PRICING LOOKBACK OPTION UNDER STOCHASTIC VOLATILITY

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DECLARATION

I, Teferi Dereje Wirtu, declare that this thesis entitled: "Pricing lookback option under stochastic volatility" is my original work and has not been presented for a degree in any University. I also confirm that this work was done wholly or mainly while in candidature for a research degree at this noble University. The work was done under the guidence of both Dr. Philip Ngare and Dr. Ananda Kube.

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DEDICATION

To my mother Yashi Mokonon, my father Dereje Wirtu, my brother Sekata Asefa, Fikire Dereje and Tesfaye Dereje my sister Aberash Dereje my friends Megersa Tadesse, Solomon Teshome, Abriham Mengesha and Yosef Hamba.

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List of Notations

SDE - Stochastic differential equation

Q - Risk neutral martingale measure

N (0, 1) - Normal distribution with mean 0 and variance 1

K - Strike price r -risk free interest rate

σ - volatility of variance

Θ - the long term mean variance process

k - the rate of mean reversion

S - stock price

PDE - Partial differential equation

$\Pi(S;t)$ - Derivative value, with underlying asset S at time t.

Δ - unit of stocks

H^f - payoff

x^* - maximum asset price

x - small asset price

v_t - variance at time t

T - time to maturity

Abstract

Derivative securities, when used correctly, can help investors to increase theirs expected returns and minimize theirs exposure to risk. Options offer influence and insurance for risk-averse investors. The pricing problems of the exotic options in finance do not have analytic solutions under stochastic volatility model and so it is difficult to calculate option prices or at least it requires a lot of time to compute them. Also the effect of stochastic volatility model resolving shortcoming of the Black-Scholes model its ability to generate volatility satisfying the market observations and also providing a closed-form solution for the European options. This study provides the required theoretical framework to practitioners for the option price estimation. This thesis focuses on pricing for floating strike lookback put option and testing option pricing formulas for the Heston stochastic volatility model, which defines the asset volatility as the stochastic process. Euler Maruyama method is the numerical simulation of a stochastic differential equation and generate the stochastic process way approximation. To simulate stock price and volatility stochastic processes in Heston's model the Euler discretization can be used to approximate the paths of the stock price and variance processes on a discretize grid. The pricing method depends on the partial differential equation approach on Heston stochastic volatility model and homotopy analysis method. Heston model has received the most attention then it can give a acceptable explanation of the underlying asset dynamics. The resulting formula is well connected to a Black-Scholes price that is the first term of the series expansion, which makes computing the option prices fairly efficient.

Chapter 1

Introduction

1.1 Background

Options are a type of financial derivative. This means that their price is not based directly on an assets price. Instead, the value of an option is based on the likelihood of change in an underlying assets price. More specifically, an option is a contract between a buyer and a seller. This contract gives the holder the right but not the obligation to buy or sell an underlying asset for a specific price (strike price) within a specific amount of time. The date at which the option expires is called the date of expiration.

European lookback options are kind of exotic option with path-dependency, introduced at first by Goldman (1997) having their settlement based on the minimum or the maximum value of an underlying index as registered during the lifetime of the option. At maturity, the holder can look back and select the most convenient price of the underlying that occurred during this period: therefore they offer investors the opportunity at a price of buying a stock at its lowest price and selling a stock at its highest price. Since this scheme guarantees the best possible result for the option holder, he or she will never regret the option payoff. As a consequence, a look back option is more expensive than any other option with similar payoff function. For the options traders this is clearly a major advantage, as look back options can be used to solve one of the major problems they face:

market timing. It is very important for the investor. It can be calls or puts, therefore it is likely to speculate on either the price of the underlying security increasing in value or decreasing in value. Look back options are classified into two. These are fixed strike lookback option and floating strike lookback option. Fixed strike is an option in which its strike price is fixed at the purchase. The payoff is the maximum difference among the optimal underlying asset price and the strike. In the case of call option the holder can look back above the life of the option and the option can exercised at assets highest price. In the case of put options, the option exercise at the lowest asset price. The options settle at the certain previous market price. Floating strike is an option in which its strike price is floating and determined at maturity. It is the optimal value of the underlying asset's price during the option life.

In this thesis the option pricing model under Heston more general setting than in the Black-Scholes framework. The model is the stochastic volatility model, in which not only let the stock price vary randomly, but also let the volatility of these random fluctuation be random. It investigate that this model can better reflect the market than the Black-Scholes model. It also assess how practically implementable this model is and try to draw conclusions on how options behave in a Heston framework.

1.2   Stochastic volatility

Stochastic volatility models are those in which a variance of a stochastic process is itself randomly distributed. They are used in the field of the mathematical finance to calculate derivative securities, such as options. The name derives from the models treatment of the underlying security's volatility as a random process,

governed by state variables such as the price level of the underlying security, the tendency of volatility to revert to some long-run mean value, and the variance of the volatility process itself, among others. Stochastic volatility models are one of the approach that resolve a shortcoming of a BlackScholes model. In particular, models based on the Black-Scholes assume that the underlying volatility is constant over the life of a derivative, and not affected by a changes of the price level of a underlying security. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to model derivatives more accurately.

Among the stochastic volatility model the popular Heston model is a commonly used stochastic volatility model, in which the randomness of the variance process varies as the square root of variance. It is a type of stochastic volatility model developed by associate finance professor Steven Heston in (1993) for analyzing bond,stock and currency options. It has also a closed-form solution for pricing options that seeks to overcome the shortcomings in the Black-Scholes option pricing model related to return skewness and strike-price bias. This model is a tool for advanced investors.

1.3   Statement of the problem

The assumption of constant volatility in the Black-Scholes formula is inappropriate for pricing lookback options. which means that the assumptions of the Black-Scholes model are unrealistic due partly to its inability to generate the volatility smile and the skewness in the distribution of the return. In particular, traders who use the BlackScholes model to hedge must continuously change the volatility assumption in order to match market prices. Their hedge ratios change accordingly in an uncontrolled way. More interestingly for us, the prices of lookback options given by models based on Black-Scholes assumptions can be wildly wrong and dealers in such options are motivated to find models which can take the volatility smile into account when pricing these. However many suggestions have been put for the use of stochastic volatility are appeared to the standard conditional volatility by several authors instead of the assumption of the Black-Scholes model. Among them, the Heston stochastic volatility model has been popular in modeling option price. The main aim of this study is to drive the appropriate pricing formula for floating strike lookback put option under the Heston stochastic volatility model. In particular, Heston proposed a stochastic volatility model with a closed-form solution for the price of the European lookback put option when the underlying assets are correlated with a latent volatility stochastic process.

1.4 Objective

1.4.1 General Objective

The main aim of the study is to develop a lookback option pricing model using stochastic volatility model. This will be achieved through the following specific objectives.

1.4.2 Specific objectives

i. To derive pricing formula for an European lookback put option with floating strike in relation to Heston model.

ii. To perform a simulation on the theoretical pricing model in (i).

iii. Fit the pricing model on empirical data to verify the theoretical stability of results.

1.5 Significance of the study

This study provides the required theoretical framework to practitioners for the lookback option price with floating strike price estimation and risk management in the markets which are inherently ineffective and hence it shall justify and validate the markets model to hedger in the real financial market and act as platform for additional research on related problems. So this research will contribute a model for the market and to markets participants in the sense that the financial participant or the investors have equal chance to participate in the market without insider information in the country. It uses for investor to buy your stocks at a lower price, reduce your cost basis and generate additional income.

1.6 The scope of the study

This study will focus on pricing derivatives for European look back put option with floating strike under stochastic volatility model. The derivatives of the pricing formula and present the numerical procedures used to construct the pricing formula under a market which consists of one risk less asset and one non-dividend paying risky asset (the stock) with price process to be formulated by Heston stochastic volatility option price model.

Chapter 2

Literature review

In this section we review some studies on look back option price. Higham (2004) the basic types of options are the European lookback Call and Put options. A European Call option gives the right for the buyer but not the obligation, to purchase from a seller a prescribed asset for agreed price at the agreed time in the future. On the other hand, a European Put option gives for the holder the right, but not the obligation, to sell to the writer an agreed asset for an agreed price at a prescribed time in the future.

The academic literature proposes several methods that accommodate the path dependency of these options. For example, see Hull and White (1993), Cheuk and Vorst (1997) and Wilmott, Howison, and Dewynne (1995).Their numerical approaches deal with the path-dependency feature of look back options either by explicitly including the historical minimum or maximum underlying asset values as an additional dimension or by tracing the possible minimum or maximum values in the underlying asset price process. A standard look back put gives the right to sell at the highest price. These options were first studied by Goldman et al. (1979), who derived closed-form pricing formulas under the geometric Brownian motion. Conze and Vishwanathan (2007) explain option price in case of calls on the maximum and put on the minimum. A call on the high pays off the difference among the realized high price and some specified strike, each is greater. The difference between the strike price and the take in the minimum price or it

may be zero, is the put on the minimum pays off. Those options are well-known as the European fixed-strike look back option. In contrast, the normal look back options are also known as European floating-strike look back option, since the floating terminal underlying asset price helps as the strike in the normal look back options. Tian and Boyle (1999) have initiated a study of barrier and European look back options. They introduce European Look back call options give the right for the investors to buy at the minimum price during the life of the options while European Look back put options would permits investors to sell at the maximum price. Bacinello and Ortu (1993) study about the stochastic interest rates, those explain on pricing and they focus on a complete financial market which leads to the existence of a unique or single corresponding martingale measure for the fairness price.

Fischer and Myron(1973) state that the mathematical framework for evaluation of option price for the basic vanilla European style option. Basic vanilla European calls and puts have a systematic closed form solution and state the two most usually used techniques: European fixed strike lookback option price and European float strike lookback option price. Guarino, (2010) suggests that a developing market is the term which discusses to a country that has accepted change in the political system, economic systems and experienced fast economic development. (Arnold and Quelch, 1998) explains developing markets represent countries and markets playing fastening up: nations with new or an undeveloped industrial base and infrastructure on the one hand, but a rapidly rate of growth or usually above that of developed nations on the other hand. Fischer and Myron (1997) describe about European look back option pricing. Option pricing is important to almost each area of the finance. European lookback Option pricing theory takes a long and well-known history.Black-Scholes(1979) describe the first acceptable

equilibrium option pricing model. In the similar way, Robert Merton (1973) long their model in numerous important ways. Also proposed a mathematical model for the pricing stock price options. Black-Scholes PDE is the method for the pricing options in which the underlying stock is priced using the Black-Scholes model. The Black Scholes formula is the formula that use for the pricing the European lookback Call and Put options using a Black-Scholes PDE. The BS model is depending on the assumption that the stock price follows the Brownian motion, using a risk neutral probability.

In this part we review certain studies on stochastic volatility model Hestons stochastic volatility model. There are a number of the methods that can use to model volatility stochastically. Hull and White (1987) model the variance using a geometric Brownian motion, in addition to an Ornstein-Uhlenbeck process with the mean-reversion related to a volatility. In the general case, mean-reversion is considered to be an important feature of observed volatility, and thus all plausible models are of the Ornstein-Uhlenbeck type. Wiggins (1987) models the logarithm of a volatility with the mean-reversion, where Scott (1987), Johnson and Shanno (1987), Heston (1993) and Stein and Stein (1991) model the variance using a square root process. Zhu (2000) also considers the double square root process, that is an extension of a basic square root process in both the drift and diffusion coefficients involves the volatility. In this thesis we focus on the Hestons square root model, under the Heston (1993) provides an analytic expression for European lookback option prices.

Under the Black and Scholes (1973) model, closed-form pricing formulas for continuously monitored lookback options were derived by Goldman et al. (1979) and Conze and Viswanathan (1991). Heynen and Kat (1995) derived the analytical

formulas for discretely monitored lookback options under the BlackScholes setting and the corresponding formulas were further generalized to the general Lvy setting by Liao (1992). In the BlackScholes model, the underlying asset price process is assumed to follow the geometric Brownian process, in which the volatility of the asset price is a constant. This assumption is inconsistent with the phenomena of implied volatility smiles observed in the market data. Thus, stochastic volatility models are proposed (see [610] for instance) to fix this problem.

The pricing of lookback options under the stochastic volatility model poses interesting mathematical challenges. Leung (2013) derived semi-analytical pricing formulas for lookback options under the two-factor stochastic volatility model and both stochastic volatility factors are driven by mean-reverting processes. Their results provide a good approximation of the price for both fixed strike and floating strike lookback options when the mean-reverting rate of one stochastic volatility factor is large and the mean-reverting rate of the second factor is small. In this work, the use the homotopy analysis method to derive the analytic pricing formulas for lookback options under Hestons stochastic volatility model. In contrast to those of Black-Scholes (1973), our formulas are free from the assumption of the relative magnitudes of all the model parameters. The homotopy analysis method, as initially suggested by Ortega and Rheinboldt (1970), has been used by Liao (1992) to solve many nonlinear problems. Recently, Zhu [18] was the first to apply this method to derive the closed-form solution for the valuation of European options and his work has been further extended to different asset price processes and different types of options.

As largely discussed in the specialized literature by Johnson (1999), the effect of stochastic volatility model resolving shortcoming of the Black-scholes model in its

ability to generate volatility satisfying the market observations, and also providing a closed-form solution for the European lookback options. However, it has not yet been established whether closed form solution introduced by stochastic volatility are suitable for the purpose of matching market prices than the Black-scholes model.

Chapter 3

The Black-Scholes Options Pricing model

The Black-Scholes formula (also called Black-Scholes-Merton) was the first widely used model for option pricing. It's used to calculate the theoretical value of European-style options using current stock prices, the option's strike price, expected interest rates, time to expiration and constant volatility. The Black-Scholes model make some assumptions:

1. The underlying assets follows the geometric Brownian motion with a constant volatility

2. The price of a stock is log-normally distributed with the mean μ,and also standard deviation σ

3. There is the constant risk-free rate

4. Market participant either borrow or lend at a risk-free rate

5. There are no transaction costs in buying the option, and No dividends are paid out during the life of the option.

The Black-Scholes Model makes a assumption of a underlying asset, following the geometric Brownian motion under some risk neutral measure as:

$$ds_t = rs_t dt + \sigma s_t dW_t \tag{3.1}$$

where r is the interest risk free rate and volatility σ. Equation (3.1) which is a short form of the following equation given as:

$$S_t = S_0 + \int_0^t r S_z dz + \int_0^t \sigma S_z dW_z \qquad (3.2)$$

Assuming that the daily asset returns follows a log normal distribution and this by introducing:

$$Y_t = log\left[\frac{S_t}{S_0}\right]$$

By applying Ito's formula receive the following expression

$$dY_t = \left[r - \frac{1}{2}\sigma^2\right]dt + \sigma dW_t$$

by integrating both sides we can get

$$Y_t = \left[r - \frac{1}{2}\sigma^2\right]t + \sigma W_t$$

substituting the value of

$$Y_t$$

into above equation, finally, we can get

$$S_t = S_0 exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t\} \qquad (3.3)$$

S_t is log normally distributed, volatility is constant and there by does the following holds

$$log\left[\frac{S_t}{S_0}\right] \sim N\left[(r - \frac{1}{2}\sigma^2)t, \sigma^2 t\right] \tag{3.4}$$

Definition 3.1: Let S_t be maximum asset price observed during the options life and S_T be stock price at the maturity time T. The payoff floating strike lookback put option is the difference between maximum asset price observed during the life the option and stock price observed at the expiration time T.

$$F_{lb} = max(S_t - S_T)|0 \le t \le T \tag{3.5}$$

The theory of a pricing given by the arbitrage free price gives us the following expression for a price of the floating strike lookback put option at the expiration time

$$F_{lb} = e^{-r(T-t)}E^Q\left[max(S_t - S_T)\right]|0 \le t \le T \tag{3.6}$$

where Q is the equivalent risk-neutral measure under which S_t is a geometric Brownian motion. The analytical solution has been found, by Goldman et al. (1979), the Black-scholes formula for lookback put option with floating strike and with constant volatility σ given as:

$$F_{lb} = Se^{-r(T-t)}N(-d_2) - Me^{T-t}N(d_1)$$
$$+Me^{-(T-t)}\frac{\sigma^2}{2r}\left[e^{T-t}N(d_1) + \left(-\frac{M}{S}\right)^{\frac{-2r}{\sigma^2}}N(d_1) - \left(\frac{2r}{\sigma}\right)\sqrt{T-t}\right] \tag{3.7}$$

$$\text{where } d_1 = \frac{log\left(\frac{M}{S}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

with N(.) being the standard normal cumulative distribution function.

3.1 Time dependent volatility in the Black-Scholes model

In this part we introduced the time dependent volatility in the model of the Black-Scholes. These extension will be used for the bridge to gap between the original Black-Scholes model and the one with the stochastic volatility.

3.1.1 Time dependent volatility

Let a stock price be modelled under a risk-neutral measure Q as

$$ds_t = rs_t dt + \sigma(t)s_t dW_t, S_0 = S \tag{3.8}$$

where $\sigma : [0, T] \longrightarrow (0, \infty)$ is the deterministic function. Applying Ito's formulas the solution to SDE governing the dynamics of a stock price is given by

$$S_t = S_0 exp \left[\int_0^t \{r - \frac{1}{2}\sigma^2(s)\}ds + \int_0^t \sigma(s)dWs \right] \tag{3.9}$$

and let us define

$$\tilde{\sigma}^2(t, T) = \frac{1}{T - t} \int_t^T \sigma(s)ds$$

then the solution of the SDE can be written as

$$S_t = S_0 exp \left[\{r - \frac{1}{2}\tilde{\sigma}^2(0, t)\}t + \int_0^t \sigma(s)dWs \right] \tag{3.10}$$

and the solution of the SDE at the maturity is:

$$S_T = S_t exp \left[\{r - \frac{1}{2}\tilde{\sigma}^2(t,T)\}(T-t) + \int_t^T \sigma(s)dWs \right]$$

and the distribution of $log \left[\dfrac{S_T}{S_t} \right]$ conditioned on S_t is given by

$$log \left[\frac{S_T}{S_t} \right] \sim N \left[\{r - \frac{1}{2}\tilde{\sigma}^2(t,T)(T-t), \tilde{\sigma}^2(t)(T-t)\} \right] \qquad (3.11)$$

Thus, we comparing this equation(3.11) with (3.4), that we can use the same pricing formula as in the standard Black -Scholes case. We only have to replace σ^2 by $\tilde{\sigma}^2$ with and doing so we arrive at, for $t \leq T$.

$$V(t) = V^{BS}(t; \sqrt{\tilde{\sigma}^2}(t,T)) \qquad (3.12)$$

where as usual $V^{Bs}(t)$ denotes the price of the option at time t. In this case, the volatility is not merely the number, but the whole function. With a approach of a deterministic but the time-dependent volatility. We have moved away from a constant volatility model of the Black-Scholes. But from Equation (3.11) that the returns still will be normally distributed. Since this is emperically not fact, we must move on, trying to find the model where the returns are not normally distributed.

3.1.2   Finding time dependent from the implied volatility

By fixing a strike price, let look at the implied volatility as the function of the time to the maturity only. It will be dependent on a observed options prices, but since these are given by a market, and not possible to choose, As parameters and suppress their dependence on the implied volatility. To conclude, let I(T) denote the implied volatility given by the observed price of some European option with

given strike price K and time to maturity T. By observing a implied volatility at the some fixed time t_0 as it changes over times to the maturity T, we recover the time-dependent volatility σ_t for $t_0 \leq t$, We can make the assumption that there exists an option with the maturity time T for every $t_0 \leq T$. The idea is to equate the theoretical volatility under the model given by Equation (3.1), the LHS in the next equation, with the observed implied volatility:

$$I(T) = \frac{1}{T-t}\sqrt{\int_{t_0}^{T} \sigma(s)ds}$$

which gives as:

$$\int_{t}^{T} \sigma(s)ds = I^2(T)(T - t_0)$$

Differentiating both sides with respect to T and fixed t_0

$$\sigma^2(T) = 2I(T)I'(T)(T - t) + I^2(T)$$

By converting $T \longrightarrow t$ and we get

$$\sigma(t) = \sqrt{2I(T)I'(T)(t - t_0) + I^2(T)}$$

Therefore, what we have accomplished is an explicit formula, showing how to extract a volatility function $\sigma(t)$ from the observed implied volatilities. From the assumption there exists an option that mature at given time $T \geq t_0$ is not realistic.

3.2   The model

Let (Ω, F, Q) be a price probability space with the filtration $(F_t)_{t \in [0,T]}$ which is generated by the Wiener processes W_t^1 and W_t^2 with correlated coefficient p and Q is a risk-neutral measure. Assume that the underlying asset S_t in risk-neutral and variance follow the following model:

$$dS_t = rS_tdt + S_t\sqrt{v_t}dW_t^1 \tag{3.13}$$

$$dv_t = k(\Theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \tag{3.14}$$

$$dW_t^1 dW_t^2 = pdt, p \in [-1, 1] \tag{3.15}$$

where r, k, Θ and σ are constant. The variance v_t is defined on the interval $[0, +\infty] \times \Re$. The Heston asset process has a variance v_t that follows Cox-Ingersoll-Ross(1998) process described in equation in(3.14). For the square root process in equation (3.14) the variance is always positive and if $2k\Theta > \sigma^2$ then it cannot reach zero. Note that the deterministic part of process (3.14) is asymptotically stable if $k > 0$.

Theorem 4.1. Consider the Cox-Ingersoll-Ross (CIR) interest rate model

$$dv_t = k(\Theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \tag{3.16}$$

then the exact solution is

$$v_t = v_0 e^{-kt} + \Theta(1 - e^{-kt}) + \sigma e^{-kt} \int_o^t e^{ks}\sqrt{v_s}dW_s \tag{3.17}$$

Proof: Equation(4.3) can be changed to the following form.

$$dv_t + kv_t dt = k\Theta dt + \sigma\sqrt{v_t}dW_t \tag{3.18}$$

Multiplying both sides of the relation(4.5) by e^{kt} result in

$$e^{kt}dv_t + ke^{kt}v_t dt + k\Theta e^{kt}dt = \sigma e^{kt}\sqrt{v_t}dW_t$$

$$d(e^{kt}v_t) = k\Theta e^{kt}dt + \sigma e^{kt}\sqrt{v_t}dW_t \tag{3.19}$$

Now, integrating both sides of the relation (4.6) on [0,t] gives us

$$e^{kt}v_t - v_0 = k\Theta \int_0^t e^{ks}ds + \sigma e^{kt}\int_0^t e^{ks}\sqrt{v_t}dW_s$$

$$v_t = e^{-kt}v_0 + \Theta(1 - e^{-kt}) + \sigma e^{-kt}\int_0^t e^{ks}\sqrt{v_t}dW_s$$

According the theorem of the CIR model has no general explicit solution

The parameters used in the Heston model are as follows:

S_t is the price of the underlying asset at the time t

K is the rate of the mean reversion

r is the risk free interest rate

Θ is the long term mean variance

v_t is the variance at the time t

σ is the volatility of the variance process.

Therefore, under a Heston model, the underlying asset follow an evolution process which is similar to a Black-Scholes model model, but it also explains a stochastic behavior for a volatility process. In particular, Heston makes a assumption that a asset variance V_t follows the mean reverting Cox-Ingersoll-Ross process. Consequently, a Heston model provides the versatile modelling framework which can accommodate many of a specific characteristics which are typically observed in the behavior of the financial assets. In particular, a parameter σ controls the kurtosis of a underlying asset return distribution, while p sets its asymmetry.

3.3 The PDE of the Heston model for the option price

In this section we derive the PDE that the price of a derivative must solve, where the tradeable security as well the volatility of the tradeable security follows general stochastic processes. The PDE that governs the prices of derivatives written on a tradeable security with stochastic volatility is derived and describe how to derive the PDE for the Heston model. This derivation is a special case of a PDE for general stochastic volatility models which is described by Leung (2013). Heston model is one of the most popular option pricing models. This is due in part to the fact that the Heston model produces either call or put prices that are in closed form, up to an integral that must evaluated numerically. In order to price options in a stochastic volatility model, it is possible use the risk-neutral valuation method.

To derive Heston PDE let form a portfolio consisting of one option being priced, denoted by the value $V = V(s, v, t)$, Δ units of the stock S, ψ of another option $U = U(S, V, T)$ that is used to hedge the volatility. The portfolio has value

$$\Pi = V + \Delta S + \psi U \tag{3.20}$$

Assuming the portfolio is self financing , the change in portfolio value is

$$d\Pi = dV + \Delta dS + \psi dU \tag{3.21}$$

Apply ito's Lemma to dV and differentiate with respect to the variables t, S, v we get

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}dt + \sigma vp\frac{\partial^2 V}{\partial v \partial S}dt \tag{3.22}$$

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Applying Ito's Lemma to dU produces the identical result, but in U. Combining these two expressions, we can write the change in portfolio value as:

$$d\Pi = dV + \Delta dS + \psi dU \tag{3.23}$$

$$
\begin{aligned}
d\Pi = &\left\{ \frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + p\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2} \right\} dt \\
&+\psi\left\{ \frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + p\sigma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} \right\} dt \\
&+\left\{ \frac{\partial V}{\partial S} + \psi\frac{\partial U}{\partial S} + \Delta \right\} dS + \left\{ \frac{\partial V}{\partial v} + \psi\frac{\partial U}{\partial v} \right\} dv
\end{aligned}
\tag{3.24}
$$

In order for the portfolio to be hedged against movements in the stock and against volatility, the last two terms in Equation (3.23) involving dS and dv must be zero. This implies that the hedge parameters must be

$$\psi = -\frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}}, \Delta = -\psi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S} \tag{3.25}$$

Moreover, the portfolio must earn the risk free rate. Hence $d\Pi = r\Pi dt$. Now with the values of Δ and ψ from Equation (3.24) the change in value of the riskless portfolio is

$$
\begin{aligned}
d\Pi = &\left\{ \frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + p\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2} \right\} dt \\
&+\psi\left\{ \frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + p\sigma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} \right\} dt
\end{aligned}
\tag{3.26}
$$

In risk less portfolio we have

$$d\Pi = r\Pi dt \tag{3.27}$$

which write as:

$$d\Pi = X + \psi Y$$

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$$X + \psi Y = r(V + \Delta S + \psi U)$$

Substituting for ψ and re-arranging, produces the equality

$$\frac{X - rV + rS\dfrac{\partial V}{\partial S}}{\dfrac{\partial V}{\partial v}} = \frac{Y - rU + rS\dfrac{\partial U}{\partial S}}{\dfrac{\partial U}{\partial v}} \tag{3.28}$$

The left-hand side of Equation (3.24) is a function of V only, and the right-hand side is a function of U only. This implies that both sides can be written as a function f(S, v, t) of S, v, and t. Heston, specify this function as $f(S, v, t) = -K(\theta - v) + \lambda(t, s, v)$ where $\lambda(t, s, v)$ is market price of volatility risk. Substitute f(S, v,t) into the left-hand side of Equation (3.24), substitute for Y and rearrange to produce the Heston PDE for the option U expressed in terms of the price S

$$\begin{aligned}
\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + p\sigma v S\frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2} - rU \\
+rS\frac{\partial U}{\partial S} + [K(\theta - v) - \lambda(t, s, v)]\frac{\partial U}{\partial v} = 0
\end{aligned} \tag{3.29}$$

Equation(3.29) is the partial differential equation of the Heston models governing the option price under this model.

3.4 The PDE in Terms of the Logarithm Price

In this section, the closed form of a partial differential equation in terms logarithm price. Let $x = lnS$ and describe the Heston partial differential equation in terms of x, v and t instead of S, v, and t. This leads to a simpler form of the PDE and need the following derivatives, which are direct to derive

$$\frac{\partial U}{\partial S}, \frac{\partial^2 U}{\partial v \partial s}, \frac{\partial^2 U}{\partial S^2}$$

Insert into the Heston PDE Equation (3.29). All the S terms eliminated and obtain the Heston PDE in terms of the logarithm price $x = lnS$ and Heston assumes that the price are risk-neutral. The reason for this term is that in reality most investors are found to be risk averse in experimental settings (Holy (2002)). Moreover, Lamoureux and Lastrapes find evidence from observed option prices that the efficient-market hypothesis and investor risk-neutrality cannot hold simultaneously. Often λ is assumed zero, so the price is given under the risk-neutral measure, i.e, under the assumption that investors are risk-neutral.

$$\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + p\sigma v\frac{\partial^2 U}{\partial v \partial x} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2} - rU + (r - \frac{v}{2})\frac{\partial U}{\partial x}$$
$$+ [K(\theta - v)]\frac{\partial U}{\partial v} = 0 \tag{3.30}$$

3.5   Floating strike lookback put option price

The payoffs of the floating strike lookback put options depend on the maximum asset price reached during the life of the option and underlying asset price observed at the maturity. Based upon the fundamental theorem of asset pricing (Shreve (2000)), the no-arbitrage price of a European lookback put option with floating strike is given by

$$U^f(t, x, x^*, v) = E^Q[e^{-r(T-t)}H^f(X_T, X_T^*)|X_t = x, X_t^* = x^*, V_t = v] \tag{3.31}$$

where X_T^* is the maximum asset price observed during the life of the option and H^f is the payoff of the put option. In this section we consider a floating strike lookback put option, where the underlying asset price is assumed to follow the SDE. Then $H^f(X_T, X_T^*) = X_T^* - X_T$ (payoff put option), the risk-neutral price of the floating strike lookback put option, denoted by $U^f(t, x, x^*, v)$ at the time

t, $t \in [0, T]$ for $X_t = x$, $X_t^* = x^*$ and $V_t = v$ is given as

$$U^f(t, x, x^*, v) = E^Q[e^{-r(T-t)}H^f(X_T^* - X_T)|X_t = x, X_t^* = x^*, V_t = v] \quad (3.32)$$

In this section $H^f = (X_T^* - X_T)$ payoff chosen. Transforming the governing equation (3.30) in terms of the differential operators as follows:

$$\pounds_1 = p\sigma v \frac{\partial}{\partial v} + p\sigma v \frac{\partial^2}{\partial x \partial v} + k(\Theta - v)\frac{\partial}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2}{\partial v^2}$$

$$\pounds_2 = \frac{\partial}{\partial t} + \frac{1}{2}v\frac{\partial^2}{\partial x^2} + (r - \frac{v}{2})\frac{\partial}{\partial x}$$

and the problem(3.32) can be transformed into the PDE problem as follows. since both x_t^* and x_t are continuous and non decreasing. So that the quadratic variance and covariance of the $[x_t^*, x_t^*]$ and $[x_t^*, x_t]$ satisfy the following conditions

$$[x_t^*, x_t^*] = \lim_{\Pi \to 0} \Sigma_{i=0}^m([x_{ti+1}^* - x_{ti}^*])^2 \le x_t^* \lim_{\Pi \to 0} max(x_{ti+1}^* - x_{ti}^*) = 0$$

and

$$[x_t, x_t^*] = \lim_{\Pi \to 0} \Sigma_{i=0}^m(x_{ti+1} - x_{ti})^2(x_{ti+1}^* - x_t^*)$$

$$\le \lim_{\Pi \to 0} max(x_{ti+1} - x_{ti}) = 0$$

for any partition $\Pi = \{0 = t_0, t_1... = t\}$. This implies the integral involved with the $dx_t dx_t^*$ and $dx_t^* dx_t^*$ will be zero. So from ito's formula we can get

$$d(e^{-rt}U^f) = e^{-rt}(\pounds_1 + \pounds_2)U^f dt + U_x^f dx_t$$

Then have

$$E\left[\int_t^T e^{-rs}(\pounds_1 + \pounds_2)U^f ds + \int_t^T U_x^f dx_s | x_t = x, x_t^* = x^*\right]$$

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for $t, T \in [0, \infty]$ since $e^{-rt}U^f$ is a martingale and the second part of the conditional expectation is zero on the $0 < x < x^*$ so that the PDE for the U^f can obtained on the interval $0 < x < x^*$ and by using mean value theorem and taking the $T \to t$ we can get

$$(\pounds_1 + \pounds_2)U^f(t, x, x^*, v) = 0 | 0 \le t \le T | 0 \le x \le x^*$$
$$U(T, x, x^*, v) = x^* - x \qquad (3.33)$$
$$\frac{\partial U}{\partial x^*}(t, x, x^*, v) \mid_{x=x^*} = 0$$

Here, the final condition follows from the definition (3.33) directly and the assumption on the continuity of partial derivatives leads to the boundary condition. As compared with (3.30) and (3.33) is much simpler to solve because its dimension is reduced by 1.

Definition 3.5.1: zeroth-order deformation equation. Let $p \in [0, 1]$ denote the embedding parameter and $U_0(t, x, x^*, v)$ be the initial approximation of the $U(t, x, x^*, v)$ such that as p increases from 0 to 1 , $U(t, x, x^*, v)$ varies continuously from the initial approximation $U_0(t, x, x^*, v)$, such kind of the continuous variation or deformations are defined by the zero order deformation equation. Applying the definition and Following the same vein as Park (2011) method, the homotopy analysis method is to solve $U(t, x, x^*, v)$ from (3.33) can construct a homotopy of the of (3.33). To construct let us consider $U(t, x, x^*, v, p)$ denoting the solution of a PDE problem given by $H(t, x, x^*, v, p) = 0$ with the final and boundary condition of (3.33),where H, called a homotopy, is defined by

$$H(t, x, x^*, v, p) = (1 - p)(\pounds_2 U(t, x, x^*, v, p) - \pounds_2 U_0(t, x, x^*, v))$$
$$+ p(\pounds_1 + \pounds_2)U(t, x, x^*, v, p), p \in [0, 1] \qquad (3.34)$$

Here $U_0(t, x, x^*, v)$ is the initial value approximation from Black-Scholes formula

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for the lookback put option price with the constant volatility. The Black-Scholes formula is well-known and, for instance, see Wilmott (2006). By this choice of U_0, the homotopy problem becomes

$$H(t, x, x^*, v, p) = \pounds_2 U(t, x, x^*, v, p) + p\pounds_1 U(t, x, x^*, v, p) = 0 \qquad (3.35)$$

with the final and the boundary condition of (3.33) apply the homotopy analysis method by the considering a Taylor series

$$U(t, x, x^*, v, p) = \sum_{n=0}^{\infty} U_n(t, x, x^*, v,)p^n \qquad (3.36)$$

where U_n denote a Taylor coefficient. Note that floating strike lookback put option price U_f is then given by

$$U_f(t, x, x^*, v,) = \lim_{p \to 1} U(t, x, x^*, v, p) = \sum_{n=0}^{\infty} U_n(t, x, x^*, v) \qquad (3.37)$$

Inserting equation (3.36) into (3.35) and using a standard perturbation argument, obtain formally a hierarchy of PDE problem as follows get

$$\pounds_2 U_n(t, x, x^*, v) + \pounds_1 U_{n-1}(t, x, x^*, v) = 0,$$
$$U_n(T, x, x^*, v) = 0, \qquad (3.38)$$
$$\frac{\partial U}{\partial x^*}(t, x, x^*, v) \mid_{x=x^*} = 0$$

for all n=1,2,3,....

To find the solution of the equation (3.38) use two lemmas: i.e a lemma about a Feynman-kac formula for floating strike lookback put option price and lemma about the joint probability density of the two Gaussian processes. For the convenience, use the notation

$$E^{x,x^*}[.] := E^Q[.|S_t = x, S_t^* = x^*]$$

where S_t and S_t^* are the solution given by

$$S_t = rS_t dt + \sqrt{V_t} S_t dW_t^t$$

$$S_t^* = max_{u \leq t} S_u$$

respectively, for some $\sqrt{V_t} \in R^+$.

Lemma 3.5.1 If $Z(t, x, x^*, v) \in U_b^{1,2}(R^+ \times R^3)$ and solve the PDE problem and also $U_b^{1,2}(R^+ \times R^3)$ is the function space of bounded functions continuously differentiable with respect to $t > 0$ and twice continuously differentiable with respect to $(t, x, x^*, v) \in R^3$.

$$£_2 Z(t, x, x^*, v) = g(t, x, x^*, v), 0 \leq t \leq T, 0 < x \leq x^*$$

$$Z(T, x, x^*, v) = h(x, x^*)$$

,

$$\frac{\partial Z}{\partial x^*}(t, x, x^*, v) \mid_{x=x^*} = 0$$

where g and h satisfy the conditions $g + h = o(e^{x^2+x^*})$ as x and $x^* \to \infty$ then

$$Z(t, x, x^*, v) = E^{x,x^*} \left[e^{-r(T-t)} h(S_T, S_T^*) - \int_t^T e^{r(t-s)} g(s, S_s, S_s^*, v) ds \right]$$

Proof. The solution of SDE (3.13) is well known geometric Brownian motion and infinitesimal series solution as

$$A_s = \frac{1}{2} v^2 x^2 \frac{\partial^2}{\partial x^2} + rx \frac{\partial}{\partial x}$$

If we define a joint process $G_t = (t, S_t, R_t, S_t^*)$ with $R_t = -rt$, then G_t is an Ito's diffusion with generator given by

$$A_G U = A_S U + \frac{\partial U}{\partial t} - r\frac{\partial U}{\partial R} + \lim_{t \to 0}\frac{1}{t}E^x\left[\int_0^t \left(\frac{\partial U}{\partial S^*}dS_s^* + \frac{1\partial^2 U}{2\partial(S^*)^2}(dS_s^*)^2 + \frac{\partial^2 U}{\partial S\partial S^*}dS_s dS_s^*\right)\right]$$

Since S_t is continuous, we have $(dS_t^*)^2 = dS_s dS_s^* = 0$ symbolically. More over, the integral

$$\int_0^t \frac{\partial U}{\partial S^*}dS_s^*$$

is also zero due to $\frac{\partial U}{\partial x^*}|_{x=x^*} = 0$ and $dS_s^* = 0$ for $S_t \neq S_t^*$. So that obtain the identity $A_G(e^{-rt}U) = e^{-rt}\pounds_2 U$. Then using a function defined by ϕ defined by $\phi(s, x, -rt, x^*, v) = e^{-rt}U(s, x, x^*, v)$ and a stopping time τ_n defined by $\tau_n = inft : S_t^* \leq n$, dynkin's formula (for example, in Oksendal (2003)) leads to

$$E^{x,x^*}[\phi(G_{T\Lambda\tau_n})] = \phi(t, x, -rt, x^*, v) + E^{x,x^*}\left[\int_t^{T\Lambda\tau_n} A_G(\phi(G_s))dS\right]$$

$$= \phi(t, x, -rt, x^{E^{x,x^*}}, v) + E^{x,x^*}\left[\int_t^{T\Lambda} e^{-rs}g(S, S_t, S_t^*, v)dS\right]$$

and so

$U(t, x, x^*, v) = e^{rt}\phi(t, x, -rt, x^*, v)$

$$= E^{x,x^*}\left[e^{-r(T\Lambda\tau_n - t)}U(T\Lambda\tau_n, S_T\Lambda\tau_n, S_T^*\Lambda\tau_n, v) - \int_t^{T\Lambda\tau_n} e^{r(t-s)}g(S, S_s, S_s^*, V)dS\right]$$

Finally, the well-known dominated convergence theorem (in Royden (2010) for instance in real analysis yields the theorem by taking n $\to \infty$.

Lemma 3.5.2 If H_t and H_t^* are the two Gaussian processes defined by

$$H_t = (r - \frac{1}{2}\sigma^2)t + \sigma W_t$$

and

$$H_t^* = max\{(r - \frac{1}{2}\sigma^2)s + \sigma W_s\}$$

then the joint probability density of a processes (H_t, H_t^*) is given as

$$Q(H_t \in db, H_t^* \in dc) = \frac{2(2c-a)}{\sigma^3 t^{\frac{3}{2}}\sqrt{2\Pi}}e^{\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}b - \frac{(r-\frac{1}{2}\sigma^2)^2}{2\sigma^2}t - \frac{(2c-b)^2}{2\sigma^2 t}}dbdc$$

Proof. If define a martingale Z and probability measure P by

$$Z(t) = e^{-\frac{r-\frac{\sigma^2}{2}}{\sigma}W_t - \frac{1}{2}(\frac{r-\frac{\sigma^2}{2}}{\sigma})^2 t}$$

$P(A) = \int_A Z(t)dQ$ for all $A \in F$

Respectively, then the process $\frac{1}{\sigma}H_t$ is Brownian motion under P by Girsanov's theorem (for instance , in oksendal (2003)). Then the joint density function of $(H_t, H_t^*$ under P is well known. Refer to shreve (2000) or Wilmott(2006). In the present context, is given by

$$P(H_t \in db, H_t^* \in dc) = \frac{2(2c-2b)}{\sqrt{2\Pi}\sigma^3 t^{\frac{3}{2}}}e^{-\frac{1}{2\sigma^2 t}(2c-b)^2}dbdc$$

So, by applying the identity $Q = \frac{1}{Z(t)}P$, obtain the lemma.

Using the above two lemmas we can approximate the solution of the PDE problem (3.38).

Theorem 3.5.1 Assume that the floating strike lookback put option price $U^f(t, x, x^*, v)$ is represented as

$$U^f(t, x, x^*, v) = \sum_{n=0}^{\infty} U_n^f(t, x, x^*, v),$$

then $U_n^f(t, x, x^*, v)$ is given by

$$U_0^f(t,x,x^*,v) = \left(1+\frac{\sigma^2}{2r}\right)xN(\delta+(T-t,\frac{x}{x^*})) + e^{-r(T-t)}x^*N(-\delta-(T-t,\frac{x}{x^*}))$$

$$-\frac{\sigma^2}{2r}e^{-r(T-t)}x\left(\frac{x^*}{x}\right)^{\frac{2r}{\sigma^2}}N(-\delta-(T-t,\frac{x^*}{x}))-x$$

for n=0 and $U_n^f(t,x,x^*,v) = \int_t^T \int_{ln(\frac{x^*}{x})}^{\infty}\int_{-\infty}^{\infty}\frac{2(2c-b)\pounds_1 U_{n-1}(s,xe^b,xe^c,v)}{\sigma^3(s-t)^{\frac{3}{2}}\sqrt{2\Pi}}.exp\{r(t-$

$s)+\frac{r-\frac{\sigma^2}{2}}{\sigma^2}b-\frac{(r-\frac{\sigma^2}{2})^2}{2\sigma^2}(s-t)-\frac{(2c-b)^2}{2\sigma^2(s-t)}\}dbdcds,$

where $\pounds_1$ is a differential operator given above , for $n \geq 1$, where N denotes the

usual cumulative normal distribution, $\delta \pm (t,x) = \frac{1}{\sigma\sqrt{t}}(lnx+(r\pm\frac{1}{2}\sigma^2)t)$ and

$\sqrt{v} = \sigma$

Proof. Since U_0^f is the Black-scholes put option price. Thus the PDE of (4.17) for

U_1 satisfies the required conditions of lemma and also U_n for $n > 1$ are smooth

to be U_o^2 due to $Q(H_t \in db, H_t^* \in dc) = o(e^{-b^2-c^2})$. Then from both lemmas we

can obtain

$$U_n(t,x,x^*,v) = E^{x,x^*}\left[\int_t^T e^{r(t-s)}\pounds_1 U_{n-1}(s,S_s,S_s^*,v)ds\right]$$

$$= E^{x,x^*}\left[\int_t^T e^{r(t-s)}\pounds_1 U_{n-1}(s,xe^{H_{s-t}},xe^{H_{s-t}^*},v)ds\right]$$

$$= \int_{ln(\frac{x^*}{x})}^{\infty}\int_{-\infty}^{\infty}\left(\int_t^T e^{r(t-s)}\pounds_1 U_{n-1}(s,xe^b,xe^c,v)ds\right)Q(H_{s-t}\in db, H_{s-t}^*\in dc)$$

$$= \int_t^T\int_{ln(\frac{x^*}{x})}^{\infty}\int_{-\infty}^{\infty}e^{r(t-s)}\pounds_1 U_{n-1}(s,xe^b,xe^c,v)Q(H_{s-t}\in db, H_{s-t}^*\in dc)ds$$

$$= \int_t^T\int_{ln(\frac{x^*}{x})}^{\infty}\int_{-\infty}^{\infty}\frac{2(2c-b)\pounds_1 U_{n-1}(s,xe^b,xe^c,v)}{\sigma^3(s-t)^{\frac{3}{2}}\sqrt{2\Pi}}.exp\{r(t-s)+\lambda\}dbdcds$$

where

$$\lambda = \frac{r - \frac{1}{\sigma^2}}{\sigma^2}b - \frac{(r - \frac{\sigma^2}{2})^2}{2\sigma^2}(s - t) - \frac{(2c - b)^2}{2\sigma^2(s - t)}$$

this proves the theorem

Chapter 4

Numerical method for stochastic approximations

4.1 SDE Approximation

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs contain a variable which represents a random white noise calculated as the derivative of Brownian motion or the Wiener process. To be able to approximate the SDEs solution the process needs to be discretizised. This is achieved by dividing the process into small grids between an interval [a: b]

$$a = t_0 < t_1 < ... < t_n = b$$

where a and b are the partition interval.

4.1.1 Euler-Maruyama Method

The numerical method for solving of SDEs is a stochastic Euler scheme (also called Euler-Maruyama scheme). The Euler-Maruyama scheme is the simple and natural guessing method for the solution of different types of stochastic differen-

tial equations. The Euler- Maruyama method (EM method) is to approximate a numerical solution of SDEs. It is the simple generalization of the Euler method for the ordinary differential equations to stochastic differential equations. Euler-Maruyama method is the numerical simulation of a stochastic differential equation and generate the stochastic process way approximation. This method is demonstrated by the following stochastic differential equation:

$$dy = \mu y dt + \sigma y dW_t \tag{4.1}$$

let

$$w_0 = y_0$$

the Euler-Maruyama Method is defined as:

$$W_{i+1} = W_i + \mu W_i(\Delta t_i) + \sigma W_i(\Delta W_t) \tag{4.2}$$

and ΔW_t is calculated as:

$$\Delta W_t = Z_i \sqrt{\Delta t_i} \tag{4.3}$$

and Z_i is a standard Gaussian random variable.

Using this Method to the Heston Model from equation (3.13) and (3.14) gives the following discrete relationship.

$$S_t = S_{t-1} + \mu S_{t-1} dt + \sqrt{V_{t-1}} S_{t-1} \sqrt{dt} Z_t^S$$
$$V_t = V_{t-1} + K(\Theta - V_{t-1}) dt + \sigma \sqrt{V_{t-1} dt} Z_t^v \tag{4.4}$$

where Z_t^S and Z_t^v is chosen from N $(0,1)$ and are independent and identically distributed.

4.2   Parameter Estimation

In order to approximate the option prices using the Heston model, it need to input parameters that are not observable from a market data. The change for each parameter will gives a big impact for the exactness of the model to fit the observed data. A variety of the method can be chosen. For example, one can observe that the real market data, and use the statistical tools to fit the data in the Heston model (Ait-Sahila, Kimmel, 2005). The method used in this thesis is called optimization, which means that we want to get the data from a real market first, and then estimate parameters by known data. To get Heston model parameter's minimize the error difference between the Heston model prices and real market price. A simple and direct approach is to reduce the mean sum of squared differences. For a put option that is calculated from a Heston model, the optimization problem can be defined as

$$MinS(\alpha) = Min(\alpha) \sum_{j=1}^{n} \left(p^h(k, T, \alpha)_j - p_j^m(k, T) \right)^2 \qquad (4.5)$$

where p_j^h and p_j^m are the jth put option price, respectively, calculated by a Heston model with time to the maturity T and collected from a real market, n is the number of the options price which are used to calibrate a model. The function S is an objective function of a optimization. When calibrating the Heston model from market data, we follow the equation (4.5). That is to choose the parameters that produces the best fit of the theoretical prices compared to the corresponding market price. Manually evaluating the optimal solutions produced by all optimizations yielded the following parameter set $S = (V_0, \theta, \kappa, \sigma, P)$ as the best fit are:

$V_0 = 0.069545829$

$\theta = 0.053565543$

$\kappa = 2.040210844$

$\sigma = 0.467514601$

$P = 0.50903932$

Note that the condition $2\kappa\theta > \sigma^2$ which ensures that Vt is strictly positive is fulfilled.

4.3 Numerical Results and Discussions

Stochastic volatility models are gradually important in the practical derivatives pricing applications. European put option on Google Inc.goog shares listed on the NASDAQ was used as the market data. The data is recorded on september 2016 to January 2017. To simulate stock price and volatility stochastic processes in Heston's model the Euler discretization can be used to approximate the paths of the stock price and variance processes on a discretize grid. Let $0 = t_o < t_1 < t_3 < ... < t_n = T$ be a partition of a time interval into n equal segments of length Δt i.e $\Delta t = \frac{T}{n}$. Then simulate Δt using Euler discretized method. Figure (4.1) and (4.2) show the simulation results for the both stock price and volatility are stochastic process using Euler discretization methods.

On the Figures (4.1) and (4.2) simulate the stock price and volatility stochastic processes in Heston's model using the Euler-Maruyama method, using the parameters $S_0 = 25, \mu = 0.2, T = 100days, K = 2, \Theta = 0.04, v_0 = 0.022, p = -0.5$, m=1000 and $\sigma = 0.1$.

To simulate the option price i.e. the European lookback put option price with floating strike from the Heston model, have to compute a integrals given by a Theorem (3.5.1). This is done by using the Mat-lab function quadl(@fun,a,b) which approximates the integral of the function (@fun, from a to b) using an adaptive Gauss Lobatto quadrature rule.
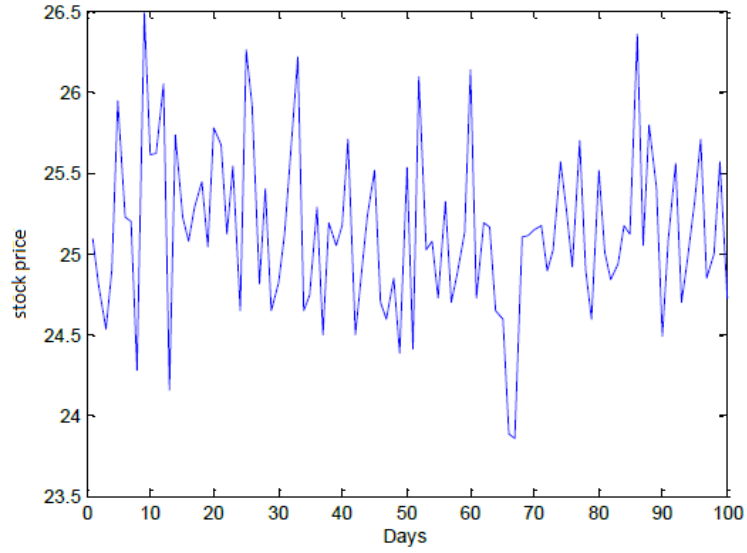
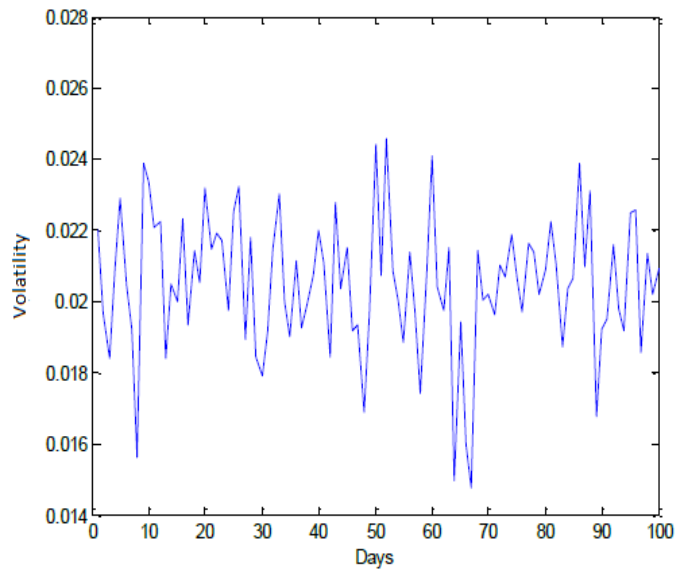Figure 4.1: Stock price dynamics in the Heston model

Figure 4.2: Volatility dynamics in the Heston model

Put option prices using both the Black-Scholes formula and Heston approximation is given in Figure (6.3). From the figure (6.3) put option prices with the Heston approximation value are higher than Black-Scholes prices. Also observed

that in the option values between our pricing formula under Heston model and Black-Scholes formula which is a red and green line under Heston approximation and Black-Scholes which is blue line. In all numerical simulations results the graph shows that put option price decreases in both the Black-Schole formula and the Heston approximation decline as the stock price increase and they are approaching zero when S is too increase. Also the Heston approximations have higher price than the Black-Scholes option as expected.

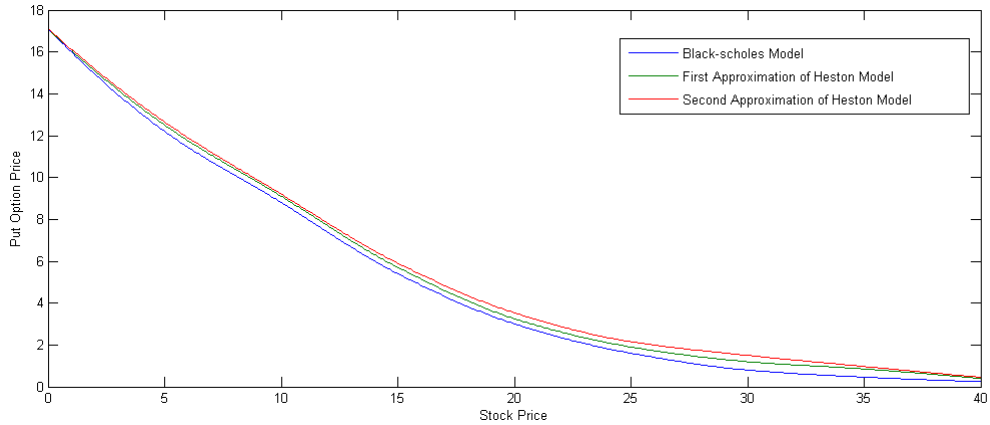From Figure (4.3),(4.4),(4.5) and(4.6) sample analysis of the two models shows

Figure 4.3: put option price in Heston approximation and Black-scholes when $x^* = 40, r = 0.05, K = 20, v_0 = 0.73, \Theta = 0.03, \sigma = 0.11, p = 0.5, k = 0.32$ and T=1 years

that the Heston model performs better than the Black-Scholes model. The results provided in the all Figure depict that the performance of the Heston stochastic model is superior to that of the Black-scholes model. Now fit approximations of the Heston model and with a real data as shown on the Figure (4.4) and (4.5) respectively. A graph showing the comparison of Heston approximation, Black-Scholes and real data with at times to maturity $T = \dfrac{3}{12}, T = \frac{2}{12}$ and $T = \frac{1}{12}$ years respectively. From Figure (4.6), it can be seen that, a smaller time to maturity produces the better fit for the approximate value of option price in the Heston model to the real data than the Black-scholes model.
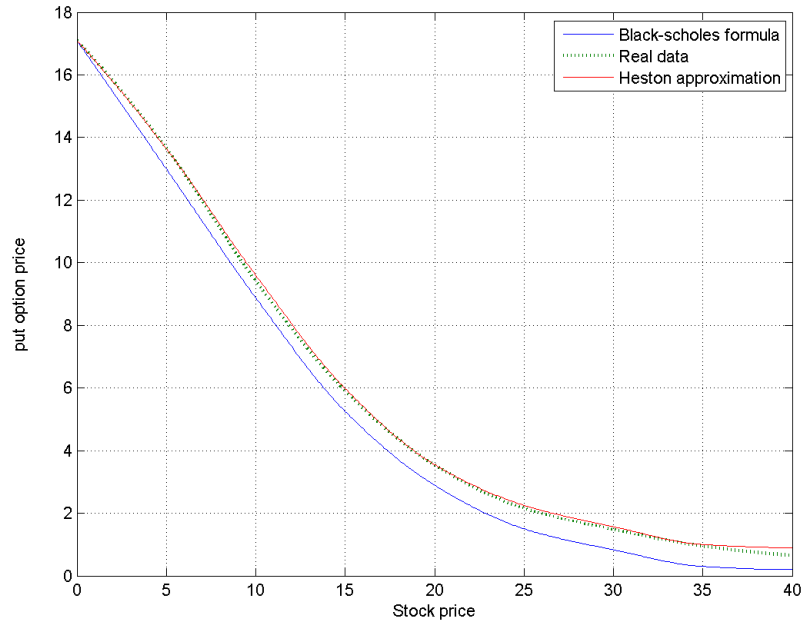
Figure 4.4: put option price in Heston approximation,Black-scholes and exact value

when $x^* = 40, r = 0.05, k = 2.040210844, v_0 = 0.069545829, \Theta = 0.053565543, \sigma = 0.467514601, p = 0.50903932$
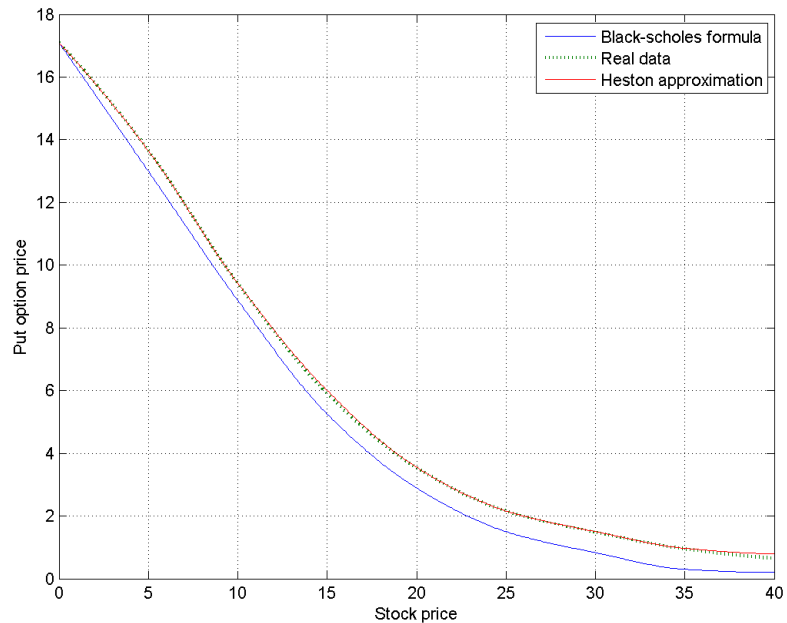

Figure 4.5: put option price in Heston approximation,Black-scholes and market value

when $x^* = 40, r = 0.05, k = 2.040210844, v_0 = 0.069545829, \Theta = 0.053565543, \sigma = 0.467514601, p = 0.50903932$
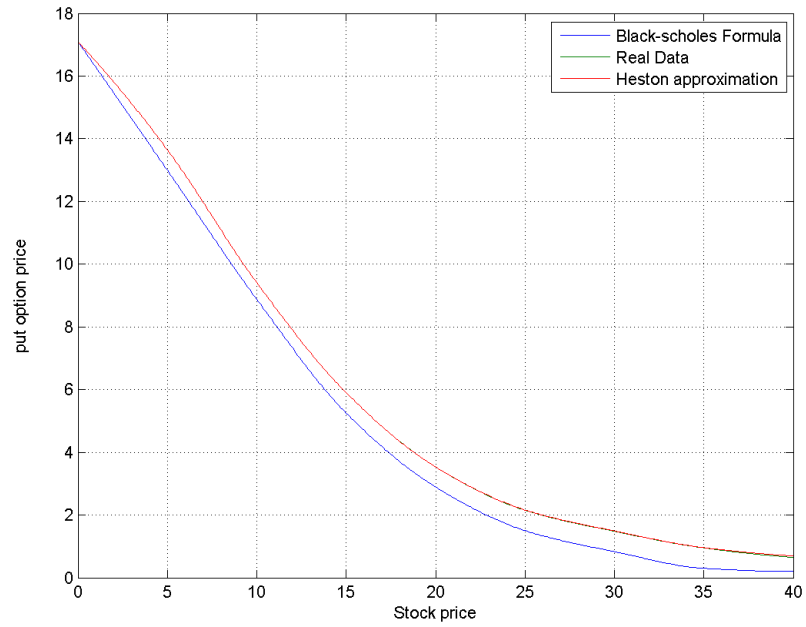
Figure 4.6: put option price in Heston approximation,exact value and real data when $x^* = 40, r = 0.05, k = 2.040210844, v_0 = 0.069545829, \Theta = 0.053565543, \sigma = 0.467514601, p = 0.50903932,$

Chapter 5

Conclusions and Recommendations

The study established a technique for the constructing analytic formula for lookback put option with the floating strike price by violating one of the assumptions in the Black-Scholes model, constant volatility. Stochastic volatility models tackle one of the most restrictive hypotheses of the Black-Scholes model framework, which assumes that volatility remains constant during the options life. However, by observing financial markets it becomes apparent that volatility may change dramatically in time.

The homotopy analysis method used in this study provides a analytic method for pricing options under Heston stochastic volatility model. The price is given by an infinite series whose value can be determined once the initial term is given well. The conclusion drawn from the simulations made in this thesis is the the Black-Scholes market model does not give a good description of actual market behavior. The Heston framework provide remedy for this, and allows for market shaped smiles as empirically shown in the thesis. The numerical simulation results shows that the Heston approximation works better than the Black-scholes formula. The stochastic volatility model in the literature like to be a Heston model, as it generates the exact value to the derivative prices. From the graph, it is observed that the numerical simulation results of Heston stochastic model has empirically perform more than the Black-scholes, and also derived the simple analytic approximations for a solution for the lookback put option.

Availability of closed-form valuation formulas is particularly important for the calibration process. In our tests, although the objective function is not necessarily convex, both local and global optimization methods provide reasonable results within a relatively timeframe. However, in cases where the objective function may exhibit several local minima, local optimization may underperform a global search. Once the model parameters have been calibrated to fit market prices, the Heston dynamics can be used to price other products that are not actively traded in the market.

The result shows that the Heston approximation works really well and only face problems when options with high time to maturity are be priced. As one could observe from the results above is that the Heston approximation loses its accuracy as the time to maturity increases, but Black-Scholes is also facing the same type of problem. However the Heston model is build on the assumption on non constant volatility showed an improvement of modeling stocks and receiving smile consistent option prices. The effect of stochastic volatility model resolving shortcoming of the Black-scholes model in its ability to generate volatility satisfying the market observations, and also providing a closed-form solution for the European options.

5.1 Future work

The suggested future research work on the calibration of a stochastic volatility parameterization, and fixed explicit parameter restrictions so as to construct a volatility surface which is suitable for the option price. Moreover, propose to investigate a calibration of the Heston model in a clean space, and test a local volatility model on the different sets of the market data. For further improvement in the further studies could been done by the presenting better variance decline

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techniques for a Monte Carlo simulation resulting in a better option price. As further matter, propose to investigate a calibration of the Heston model in a clean space, and test a local volatility model on the differenterent sets of the market data.

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Appendix

The simulation MATLAB code is shown below: APPENDIX 1: Mat-lab code for simulation of stock price function stochastic(S_0, V_0,mu,kappa,theta,sigma,delt,rho) times=10000

for i=1:times random1=randn(1,times); random2=randn(1,times);

S=zeros(1,times);

V=zeros(1,times);

$V = V_0$+kappa*(theta-V0)*T+sigma*sqrt(V0)*random1*sqrt(delT);

$S = S_0 + mu * S0 * delT + S0 * . * (rho * random1 + sqrt(1 - rho)*$

random2) $\sigma * sqrt(T)$;

figure; plot(S)

figure; plot(V)

end

Script 1: Heston parameter estimation using Matlabs lsqnonlin (Heston-local.m)

clear all

global data; global cost; global finalcost;

load data.txt x0 = [.5,.5,1,-0.5,1]; lb = [0, 0, 0, -1, 0]; ub = [1, 1, 5, 1, 20];

tic; x = lsqnonlin(@costf,x0,lb,ub); toc; Heston-sol= $[x(1), x(2), x(3), x(4), (x(5)+ x(3)^2)/(2 * x(2))]$

x

min = final cost

APPENDIX 1: Mat-lab code for simulation of stock price function stochastic(S_0, V_0,mu,kapp) times=10000

for i=1:times random1=randn(1,times); random2=randn(1,times);

S=zeros(1,times);

V=zeros(1,times);

$V = V_0$+kappa*(theta-V0)*T+sigma*sqrt(V0)*random1*sqrt(delT);

$S = S_0 + mu * S0 * delT + S0 * . * (rho * random1 + sqrt(1 - rho)*$

random2) $\sigma * sqrt(T)$;

figure; plot(S)

figure; plot(V)

end

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clear all

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tic; x = lsqnonlin(@costf,x0,lb,ub); toc; Heston-sol= $[x(1), x(2), x(3), x(4), (x(5)+$
$x(3)^2)/(2*x(2))]$

x

min = final cost

Appendix 2: Mat-lab code for Simulation of put option price

Script 2: Heston global calibration using ASA (Heston calibration global.m)

global data; global cost; global final cost; load data.txt

x0 = [.5,.5,1,-0.5,5];

lb = [0, 0, 0, -1, 0];

ub = [1, 1, 6, 1, 20];

Optimization: put function $costf_2.m$:

asamin('set', 'testin-cost-func', 0);

xtype = [-1;-1;-1;-1;-1]; tic;

$[f, x_0$ option, grad, hessian, state] = asamin ('minimize','$cost f_2'$, x_0',lb',ub', xtype)

toc;

  Heston sol= $[x(1), x(2), x(3), x(4), (x(5) + x(3)^2)/(2 * x(2))]$

x

min= finalcost

  Stochastic process stock simulation in matlab for stock X

Z in -rnorm(1479,0,1) Random normally distributed values, mean = 0, stdv = 1

u =- 0.3 Expected annual return (30sd =- 0.2 Expected annual standard

deviation (20s = 30 Starting price

price =-p(s) Price vector

a =- 2 See below

t =- 1years stock price to put on the x axis

  for(i in Z)

S = s + s*(u/1479 + sd/sqrt(1479)*i)

price[a] =- S

$S = S_0$

a = a + 1

plot(t,price,main="Time series stock

X",xlab="time",ylab="price", type="l",col="blue")

summary(price)

statistics-

c(sd(price),mean(price),(price[1480]-price[1])/price[1]*100)

names(statistics) - p("Volatility","Average price)","Return print(statistics)

plot(RealData, type = "l", col = "green", lwd = 2, xlab ="time", ylab = "Put

option Price")

lines(price,type = "l", col = "blue", lwd = 2, xlab ="stock price", ylab = "Put option price

Add Legend

legend("topright",legend=c("Simulated","Real Data"),

text.col=c("green","blue",red),pch=c(16,15),col=c("green","blue",red))

title (main = "black-scholes,Real and Simulation Data on Heston model")

MATLAB code for European lookback Put option: Black-Scholes Formula Black Scholes Price function

p=bsm-price(S,K,r,t,sigma)

$d1 = (lnx + (r + 0.5. * sigma.^2). * t)./(sigma. * sqrt(t)); d2 = d1 - sigma. * sqrt(t);$

p=normcdf(d1)*St-normcdf(d2)*exp(-r*t)*K;

load data2.txt

fori=1:length(data2)

$y(i) = bsmputprice(data2(i, 1), data2(i, 2), data2(i, 3), data2(i, 4), 0.1706);$

end

Matlab Function : lookback put option price in the Heston model (put-heston-pf.m)

function v = lookback put option heston pf(s0, v0, vbar, a, vvol, r, rho, t, k)

Inputs: s0: stock price

v0: initial volatility ($v0$ initial variance)

vbar: long-term variance mean experiences

a: variance mean-reversion speed

vvol: volatility of the variance process

r: risk-free rate

rho: correlation between the Weiner processes of the stock price and its variance

t: time to maturity

k: strike price

chfun-heston: Heston characteristic function

1st step: calculate

Inner integral 1

int1 = triplequad(fun,xmin,xmax,ymin,ymax,zmin,zmax)

$((2(2c-b))/(\sigma^3 2)exp(r+(r-1/\sigma^2)/\sigma^2-(r-\sigma^2/\sigma^2$

$)/(\sigma^2)-(2c-b)/(2\sigma^2))$dbdcd).*chfun-heston(s0, v0, vbar, a, vvol, r, rho, t,

).chfun-heston(s0, v0,

vbar,

a, vvol, r, rho, t,))); inner integral1

int1 = integral(@(w)int1(w,s0, v0, vbar, a, vvol, r,

rho, t, k),0,100); numerical integration

Inner integral 2:

int2 = triplequad(fun,xmin,xmax,ymin,ymax,zmin,zmax)

$((2(2c-b))/(\sigma^3 2)exp(r+(r-1/^2)/^2-(r-\sigma^2/\sigma^2)/(2\sigma^2)-(2c-b)/(2\sigma^2))$dbdcd).*chfun-heston(s0, v0, vbar, a, vvol, r, rho, t, w).);

int2 = integral(@(w)int2(w,s0, v0, vbar, a, vvol, r, rho, t, k),0,100);int2 = real(int2);

$V = k*exp(-r*T)*(normcdf(d2)-normcdf(d2)-(Sd./So).^{(-1+(2*r/\sigma^2))}.*$
(normcdf(d1)-normcdf(d1)))-(So.*(normcdf(d1)-normcdf(d2)-... $(Sd./So).^{(1+(2*}$
$r/\sigma^2)).*(normcdf(d1)-normcdf(d2)))$;

commands use to plot option price vs stock price plot(So,V,'–rs')

title('Option Price Vs Stock ')

xlabel('Stock Price')

ylabel('Option Price')

end

Heston solution for lookback option Matlab code.

Driver uses to put Heston function to compute lookback put option.

function [Vput-store]=HesExact-Driver

global kappa lamda theta v0 rho sigma r u1 u2 a b1 b2 global S0 K T x

para=[0.1237 0.0677 0.3920 1.1954 -0.6133];

Initial Values S0=20;

K=30; T=1;

Heston's Parameters

kappa=para(4);

lamda=0;

theta=para(2);

v0=para(1);

rho=para(5);

sigma=para(3);

r=0.02;

change of variable to avoid negative stock prices.

x=log(S0);

put option Heston function to compute put option. store=size(K,2);

for w=1:store

Vput = Heston-Exact(S0,K(w),T,σ,r,v_0);

Vput-store(w)=Vput option

(end);

plot(So,V,'–rs')

title('Option Price Vs Stock Price')

```
xlabel('Stock Price')

ylabel('Option Price')

end
```