library(Matrix)
lambda=lambda
alpha=alpha
beta=beta
c=c
d=d
n1=c(25,50,75,100,200,300,600)
for(j in 1:length(n1)){
n=n1[j]
N=1000
mle_lambda<-c(rep(0,N))
mle_alpha<-c(rep(0,N))
mle_beta<-c(rep(0,N))
mle_c<-c(rep(0,N))
mle_d<-c(rep(0,N))
```

```
LC_lambda<-c(rep(0,N))
UC_lambda<-c(rep(0,N))
LC_alpha<-c(rep(0,N))
UC_alpha<-c(rep(0,N))
LC_beta<-c(rep(0,N))
UC_beta<-c(rep(0,N))
LC_c<-c(rep(0,N))
UC_c<-c(rep(0,N))
LC_d<-c(rep(0,N))
UC_d<-c(rep(0,N))
count_lambda=0
count_alpha=0
count_beta=0
count_c=0
count_d=0
temp=1
HH1<-matrix(c(rep(2,25)),nrow=5,ncol=5)
HH2<-matrix(c(rep(2,25)),nrow=5,ncol=5)
for(i in 1:N)
{
print(i)
flush.console()
repeat{
x<-c(rep(0,n))
# Generate a random variable from uniform distribution
```

```
u<-0
u<-runif(n,min=0,max=1)
for(k in 1:n){
x[k]<-quantile(lambda,alpha,beta,c,d,u[k])
}
#Maximum likelihood estimation
mle.result<-nlminb(c(lambda,alpha,beta,c,d),EGHLBX_LL,lower=0,upper=Inf)
temp=mle.result$convergence
if(temp==0){
temp_lambda<-mle.result$par[1]
temp_alpha<-mle.result$par[2]
temp_beta<-mle.result$par[3]
temp_c<-mle.result$par[4]
temp_d<-mle.result$par[5]
HH1<-hessian(EGHLBX_LL, c(temp_lambda,temp_alpha,temp_beta,temp_c,temp_d))
if(sum(is.nan(HH1))==0&(diag(HH1)[1]>0)&(diag(HH1)[2]>0)&(diag(HH1)[3]>0)
&(diag(HH1)[4]>0)&(diag(HH1)[5]>0)){
HH2<-solve(HH1)
#print(det(HH1))
}
else{
temp=1}
}
if((temp==0)&(diag(HH2)[1]>0)&(diag(HH2)[2]>0)&(diag(HH2)[3]>0)&
(diag(HH2)[4]>0)&(diag(HH2)[5]>0)&(sum(is.nan(HH2))==0)){
```

```
break
}
else{
temp=1}
}
temp=1
mle_lambda[i]<-mle.result$par[1]
mle_alpha[i]<-mle.result$par[2]
mle_beta[i]<-mle.result$par[3]
mle_c[i]<-mle.result$par[4]
mle_d[i]<-mle.result$par[5]
HH<-hessian(EGHLBX_LL,c(mle_lambda[i],mle_alpha[i],mle_beta[i],mle_c[i],
mle_d[i]))
H<-solve(HH)
LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[1])
if((LC_lambda[i]<=lambda)&(lambda<=UC_lambda[i])){
count_lambda=count_lambda+1
}
LC_alpha[i]<-mle_alpha[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_alpha[i]<-mle_alpha[i]+qnorm(0.975)*sqrt(diag(H)[2])
if((LC_alpha[i]<=alpha)&(alpha<=UC_alpha[i])){
count_alpha=count_alpha+1
}
LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[3])
```

```
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[3])
if((LC_beta[i]<=beta)&(beta<=UC_beta[i])){
count_beta=count_beta+1
}
LC_c[i]<-mle_c[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_c[i]<-mle_c[i]+qnorm(0.975)*sqrt(diag(H)[4])
if((LC_c[i]<=c)&(c<=UC_c[i])){
count_c=count_c+1
}
LC_d[i]<-mle_d[i]-qnorm(0.975)*sqrt(diag(H)[5])
UC_d[i]<-mle_d[i]+qnorm(0.975)*sqrt(diag(H)[5])
if((LC_d[i]<=d)&(d<=UC_d[i])){
count_d=count_d+1
}
}
# Calculate Average Bias
ABias_lambda<-sum(mle_lambda-lambda)/N
ABias_alpha<-sum(mle_alpha-alpha)/N
ABias_beta<-sum(mle_beta-beta)/N
ABias_c<-sum(mle_c-c)/N
ABias_d<-sum(mle_d-d)/N
print(cbind(ABias_lambda,ABias_alpha,ABias_beta,ABias_c,ABias_d))
# Calculate RMSE
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/N)
```

```
RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)
RMSE_c<-sqrt(sum((c-mle_c)^2)/N)
RMSE_d<-sqrt(sum((d-mle_d)^2))/N
print(cbind(RMSE_lambda,RMSE_alpha,RMSE_beta,RMSE_c,RMSE_d))
}
### Simulation Code for EGLIE Distribution ####
##### Specifying EGLIE Qunatile function #####
quantile(lambda,c,d,gamma,u){
Z<-(1-(((1-lambda)^(1-u))))/lambda
fxn<-((-1/gamma)*log(1-(1-(1-Z)^(1/c))^(1/d)))^(-1)
return(fxn)
}
#### Negative Log-Likelihood function for EGLIE Distribution ####
EGLIE_LL<-function(par){
A<-1-exp(-par[4]*(x^(-1)))
B<-exp(-par[4]*(x^(-1)))
fx<-(par[1]*par[2]*par[3]*par[4]*(x^(-2))*B*(A^(par[3]-1))*((1-A^par[3])
^(par[2]-1)))/((log(1-par[1]))*(par[1]*(1-((1-A^par[3])^(par[2])))-1))
fxn<- -sum(log(fx))
return(fxn)
}


### Algorithm for Monte Carlo Simulation for EGLIE Distribution ###
library(numDeriv)
library(Matrix)
```

```
lambda=lambda
c=c
d=d
gamma=gamma
n1=c(25,50,75,100)
for(j in 1:length(n1)){
n=n1[j]
N=1500
mle_lambda<-c(rep(0,N))
mle_c<-c(rep(0,N))
mle_d<-c(rep(0,N))
mle_gamma<-c(rep(0,N))
LC_lambda<-c(rep(0,N))
UC_lambda<-c(rep(0,N))
LC_c<-c(rep(0,N))
UC_c<-c(rep(0,N))
LC_d<-c(rep(0,N))
UC_d<-c(rep(0,N))
LC_gamma<-c(rep(0,N))
UC_gamma<-c(rep(0,N))
count_lambda=0
count_c=0
count_d=0
count_gamma=0
temp=1
```

```
HH1<-matrix(c(rep(2,16)),nrow=4,ncol=4)
HH2<-matrix(c(rep(2,16)),nrow=4,ncol=4)
for(i in 1:N)
{
print(i)
flush.console()
repeat{
x<-c(rep(0,n))
# Generate a random variable from uniform distribution
u<-0
u<-runif(n,min=0,max=1)
for(k in 1:n){
x[k]<-quantile(lambda,c,d,gamma,u[k])
}
# Maximum likelihood estimation
mle.result<-nlminb(c(lambda,c,d,gamma),EGGIE_LL,lower=c(0,0,0,0),
upper=c(1,Inf,Inf,Inf))
temp=mle.result$convergence
if(temp==0){
temp_lambda<-mle.result$par[1]
temp_c<-mle.result$par[2]
temp_d<-mle.result$par[3]
temp_gamma<-mle.result$par[4]
HH1<-hessian(EGGIE_LL, c(temp_lambda,temp_c,temp_d,temp_gamma))
if(sum(is.nan(HH1))==0&(diag(HH1)[1]>0)&(diag(HH1)[2]>0)&(diag(HH1)[3]>0)
```

```
&(diag(HH1)[4]>0)){
HH2<-solve(HH1)
#print(det(HH1))
}
else{
temp=1}
}
if((temp==0)&(diag(HH2)[1]>0)&(diag(HH2)[2]>0)&(diag(HH2)[3]>0)
&(diag(HH2)[4]>0)&(sum(is.nan(HH2))==0)){
break
}
else{
temp=1}
}
temp=1
mle_lambda[i]<-mle.result$par[1]
mle_c[i]<-mle.result$par[2]
mle_d[i]<-mle.result$par[3]
mle_gamma[i]<-mle.result$par[4]
HH<-hessian(EGGIE_LL,c(mle_lambda[i],mle_c[i],mle_d[i],mle_gamma[i]))
H<-solve(HH)
LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[1])
if((LC_lambda[i]<=lambda)&(lambda<=UC_lambda[i])){
count_lambda=count_lambda+1
```

```
}
LC_c[i]<-mle_c[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_c[i]<-mle_c[i]+qnorm(0.975)*sqrt(diag(H)[2])
if((LC_c[i]<=c)&(c<=UC_c[i])){
count_c=count_c+1
}
LC_d[i]<-mle_d[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_d[i]<-mle_d[i]+qnorm(0.975)*sqrt(diag(H)[3])
if((LC_d[i]<=d)&(d<=UC_d[i])){
count_d=count_d+1
}
LC_gamma[i]<-mle_gamma[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_gamma[i]<-mle_gamma[i]+qnorm(0.975)*sqrt(diag(H)[4])
if((LC_gamma[i]<=gamma)&(gamma<=UC_gamma[i])){
count_gamma=count_gamma+1
}


}
# Average Estimate
AV_lambda<-sum(mle_lambda)/N
AV_c<-sum(mle_c)/N
AV_d<-sum(mle_d)/N
AV_gamma<-sum(mle_gamma)/N
print(cbind(AV_lambda,AV_c,AV_d,AV_gamma))
# Calculate RMSE
```

```
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)

RMSE_c<-sqrt(sum((c-mle_c)^2)/N)

RMSE_d<-sqrt(sum((d-mle_d)^2)/N)

RMSE_gamma<-sqrt(sum((gamma-mle_gamma)^2)/N)

print(cbind(RMSE_lambda,RMSE_c,RMSE_d,RMSE_gamma))

}
```

Appendix A5

List of Publications

- Nasiru, S., Mwita, P. N. and Ngesa, O. (2017). Exponentiated Generalized *T-X* Family of Distributions. Journal of Statistical and Econometric Methods, **6**(4): 1-17.
- Nasiru, S., Mwita, P. N. and Ngesa, O. (2017). Exponentiated Generalized Exponential Dagum Distribution. Journal of King Saud University-Science. http://dx.doi.org/10.1016/j.jksus.2017.09.009.
- Nasiru, S., Mwita, P. N. and Ngesa, O. (2018). Discussion on Generalized Modified Inverse Rayleigh Distribution. Applied Mathematics and Information Sciences, **12**(1): 113-124.
- Nasiru, S., Mwita, P. N. and Ngesa, O. (2018). Exponentiated Generalized Half Logistic Burr X Distribution. Advances and Applications in Statistics, **52**(3): 145-169.
- Nasiru, S., Mwita, P. N. and Ngesa, O. (2018). Exponentiated Generalized Geometric Burr III Distribution. To appear in the Conference Proceedings of the Machakos University 1st Annual International Conference.
- Nasiru, S., Mwita, P. N. and Ngesa, O. (2017). Exponentiated Generalized Power Series Family of Distributions. Annals of Data Science. In Review. Manuscript ID: AODS-S-17-00109.

Conference Presentation

- Nasiru, S., Mwita, P. N. and Ngesa, O. (2018). Exponentiated Generalized Geometric Burr III Distribution. The Machakos University 1st Annual International Conference, Machakos, Kenya. 17th-19th April, 2018.