ACTION OF *GL(2,q)* AND *GL(3,q)* ON NON ZERO VECTORS OVER *GF(q)*

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MASTER OF SCIENCE

(Pure Mathematics)

JOMO KENYATTA UNIVERSITY

OF

AGRICULTURE AND TECHNOLOGY

2022

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A Thesis Submitted in Partial Fulfilment of the Requirements for the Degree of Master of Science in Pure Mathematics of the Jomo Kenyatta University of Agriculture and Technology

2022

DECLARATION

This thesis is my original work and has not been presented for a degree in any other University.

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This thesis has been submitted for examination with our approval as University supervisors.

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DEDICATION

To my father John Ndichu, my mother The Late Jane Ndichu and my siblings; Sally, Mary, George and Pius.

ACKNOWLEDGEMENT

I would like to express my heartfelt gratitude to the Almighty God for His love, care, protection and good health throughout my research period and life in general.

I also extend a special recognition to my supervisors; Dr. Patrick Kimani and Dr. Peter Waweru for their first-class aid in pursuing this research. Their support not only motivated me towards undertaking this study but also made the realization of this work a success. May God bless you immensely.

I am also greatly indebted to my parents and siblings for their encouragement, support and prayers. You provided a shoulder to lean on and that made me proud.

Finally to my friends Simon Muoki, Paul Wachira, Frank Oketch and David Chepkonga; I owe you a lot for your encouragement, sacrifice and technical support in latex. To my colleague in struggle Felix Mutua, we always fought our battles together.

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SYMBOLS AND ABBREVIATIONS

$GF(q)/F_q$	Galois Field with q elements
F_q^*	Set of non zero elements in F_q
F_q^n	n-dimensional vector space over $GF(q)$
$\{0\}$	Zero vector
$GL(n,q)$	General linear group of dimension n over F_q
$PGL(2,q)$	Projective general linear group of dimension 2 over $GF(q)$
$PSL(2,q)$	Projective special linear group of dimension 2 over $GF(q)$
$TL(2,q)$	Lower triangular matrices of dimension 2 over $GF(q)$
$\|Y\|$	Number of elements in set Y
$Orb_G(x)$	An Orbit of G containing an element x
$Stab_G(x)$	The Stabilizer of x in G
Δ	An orbit of $Stab_G(x)$
$[1]^{[q-1]}$	$q-1$ suborbits each of length 1
G	A finite group
$R(G)$	The rank of a group G
Γ	The suborbital graph
S_n	Symmetric group of degree n
C_n	Cyclic group of order n
D_n	Dihedral group of order n
O	Suborbital of G
$\emptyset$	The empty set
$\cap$	Set intersection
$\cup$	Set union
α	Primitive element in F_q.

ABSTRACT

The action of the General Linear group has been studied by several researchers. Most of them concentrated in deriving the cycle index formula of $GL(n,q)$ leaving out combinatorial properties, invariants and structures of this group. This thesis determines transitivity, primitivity, ranks, subdegrees and suborbital graphs of the action of $GL(2,q)$ and $GL(3,q)$ on their non zero vectors over F_q. In this study, Orbit-Stabilizer theorem was used to determine transitivity and it was found that, both $GL(2,q)$ and $GL(3,q)$ act transitively on $F_q^2 \setminus \{0\}$ and $F_q^3 \setminus \{0\}$ respectively. The rank of $GL(2,q)$ acting on $F_q^2 \setminus \{0\}$ is q while the subdegrees are $[1]^{[q-1]}$ and q^2-q. In the action of $GL(3,q)$ on $F_q^3 \setminus \{0\}$, rank is q while the subdegrees are $[1]^{[q-1]}$ and q^3-q. The suborbital graphs of these two actions were constructed using Sims procedure. It was observed that all suborbital graphs corresponding to suborbit of length q^2-q in the action of $GL(2,q)$ on $F_q^2 \setminus \{0\}$ are: connected, undirected, regular and complete for $q=2$. The diameter is 1 where, $q=2$ and 2 for $q>2$. Also in the same action, the suborbital graphs corresponding to suborbits of length 1 are: regular, disconnected and have chromatic number as either 2 or 3. In the action of $GL(3,q)$ on $F_q^3 \setminus \{0\}$ the suborbital graphs corresponding to suborbits of q^3-q are connected, self-paired and complete for $q=2$. Also in the same action, Γ_i corresponding to Δ_i where $|\Delta_i| = 1$ is disconnected, regular and diameter is ∞. In conclusion, primitivity was determined using both the graphical method and the stabilizer as maximal subgroup approach. It was ascertained that both actions were primitive where $q=2$ and imprimitive where $q \geq 3$.

CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

This research expounds more on the group action of the General linear group in reference to a finite field. The main task is to determine the combinatorial properties, invariants and structures associated with the above mentioned action. Moreover, some of the basic terms and concepts relating to this study are explained in the following subsections.

1.1.1 Group Actions

Definition 1.1.1. For every prime power p^r, there exist a finite field usually denoted by $GF(p^r)$ or F_q, where $q = p^r$ unique up to isomorphism and contains p^r elements. This field is called Galois Field (Rose, 1978).

Definition 1.1.2. Suppose K is a field. The general linear group $GL(n,K)$ is a group of all $n \times n$ non-singular matrices over K under the usual matrix multiplication. Let $K = GF(q)$, then $GL(n,K)$ can be written as $GL(n,q)$ and $|GL(n,q)| = (q^n - 1)(q^n - q)\dots(q^n - q^{n-1})$.

Definition 1.1.3. Let m be an $n \times n$ matrix over a field F_q and $v \in F_q^n$ then $mv \in F_q^n$ $\forall v \in F_q^n$. If $v \neq \{0\}$ and m is an invertible matrix then $mv \neq \{0\}$.
Let $G = GL(n,q)$. Then G acts on $F_q^n \setminus \{0\}$, where $F_q = GF(q)$.
This action is defined by $G \times \left(F_q^n \setminus \{0\}\right) \to \left(F_q^n \setminus \{0\}\right) \qquad (m,v) \to mv$
$\forall m \in G$ and $v \in F_q^n \setminus \{0\}$ satisfying the following conditions:
$a)$ $Iv = v \ \forall v \in F_q^n \setminus \{0\}$, where I is the identity matrix.
$b)$ $(m_1 m_2)v = m_1(m_2 v) \ \forall v \in F_q^n \setminus \{0\}, \forall m_1, m_2 \in G$.

Definition 1.1.4. Suppose G acts on Y and let $y \in Y$. The collection of elements $g \in G$ which leave y unchanged under the action of G is called a stabilizer of y in G denoted as $Stab_G(y)$, that is $Stab_G(y) = \{g \in G \mid gy = y\}$ (Muriuki, 2017).

Definition 1.1.5. Let G act on a set Y and let $y \in Y$, then the collection of elements that y is mapped to under the action of G is called the orbit i.e. $Orb_G(y) = \{gy \mid g \in G\}$.

Theorem 1.1.6. *(Orbit-Stabilizer Theorem)*
Let G act on Y and $y \in Y$. Then, $\mid Orb_G(y) \mid = \dfrac{\mid G \mid}{\mid Stab_G(y) \mid}$ *(Rose, 1978).*

Definition 1.1.7. A group action that possesses a single orbit is said to be transitive.

Lemma 1.1.8. *(Cauchy-Frobenius Lemma)*
Suppose G acts on Y. Then, the number of orbits on Y is given by, $\frac{1}{\mid G \mid} \sum_{g \in G} \mid Fix(g) \mid$ *(Rotman, 2010).*

Definition 1.1.9. Suppose G acts transitively on Y and let $y \in Y$, then the $Stab_G(y)$-orbits on Y are referred to as suborbits. Rank is the total number of these $Stab_G(y)$-orbits and the cardinality of each suborbit is called subdegree (Kimani, 2019).

Theorem 1.1.10. *Suppose G acts transitively on the set Y and $y \in Y$. The number of self-paired suborbits is* $\frac{1}{\mid G \mid} \sum_{g \in G} \mid Fix(g^2) \mid$ *(Cameron, 1974).*

Theorem 1.1.11. *Let G act transitively on Y and $x \in Y$. Then G is primitive if and only if $Stab_G(x)$ is a maximal subgroup of G or equivalently G is imprimitive if and only if $Stab_G(x)$ is not a maximal subgroup of G (Mwai, 2016).*

1.1.2 Graph Theory

Definition 1.1.12. A graph $\Gamma(V, E)$ is a diagram comprising of a set V of vertices and a set E of edges.

Definition 1.1.13. A diagraph $\Gamma(V, E)$ is a diagram comprising of a set V of vertices and a set E of directed edges. It is also known as a directed graph.

Definition 1.1.14. In a graph $\Gamma(V, E)$, $(v, e) \in E$ is said to be a loop if $v = e$.

Definition 1.1.15. A graph $\Gamma(V, E)$ with multiple edges but no loops is referred to as a multigraph.

Definition 1.1.16. A simple graph $\Gamma(V, E)$ is a non-empty set V of vertices and a (possibly empty) set E of edges.

Definition 1.1.17. The minimum number of vertices of a graph Γ that can be colored such that no two adjacent vertices have same color is called the chromatic number.

Definition 1.1.18. A graph $\Gamma(V, E)$ is said to be bipartite if V can be partitioned into two subsets V_1 and V_2 such that edges join two vertices from different subsets and no edge joins vertices in the same subset. Bipartite graphs have chromatic number 2 since all its circuits are of even length.

Definition 1.1.19. Two graphs Γ_1 and Γ_2 are isomorphic, if there exists a one-to-one correspondence $\beta : V(\Gamma_1) \to V(\Gamma_2)$ such that β preserves the adjacency; that is $(u, v) \in E(\Gamma_1)$ if and only if $(\beta u, \beta v) \in \Gamma_2$. Since $|V(\Gamma_1| = |V(\Gamma_2|$, any one-to-one correspondence is equivalent to a relabeling of vertices.

Definition 1.1.20. A finite sequence $v_0 \to v_1, v_1 \to v_2, v_2 \to v_3, \ldots, v_{m-1} \to v_m$ of edges is known as a walk. The length of this walk is m. Moreover, if a walk has distinct nodes it is known as a path. A closed path is known as a cycle. In addition, the length of the smallest cycle is referred to as girth.

Definition 1.1.21. A graph Γ is connected if for each pair of vertices v_o, v_1 there exist a path from v_0 to v_1 otherwise it is disconnected.

Definition 1.1.22. The distance $d(u,v)$ from u to v is the length of the shortest path from u to v in Γ. If there is no path from u to v then $d(u,v) = \infty$. The diameter of the graph is, $\text{Max}\{d(u,v) : u,v \in V\}$.

Definition 1.1.23. Valency or degree is the number of nodes adjacent to vertex v in a graph Γ. A graph Γ is regular of degree k if the valency of each vertex is k.

1.1.3 Suborbitals and Suborbital Graphs

Definition 1.1.24. Let G act transitively on Y. Then, G also acts on $Y \times Y$ by $g(x,y) = (gx, gy)$, where $g \in G$ and $x, y \in Y$.
These G-orbits resulting from the above action are referred to as suborbitals. The suborbital of G containing (x,y) is denoted as $O(x,y)$. For any $x \in Y$,
$\Delta(x) = \{y \in Y | (x,y) \in O(x,y)\}$ and $O(x,y) = \{(gx, gy) | g \in G, y \in \Delta(x)\}$ and therefore there is a one to one correspondence between $O(x,y)$ and $\Delta(x)$ (Kimani, 2019).

Definition 1.1.25. Let G act on X transitively and $x, y \in X$. The suborbital $O(x,y)$ is either equal to or disjoint from $O(y,x)$. Now, in the latter case $\Gamma(x,y)$ is just $\Gamma(y,x)$ with arrows reversed. In this case $\Gamma(x,y)$ and $\Gamma(y,x)$ are *paired suborbital graphs*. In the case where $\Gamma(x,y) = \Gamma(y,x)$, the graph consists of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge to yield an undirected suborbital graph which is called *self-paired* (Güler *et al.*, 2015).

Proposition 1.1.26. *Let G act on X transitively. The components of any suborbital graph corresponding to a non-trivial suborbit of length 1 are either trees or cycles (Kimani, 2019).*

Theorem 1.1.27. *Let G acts transitively on Y. Then, G is said to be primitive if and only if each suborbital graph $\Gamma_j \, (j = 1, 2, \ldots, k-1)$ is connected (Sims, 1967).*

Theorem 1.1.28. *Suppose G acts transitively on Y and Γ is a suborbital graph corresponding to suborbit $\Delta(x)$. Then Γ is said to be undirected if $\Delta(x)$ is self-paired and directed if $\Delta(x)$ is not self-paired (Sims, 1967).*

Theorem 1.1.29. *Let Γ be a suborbital graph of a transitive action. Then all disconnected components of Γ are isomorphic (Jefferson* et al.*, 2019).*

Theorem 1.1.30. *Let G act on Y and let Γ_i be any undirected suborbital graph corresponding to $\Delta_i(x)$, where $x \in Y$, then the number $\lambda(\Gamma_i)$ of triangles in Γ_i is given by,*

$$\lambda(\Gamma_i) = \frac{|Y||\Delta_i(x) \cap \Delta_i(y)||\Delta_i(x)|}{6},$$

where $y \in \Delta_i(x)$ *(Kimani, 2019).*

1.2 Statement of the Problem

In linear groups, alot has been done on $PGL(2,q)$ and $PSL(2,q)$. This includes determination of transitivity, primitivity, ranks, subdegrees, derivation of cycle index formulas and construction of suborbital graphs. The cycle index formula of $GL(n,q)$ has also been derived. Again, the action of $GL(n,q)$ on the set F_q^n was found to be intransitive. Thus, the study did not determine ranks, subdegrees and suborbital graphs since all $g \in GL(n,q)$ fixes $\{0\} \in F_q^n$ and therefore $\{0\}$ exists as a trivial orbit of length 1 and the other elements that is $F_q^n \setminus \{0\}$ also exist as an orbit. Now, this study seeks to examine the action of $GL(n,q)$ on $F_q^n \setminus \{0\}$, where $n = 2,3$ by determining the combinatorial properties and invariants. In addition, structures corresponding to the above action will be constructed.

1.3 Justification

This research is valuable owing to the reason that it has many applications in the real world. It will provide important information in graph theory, computer science and also in physics.

Invertible matrices are used in cryptography. In this process, an invertible matrix is chosen in order for the sender to encode a message which is to be send through a transmission channel. The recipient then uses the inverse of the encoding matrix to decode the recieved data in order to get the actual message. In the Hill Cipher method, this concept is applied and it is very significant in data security .

In Graph Theory, suborbits are used in computing suborbitals which aid in the construction of suborbital graphs. These graphs are very useful in the determination of graphical properties such as orientation and connectedness of suborbital graphs.

Graphs are used in determining the shortest distance on the earth's surface. This is

very fundamental in the planning of infrastructural developments like roads, railway and electrification projects of an area. This is very efficient in cost reduction.
In the field of biology, graphs can be used in acquiring important information concerning wild animals by tracking and monitoring the breeding zones and spread of diseases of species such as wildbeasts in their habitats.
In designing Computer programs such as GAP which are used in Algebra. This software is used to study group structures and combinatorial properties. This research will provide additional information that can be beneficial in improving this software.

1.4 Objectives

1.4.1 General Objective

To determine transitivity, primitivity, ranks, subdegrees and suborbital graphs of the action of $GL(2,q)$ and $GL(3,q)$ on non zero vectors over $GF(q)$.

1.4.2 Specific Objectives

1. To determine transitivity and primitivity of the action of $GL(2,q)$ and $GL(3,q)$ on non zero vectors over $GF(q)$.
2. To calculate ranks and subdegrees of the action of $GL(2,q)$ and $GL(3,q)$ on non zero vectors over $GF(q)$.
3. To construct suborbital graphs of the action of $GL(2,q)$ and $GL(3,q)$ on non zero vectors over $GF(q)$.

CHAPTER TWO
LITERATURE REVIEW

This chapter provides a review of previous studies that are related to this work in regard to ranks, subdegrees and suborbital graphs.

Faradžev *et al.* (1990) computed suborbits of primitive representation of $G = PSL(2,q)$ acting on the cosets of a maximal subgroup H. It was proved that $R(G) \geq \frac{|G|}{|H|^2}$ and if $q > 100$, then $R(G) > 5$.
Marusic & Scapellato (1994) studied $G = PSL(2,q^2)$ acting on right cosets of $PGL(2,q)$ where q is odd and discovered that if $q \equiv 1$ (mod 4), then the subdegrees are $\frac{q(q-1)}{2}$, q^2-1, $[q^2-q]^{[\frac{q-5}{4}]}$ and $[q^2+q]^{[\frac{q-1}{4}]}$. It was also observed that, if $q \equiv$ -1 (mod 4) the subdegrees are $\frac{q(q+1)}{2}$, q^2-1, $[q^2-q]^{[\frac{q-3}{4}]}$, $[q^2+q]^{[\frac{q-3}{4}]}$. Again, it was discovered that all suborbits are self-paired.
Fulman (1997) developed the cycle index formula for the finite general linear group as $Z(GL(n,q)) = \dfrac{1}{|GL(n,q)|} \sum\limits_{\alpha \in GL(n,q)} \prod\limits_{\phi \neq z} x_{\phi,\lambda_\phi(\alpha)}$, where $x_{\phi,\lambda}$ are variables corresponding to pairs of polynomials and partitions.
Kamuti (2006) determined the ranks and subdegrees of $G = PGL(2,q)$ acting on the cosets of its maximal subgroups. It was proved that the rank of $G = PGL(2,q)$ acting on the cosets of $PGL(2,e)$, where $q = e^h$ and h is odd is $R(G) = \dfrac{e^{3h-2} - e^{h+3} - e^{h+1} - 3e^h - e^2 + 2e + 1}{(e^2-1)^2}$.
Lu (2007) constructed infinitely many primitive half-transitive graphs with automorphism groups S_n for n prime. It was discovered that, there exists at least 1 primitive half transitive graph of valency $2p$ for p prime which is not less than 7, where $p \neq 13$. In the study, construction of 2-arc-regular carley graphs was carried out.
Smith (2009) studied suborbital graphs of $PSL(2,p)$ acting on the cosets of its maximal subgroup D_{p-1}, where p is prime between 2 and 200. It was proved that the largest diameter is 7 when $p = 17$.
Kader *et al.* (2010) calculated the number of sides of circuit in a suborbital graph for the normalizer of $\wedge_0(n)$ in $PSL(2,\mathbb{R})$, where n takes the form of $2p^2$, p is a prime and $p \cong 1($ mod 4$)$. The research went further to give a theoretical result as $2u^2 \pm 2u + 1$ are of the form $p \cong 1(mod 4)$.
Güler *et al.* (2011) studied the relationship between elliptic elements and circuits in a graph for the normalizer $\wedge_0(N)$ in $PSL(2,\mathbb{R})$. It was determined that if $N = 2^\alpha 3^\beta p^2$ where $F(\infty, \frac{u}{p^2})$ is the subgraph of $G(\infty, \frac{u}{p^2})$,then $F(\infty, \frac{u}{p^2})$ has a circuit.

Beşenk *et al.* (2012) constructed suborbital graphs for the action of the normalizer of $\wedge_0(N)$ in $PSL(2,\mathbb{R})$ where N is of the form 2^8p^2, $p > 3$ and p is a prime. In the research, the conditions to be a forest for normalizer in the suborbital graph $F(\infty, \frac{u}{2^8p^2})$ were stated.
Kamuti *et al.* (2012) studied suborbital graph properties on the action of stabilizer of ∞ in the modular group. It was discovered that the graph $\Gamma(0,a)$ is paired with $\Gamma(0,-a)$, $\Gamma(0,a)$ has $|a|$ connected components and $\Gamma(0,a)$ connected if and only if $|a| = 1$.
Rimberia (2012) studied the action of $G = S_n$ on the set $Y^{[k]}$ and proved that if $n \geq 2k$, $R(G)$ is given by $k! + k^2(k-1)! + \dfrac{k^2(k-1)^2(k-2)!}{2!2!} + \dfrac{k^2(k-1)^2(k-2)^2(k-3)!}{3!3!} + \cdots + \dfrac{k^2(k-1)^2}{2!} + k^2 + 1$.
Amarra *et al.* (2013) studied symmetric graphs of diameter 2. The research categorized all connected graphs resulting from groups that are not subgroups of 1-dimensional affine groups and those with diameter greater than 2 were identified.
Magero (2015) computed the suborbits of $G = PSL(2,q)$ acting on cosets of P_q. It was shown that $R(G) = 2(q-1)$ if $2|q$ and $R(G) = (q-1)$ if $2 \nmid q$.
Dolfi *et al.* (2016) proved that a set of pairwise coprime non-trivial subdegree has maximal size of at most 2.
Mwai (2016) studied C_n acting on the vertices of a regular n-gon and determined that the action is transitive. In addition C_n is imprimitive for n not prime. Moreover, the rank of C_n was found to be n.
Rotich (2016) constructed suborbital graphs of $PGL(2,q)$ acting on the cosets of C_{q-1}. The connected components corresponding to non-trivial suborbit of cardinality 1 was proved to be equal to $\frac{q(q+1)}{2}$.
Goren (2017) studied the action of $GL(n,q)$ on the set F_q^n. It was proved that the action has two orbits and hence intransitive.
Muriuki (2017) studied the action of $S_n \times S_n \times S_n$ on the set $X \times Y \times Z$ and proved that the action is transitive. In addition, if $n \geq 2$ the action has 8 suborbits of lengths 1, $[n-1]^{[3]}$, $[(n-1)^2]^{[3]}$ and $(n-1)^3$.
Kimani *et al.* (2019) computed the suborbits of $PGL(2,q)$ acting on cosets of $PGL(2,e)$, where q is an even power of e. It was found that the rank is $\dfrac{e^5q - e^5 + e^4q - e^3q + e^3 - 4e^2q + 2e^2 + q^3}{e^2(e^2-1)^2}$.
From the above, it is clear that little has been done on transitivity, primitivity, ranks and subdegrees associated with the actions of $GL(2,q)$ and $GL(3,q)$ on $F_q^2 \setminus \{0\}$ and $F_q^3 \setminus \{0\}$ respectively. In addition, the suborbital graphs associated with these actions remain undetermined. This research was formulated to address this knowledge gap.

CHAPTER THREE
TRANSITIVITY, PRIMITIVITY, RANKS AND SUBDEGREES OF *GL(2,q)* AND *GL(3,q)* ON THEIR NON-ZERO VECTORS

3.1 Introduction

In this chapter, transitivity, primitivity, ranks and subdegrees of the action of $GL(2,q)$ and $GL(3,q)$ on their non-zero vectors are determined. In Section 3.2, transitivity of $GL(2,q)$ on $F_q^2 \setminus \{0\}$ is determined followed by its ranks and subdegrees in Section 3.3. Transitivity of $GL(3,q)$ on $F_q^3 \setminus \{0\}$ is determined in Section 3.5 whereas their ranks and subdegrees are calculated in Section 3.6. In addition, primitivity of both actions is determined in Section 3.4 and 3.7.

3.2 Transitivity of *GL(2,q)* on $F_q^2 \setminus \{0\}$

Proposition 3.2.1. The group G = *GL(2,2)* acts transitively on $Y = F_2^2 \setminus \{0\}$.

Proof. The field, $F_2 = \{0,1\}$ and the set $Y = \left\{ \binom{0}{1}, \binom{1}{0}, \binom{1}{1} \right\}$ and $|G| = (2^2-1)(2^2-2) = 6$ from Definition 1.1.2. Now,

$$\begin{aligned} Stab_G\left(\begin{pmatrix}0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} : \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}, ad-bc \neq 0\right\} \\ &= \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} : \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}, ad-bc \neq 0\right\} \\ &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix} : a \neq 0\right\}, \end{aligned} \tag{3.1}$$

where $a,b,c,d \in F_2$. From Equation 3.1, $a \in F_2$ is non-zero and therefore can be chosen in only 1 way while c can be chosen in 2 ways. Hence,
$|Stab_G\left(\binom{0}{1}\right)| = 2 \times 1 = 2$. Therefore by Theorem 1.1.6,
$|Orb_G\left(\binom{0}{1}\right)| = \dfrac{|G|}{|Stab_G\left(\binom{0}{1}\right)|} = \dfrac{6}{2} = 3 = |Y|$. □

Proposition 3.2.2. The group G = *GL(2,3)* acts transitively on $Y = F_3^2 \setminus \{0\}$, where α is a primitive element in F_3.

Proof. The field $F_3 = \{0, \alpha = 2, \alpha^2 = 1\}$ and the set $Y = \left\{ \binom{0}{1}, \binom{0}{2}, \binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{2}{0}, \binom{2}{1}, \binom{2}{2} \right\}$. By Definition 1.1.2, $|G| = (3^2-1)(3^2-3) = 48$ and therefore,

$$Stab_G\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} : \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}, ad-bc \neq 0\right\}$$

$$
\begin{aligned}
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ad-bc \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0 \right\}, \qquad (3.2)
\end{aligned}
$$

where $a,b,c,d \in F_3$.
From Equation 3.2, a can be chosen in 2 ways since $a \neq 0$ and c can be chosen in 3 ways. Therefore, $|Stab_G(\binom{0}{1})| = 3 \times 2 = 6$. By Theorem 1.1.6, it follows that, $\left|Orb_G\left(\binom{0}{1}\right)\right| = \dfrac{|G|}{\left|Stab_G\left(\binom{0}{1}\right)\right|} = \dfrac{48}{6} = 8 = |Y|$.

□

Proposition 3.2.3. The group G = *GL(2,5)* acts transitively on $Y = F_5^2 \setminus \{0\}$ where α is a primitive element in F_5.

Proof. The field $F_5 = \{0, \alpha = 2, \alpha^2 = 4, \alpha^3 = 3, \alpha^4 = 1\}$ and by Definition 1.1.2, $|G| = (5^2-1)(5^2-5) = 480$. Therefore,

$$
\begin{aligned}
Stab_G\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ad-bc \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ad-bc \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0 \right\}, \qquad (3.3)
\end{aligned}
$$

where $a,b,c,d \in F_5$. From Equation 3.3, $a \neq 0$ can be chosen in 4 ways whereas element c can be chosen in 5 ways. Therefore, $|Stab_G\left(\binom{0}{1}\right)| = 4 \times 5 = 20$. By Theorem 1.1.6, it follows that,
$|Orb_G\left(\binom{0}{1}\right)| = \dfrac{|G|}{|Stab_G\left(\binom{0}{1}\right)|} = \dfrac{480}{20} = 24 = |Y|$. □

Lemma 3.2.4. *Let G=GL(2,q) act on* $Y = F_q^2 \setminus \{0\}$. *Then,* $Stab_G\left(\binom{0}{1}\right) = \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0, c \in F_q \right\}$ *and* $|Stab_G\left(\binom{0}{1}\right)| = q^2 - q$.

Proof. Let $a,b,c,d \in F_q$, then

$$
\begin{aligned}
Stab_G\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ad-bc \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ad-bc \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0 \right\}. \qquad (3.4)
\end{aligned}
$$

In Equation 3.4 above, a can be chosen in $q-1$ ways since $a \neq 0$ whereas c can be chosen in q ways. Thus, $|Stab_G\left(\binom{0}{1}\right)| = q^2 - q$. □

Theorem 3.2.5. *The group G=GL(2,q) acts transitively on* $Y = F_q^2 \setminus \{0\}$.

Proof. Suppose G acts on Y and $\binom{0}{1} \in Y$. From Definition 1.1.2, $|G| = (q^2-1)(q^2-q)$ and by Lemma 3.2.4, $|Stab_G(\binom{0}{1})| = q^2 - q$. By Theorem 1.1.6, it follows that,

$$\begin{aligned}\left|Orb_G\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\right| &= \frac{(q^2-1)(q^2-q)}{q^2-q} \\ &= q^2-1 \\ &= |Y|.\end{aligned}$$

□

3.3 Ranks and Subdegrees of *GL(2,q)* on $F_q^2 \setminus \{0\}$

Proposition 3.3.1. Suppose $G = GL(2,2)$ acts on $Y = F_2^2 \setminus \{0\}$. Then, rank is 2 and the subdegrees are 1 and 2.

Proof. Let $H = Stab_G\left(\binom{0}{1}\right)$. By Lemma 3.2.4, $H = \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix} : a \neq 0, c \in F_2\right\} = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\}$. Therefore,

$$\begin{aligned}Orb_H\left(\begin{pmatrix}0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} : a \neq 0, c \in F_2\right\} \\ &= \left\{\begin{pmatrix}0\\1\end{pmatrix}\right\} \qquad (3.5)\end{aligned}$$

By Equation 3.5, $\Delta_0 = Orb_H\left(\binom{0}{1}\right) = \left\{\binom{0}{1}\right\}$ and therefore $|\Delta_0| = 1$.
Now, the suborbit Δ_1 which contains element $\binom{1}{0}$ is computed below as follows:

$$\begin{aligned}Orb_H\left(\begin{pmatrix}1\\0\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} : a \neq 0, c \in F_2\right\} \\ &= \left\{\begin{pmatrix}a\\c\end{pmatrix} : a \neq 0, c \in F_2\right\} \qquad (3.6) \\ &= \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} \qquad (3.7)\end{aligned}$$

By Equation 3.6, $a \in F_2$ can be chosen in only 1 way since $a \neq 0$ while $c \in F_2$ can be chosen in 2 ways and thus, $|Orb_H\left(\binom{1}{0}\right)| = 2$ as seen in Equation 3.7. Therefore, the suborbits are $\left\{\binom{0}{1}\right\}$ and $\left\{\binom{1}{0}, \binom{1}{1}\right\}$ and hence, rank is 2 and the subdegrees are 1 and 2. □

Proposition 3.3.2. Let *G=GL(2,3)* act on $Y = F_3^2 \setminus \{0\}$ and α be the primitive element in F_3. Then, the rank is 3 and the subdegrees are 1,1 and 6.

Proof. Let $H = Stab_G(\binom{0}{1})$. By Lemma 3.2.4, $H = \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0, c \in F_3 \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \right\}$. Thus,

$$\begin{aligned} Orb_H\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} : a \neq 0, c \in F_3 \right\} \\ &= \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \end{aligned} \tag{3.8}$$

From Equation 3.8, $\Delta_0 = Orb_H\left(\binom{0}{1}\right) = \left\{\binom{0}{1}\right\}$ and thus $|\Delta_0| = 1$.
Next,

$$\begin{aligned} Orb_H\left(\begin{pmatrix} 0 \\ \alpha \end{pmatrix}\right) = Orb_H\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} : a \neq 0, c \in F_3 \right\} \\ &= \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \end{aligned} \tag{3.9}$$

By Equation 3.9, $\Delta_1 = Orb_H\left(\binom{0}{2}\right) = \left\{\binom{0}{2}\right\}$ and thus $|\Delta_1| = 1$.
The suborbit Δ_2 which contains vector $\binom{1}{0}$ is determined as follows:

$$\begin{aligned} Orb_H\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} : a \neq 0, c \in F_3 \right\} \\ &= \left\{ \begin{pmatrix} a \\ c \end{pmatrix} : a \neq 0, c \in F_3 \right\} \qquad (3.10) \\ &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \qquad (3.11) \end{aligned}$$

By Equation 3.10, $a \in F_3$ can be chosen in 2 ways since $a \neq 0$ while $c \in F_3$ can be chosen in 3 ways; therefore $|Orb_H\left(\binom{1}{0}\right)| = 6$ as seen in Equation 3.11. Therefore, the suborbits are $\left\{\binom{0}{1}\right\}$, $\left\{\binom{0}{2}\right\}$ and $\left\{\binom{1}{0}, \binom{1}{1}, \binom{1}{2}, \binom{2}{0}, \binom{2}{1}, \binom{2}{2}\right\}$ and hence rank is 3 and the subdegrees are 1, 1, 6. ☐

Proposition 3.3.3. Suppose *G=GL(2,5)* acts on $Y = F_5^2 \setminus \{0\}$ and α is the primitive element in F_5. Then, rank is 5 and the subdegrees are 1,1,1,1 and 20.

Proof. Let $H = Stab_G\left(\binom{0}{1}\right)$. By Lemma 3.2.4, $H = \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0, c \in F_5 \right\} =$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix},$$

$\begin{pmatrix}4 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\ 3 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\ 4 & 1\end{pmatrix}\Big\}$

Thus,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} : a \neq 0, c \in F_5\right\} \\ &= \left\{\begin{pmatrix}0\\1\end{pmatrix}\right\} \end{aligned} \tag{3.12}$$

By Equation 3.12, $\Delta_0 = Orb_H\left(\binom{0}{1}\right) = \left\{\binom{0}{1}\right\}$ and hence $|\Delta_0| = 1$.
Also,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\ \alpha\end{pmatrix}\right) = Orb_H\left(\begin{pmatrix}0\\2\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} : a \neq 0, c \in F_5\right\} \\ &= \left\{\begin{pmatrix}0\\2\end{pmatrix}\right\} \end{aligned} \tag{3.13}$$

From Equation 3.13, $\Delta_1 = Orb_H\left(\binom{0}{2}\right) = \left\{\binom{0}{2}\right\}$ and $|\Delta_1| = 1$.
Also,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\ \alpha^2\end{pmatrix}\right) = Orb_H\left(\begin{pmatrix}0\\4\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix} : a \neq 0, c \in F_5\right\} \\ &= \left\{\begin{pmatrix}0\\4\end{pmatrix}\right\} \end{aligned} \tag{3.14}$$

By Equation 3.14, $\Delta_2 = Orb_H\left(\binom{0}{4}\right) = \left\{\binom{0}{4}\right\}$ and $|\Delta_2| = 1$.
Also,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\ \alpha^3\end{pmatrix}\right) = Orb_H\left(\begin{pmatrix}0\\3\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix} : a \neq 0, c \in F_5\right\} \\ &= \left\{\begin{pmatrix}0\\3\end{pmatrix}\right\} \end{aligned} \tag{3.15}$$

From Equation 3.15, $\Delta_3 = Orb_H\left(\binom{0}{3}\right) = \left\{\binom{0}{3}\right\}$ and therefore $|\Delta_3| = 1$.
Now, the suborbit Δ_4 which contains element $\binom{1}{0}$ is computed and determined as follows:

$$\begin{aligned} Orb_H\left(\begin{pmatrix}1\\0\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & 0\\ c & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} : a \neq 0, c \in F_5\right\} \\ &= \left\{\begin{pmatrix}a\\c\end{pmatrix} : a \neq 0, c \in F_5\right\} \end{aligned} \tag{3.16}$$

$$= \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix} \right\} \quad (3.17)$$

By Equation 3.16, $a \in F_5$ can be chosen in 4 ways since $a \neq 0$ while $c \in F_5$ can be chosen in 5 ways hence $|Orb_H\left(\binom{1}{0}\right)| = 20$ as seen in Equation 3.17. Thus, there are 5 suborbits, whereby 4 of them are suborbits of length 1 and the remaining suborbit is of length 20. □

Theorem 3.3.4. *Let $G = GL(2,q)$ act on $Y = F_q^2 \setminus \{0\}$. Then rank is q and the subdegrees are $[1]^{[q-1]}$ and $q^2 - q$.*

Proof. The elements of F_q can be expressed in terms of a primitive element α as $0, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{q-1} = 1$.
By lemma 3.2.4, $H = Stab_G\left(\binom{0}{1}\right) = \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0, c \in F_q \right\}$ and $|H| = q^2 - q$. Now,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\x\end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix}0\\x\end{pmatrix} : a \neq 0, x \neq 0, c \in F_q \right\} \\ &= \left\{ \begin{pmatrix}0\\x\end{pmatrix} : x \neq 0 \in F_q \right\} \end{aligned} \quad (3.18)$$

From Equation 3.18, it is observed that all vectors of the form $\binom{0}{x}$, where $x \in F_q^*$ exist as suborbits of length 1. Hence, there are $q-1$ suborbits of length 1, i.e. $\Delta_0, \Delta_1, ..., \Delta_{q-2}$. To compute Δ_{q-1}, an element such as $\binom{1}{0} \in Y$ is chosen since its not contained in any of the above $q-1$ suborbits. It follows that,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}1\\0\end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} : a \neq 0, c \in F_q \right\} \\ &= \left\{ \begin{pmatrix}a\\c\end{pmatrix} : a \neq 0, c \in F_q \right\} \end{aligned} \quad (3.19)$$

Now, from Equation 3.19, $a \in F_q$ can be chosen in $q-1$ ways since $a \neq 0$ while $c \in F_q$ can be chosen in q ways; thus $|Orb_H\left(\binom{1}{0}\right)| = q^2 - q$. In addition, the total number of suborbits (rank) is q, whereby $q-1$ of them are suborbits of length 1 while the remaining suborbit is of length $q^2 - q$. □

3.4 Primitivity of *GL(2,q)* on $F_q^2 \setminus \{0\}$

Theorem 3.4.1. *The group G=GL(2,q) acts primitively on $F_q^2 \setminus \{0\}$ for q=2.*

Proof. There are only two groups of order 6, that is C_6 and S_3. The group $GL(2,2)$ is isomorphic to S_3 and a subgroup of order 2 in S_3 is a maximal subgroup. Now, since $|GL(2,2)| = (2^2-1)(2^2-2) = 6$ and by Lemma 3.2.4, $|H| = |Stab_G(\binom{0}{1})| = 2$. It therefore follows that G is primitive since H is maximal. In addition, there exists no subgroup M of G such that $H < M < G$, whereby $|M|$ is a factor of $|G|$ and $|H|$ is a factor of $|M|$ simultaneously. Thus G is primitive. □

Theorem 3.4.2. *The group G=GL(2,q) acts imprimitively on $F_q^2 \setminus \{0\}$ for $q \geq 3$.*

Proof. By Lemma 3.2.4, $H = Stab_G(\binom{0}{1}) = \left\{\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \neq 0, c \in F_q\right\}$ and $|H| = q^2 - q$. Now, $TL(2,q)$ is a proper subgroup of $GL(2,q)$ and H is a proper subgroup of $TL(2,q)$. Since $|TL(2,q)| = (q^2-q)(q-1)$, it follows that $|H| < |TL(2,q)| < |GL(2,q)|$ and thus G is imprimitive since H is not maximal. □

3.5 Transitivity of *GL(3,q)* on $F_q^3 \setminus \{0\}$

Proposition 3.5.1. The group G = *GL(3,2)* acts transitively on $Y = F_2^3 \setminus \{0\}$.

Proof. The field, $F_2 = \{0,1\}$ and by Definition 1.1.2, $|G| = (2^3-1)(2^3-2)(2^3-2^2) = 168$. Therefore,

$$\begin{aligned} Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : \begin{pmatrix}c\\f\\i\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{pmatrix}\right\}, \end{aligned} \tag{3.20}$$

where $\begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{vmatrix} \neq 0$ and $a,b,c,d,e,f,g,h,i \in F_2$.

For the matrix in Equation 3.20 above to be invertible then $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$ must be invertible and hence corresponds to an element of $GL(2,2)$. Such elements are $|GL(2,2)|$ while elements $g,h \in F_2$ can be chosen in 2 ways each. Therefore $\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = |GL(2,2)| \cdot 2^2 = 24$. By Theorem 1.1.6, it follows that,

$$\left|Orb_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = \frac{|G|}{\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right|} = 7 = |Y|.$$

□

Proposition 3.5.2. The group G = *GL(3,3)* acts transitively on $Y = F_3^3 \setminus \{0\}$.

Proof. The field $F_3 = \{0,1,\alpha = 2\}$ where α is a primitive element in F_3 and by Definition 1.1.2, $|G| = (3^3-1)(3^3-3)(3^3-3^2) = 11,232$. It therefore follows that,

$$\begin{aligned} Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} : \begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} : \begin{pmatrix}c\\f\\i\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix}a&b&0\\d&e&0\\g&h&1\end{pmatrix}\right\}, \end{aligned} \tag{3.21}$$

where $\begin{vmatrix}a&b&0\\d&e&0\\g&h&1\end{vmatrix} \neq 0$ and $a,b,c,d,e,f,g,h,i \in F_3$.

For the matrix in Equation 3.21 above to be invertible, then $\begin{pmatrix}a&b\\d&e\end{pmatrix}$ must be invertible and hence corresponds to an element of $GL(2,3)$. Such elements are $|GL(2,3)|$ while the elements g,h can be chosen in 3 ways each. Therefore, it follows that $\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = |GL(2,3)| \cdot 3^2 = 432$. And so by Theorem 1.1.6,

$\left|Orb_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = \dfrac{|G|}{\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right|} = 26 = |Y|.$ □

Proposition 3.5.3. The group G = *GL(3,5)* acts transitively on $Y = F_5^3 \setminus \{0\}$.

Proof. The field, $F_5 = \{0,1,\alpha = 2,\alpha^2 = 4,\alpha^3 = 3\}$ where α is a primitive element in F_5 while $|G| = (5^3-1)(5^3-5)(5^3-5^2) = 1,488,000$ by Definition 1.1.2. Thus,

$$\begin{aligned} Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} : \begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} : \begin{pmatrix}c\\f\\i\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix}a&b&0\\d&e&0\\g&h&1\end{pmatrix}\right\}, \end{aligned} \tag{3.22}$$

where $\begin{vmatrix}a&b&0\\d&e&0\\g&h&1\end{vmatrix} \neq 0$ and $a,b,c,d,e,f,g,h,i \in F_5$.

For the matrix in Equation 3.22 above to be invertible, $\begin{pmatrix}a&b\\d&e\end{pmatrix}$ must be invertible

and hence corresponds to an element of $GL(2,5)$. Such elements are $|GL(2,5)|$ while $g,h \in F_5$ can be chosen in 5 ways each. Hence,
$\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = |GL(2,5)|\cdot 5^2 = 12,000$. Thus, by Theorem 1.1.6,
$\left|Orb_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = \dfrac{|G|}{\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right|} = 124 = |Y|.$ □

Lemma 3.5.4. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$. Then,*
$Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0 \text{ and } a,b,d,e,g,h \in F_q\right\}$
and $\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = q^3(q^2-1)(q-1).$

Proof. The field F_q contains q elements. Then,

$$\begin{aligned} Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} : \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} : \begin{pmatrix}c\\f\\i\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \\ &= \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix}\right\}, \end{aligned} \tag{3.23}$$

where $\begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0$ and $a,b,c,d,e,f,g,h,i \in F_q$.

For the matrix in Equation 3.23 to be invertible, then $\begin{pmatrix}a & b\\ d & e\end{pmatrix}$ must be invertible and it corresponds to an element of $GL(2,q)$. Such elements are $|GL(2,q)|$ while elements $g,h \in F_q$ can be chosen in q ways each. Thus,
$\left|Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = |GL(2,q)|\cdot q^2 = q^3(q^2-1)(q-1).$ □

Theorem 3.5.5. *The group G=GL(3,q) acts transitively on $Y = F_q^3 \setminus \{0\}$.*

Proof. By Definition 1.1.2 $|G| = (q^3-1)(q^3-q)(q^3-q^2)$. Suppose H denotes $Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ where $\begin{pmatrix}0\\0\\1\end{pmatrix} \in Y$. Then, $|H| = q^3(q^2-1)(q-1)$ by Lemma 3.5.4. Hence by Theorem 1.1.6, it follows that,

$$\left|Orb_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right| = \frac{|G|}{|H|}$$

$$
\begin{aligned}
&= \frac{(q^3-1)(q^3-q)(q^3-q^2)}{q^3(q^2-1)(q-1)} \\
&= q^3-1 \\
&= |Y|.
\end{aligned}
$$

□

3.6 Ranks and Subdegrees of *GL(3,q)* on $F_q^3 \setminus \{0\}$

Proposition 3.6.1. Suppose G = *GL(3,2)* acts on $Y = F_2^3 \setminus \{0\}$. Then, rank is 2 and subdegrees are 1 and 6.

Proof. Let
$H = Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \left\{\begin{pmatrix}a&b&0\\d&e&0\\g&h&1\end{pmatrix} : \begin{vmatrix}a&b&0\\d&e&0\\g&h&1\end{vmatrix} \neq 0, \text{ where } a,b,d,e,g,h \in F_2\right\} =$

$$
\left\{\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&1&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&1&0\\0&1&1\end{pmatrix},\right.
$$
$$
\begin{pmatrix}0&1&0\\1&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&1&0\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix},
$$
$$
\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&1&0\\1&1&1\end{pmatrix}, \begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\1&0&0\\0&1&1\end{pmatrix},
$$
$$
\left.\begin{pmatrix}1&1&0\\1&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\1&0&0\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\right\}
$$

From Proposition 3.5.1, $|H| = 24$. Therefore,

$$
\begin{aligned}
Orb_H\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a&b&0\\d&e&0\\g&h&1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} : \begin{vmatrix}a&b&0\\d&e&0\\g&h&1\end{vmatrix} \neq 0\right\} \\
&= \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \qquad (3.24)
\end{aligned}
$$

By Equation 3.24, $\Delta_0 = Orb_H\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and therefore $|\Delta_0| = 1$.

Now, the suborbit Δ_1 which contains the vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is determined as follows:

$$\begin{aligned} Orb_H\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0\right\} \\ &= \left\{\begin{pmatrix}a\\d\\g\end{pmatrix} : \begin{pmatrix}a\\d\\g\end{pmatrix} \neq \alpha^i\begin{pmatrix}0\\0\\1\end{pmatrix}\ \forall \alpha^i \in F_2^*\right\} \quad (3.25) \\ &= \left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \quad (3.26) \end{aligned}$$

By Equation 3.25, $Orb_H\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$ contains all elements that are not scalar multiples of vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Thus, $\left|Orb_H\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\right| = 6$ by Equation 3.26. Hence, there are 2 suborbits Δ_0 and Δ_1 of length 1 and 6 respectively. □

Proposition 3.6.2. Suppose *G=GL(3,3)* acts on $Y = F_3^3 \setminus \{0\}$. Then, rank is 3 and the subdegrees are 1,1 and 24.

Proof. Let $H = Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ and α be the primitive element in F_3. Then by Lemma 3.5.4, $H = \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0\right\}$, where $a,b,d,e,g,h \in F_3$. Now,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0\right\} \\ &= \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \quad (3.27) \end{aligned}$$

By Equation 3.27, $\Delta_0 = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and thus $|\Delta_0| = 1$.

Also,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}0\\0\\\alpha\end{pmatrix}\right) = Orb_H\left(\begin{pmatrix}0\\0\\2\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix}\begin{pmatrix}0\\0\\2\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0\right\} \\ &= \left\{\begin{pmatrix}0\\0\\2\end{pmatrix}\right\} \quad (3.28) \end{aligned}$$

By Equation 3.28, $\Delta_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ and as a result $|\Delta_1| = 1$.

Now, the suborbit Δ_2 that contains element $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is determined as follows:

$$\begin{aligned}
Orb_H\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{vmatrix} \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} a \\ d \\ g \end{pmatrix} : \begin{pmatrix} a \\ d \\ g \end{pmatrix} \neq \alpha^i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; \forall \alpha^i \in F_3^* \right\} \qquad (3.29) \\
&= \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \right. \\
&\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \\
&\quad \left. \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\} \qquad (3.30)
\end{aligned}$$

By Equation 3.29, $Orb_H\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$, contains all elements that are not scalar multiples of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, $\left| Orb_H\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \right| = 24$ by Equation 3.30. Thus, there are 3 suborbits Δ_0, Δ_1 and Δ_2 of length 1, 1 and 24 repectively. □

Theorem 3.6.3. *Let $G = GL(3,q)$ act on $Y = F_q^3 \setminus \{0\}$. Then rank is q and subdegrees are $[1]^{[q-1]}, (q^3 - q)$.*

Proof. The elements of F_q can be expressed in terms of the primitive element α as follows: $0, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{q-1} = 1$. By Lemma 3.5.4, it follows that,

$H = Stab_G\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{pmatrix} : \begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{vmatrix} \neq 0, \text{ where } a,b,d,e,g,h \in F_q \right\}$ and $|H| = q^3(q^2-1)(q-1)$. Therefore,

$$\begin{aligned}
Orb_H\left(\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}\right) &= \left\{ \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} : \begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{vmatrix} \neq 0 \right\} \\
&= \left\{ \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \right\} \qquad (3.31)
\end{aligned}$$

Therefore, the vectors of the form $\begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$, where $x \in F_q^*$ exist as suborbits of length 1 and hence, there are $q-1$ suborbits of length 1 that is $\Delta_0, \Delta_1 \dots \Delta_{q-2}$. In addition, since $\begin{pmatrix}1\\0\\0\end{pmatrix} \in F_q^3 \setminus \{0\}$ is not among the $q-1$ suborbits, the next step is to compute Δ_{q-1} which is the suborbit containing $\begin{pmatrix}1\\0\\0\end{pmatrix}$ as follows,

$$\begin{aligned} Orb_H\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) &= \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0\right\} \\ &= \left\{\begin{pmatrix}a\\d\\g\end{pmatrix} : \begin{pmatrix}a\\d\\g\end{pmatrix} \neq \alpha^i \begin{pmatrix}0\\0\\1\end{pmatrix} \forall \alpha^i \in F_q^*\right\} \end{aligned} \quad (3.32)$$

From Equation 3.32, $Orb_H\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$ contains all vectors that are not scalar multiples of $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Now, any vector in $F_q^3 \setminus \{0\}$ has $|F_q^*| = q-1$ non-zero scalar multiples of it and by Theorem 3.5.5, $|F_q^3 \setminus \{0\}| = q^3 - q$. Therefore,

$$\begin{aligned} \left|Orb_H\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\right| &= (q^3-1)-(q-1) \\ &= q^3 - q \end{aligned} \quad (3.33)$$

Thus, the total number of suborbits is q, whereby $q-1$ of them are suborbits of length 1 while the remaining suborbit is of length $q^3 - q$. □

3.7 Primitivity of *GL(3,q)* on $F_q^3 \setminus \{0\}$

Theorem 3.7.1. *The group G=GL(3,q) acts primitively on* $F_q^3 \setminus \{0\}$ *for q=2.*

Proof. Let $H = Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$. By Lemma 3.5.4, $|H| = 24$ and by Definition 1.1.2, $|G| = 168$. Now, there exists no subgroup M of G that is $H < M < G$, whereby $|M|$ is a factor of $|G|$ and $|H|$ is a factor of $|M|$ concurrently. Hence H is maximal and G is primitive. □

Theorem 3.7.2. *The group G=GL(3,q) acts imprimitively on* $F_q^3 \setminus \{0\}$ *for* $q \geq 3$.

Proof. By Lemma 3.5.4, $H = Stab_G\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & 1\end{vmatrix} \neq 0\right\}$, where $a,b,d,e,g,h \in F_q$ and $|H| = q^2(q^2-1)(q^2-q)$. Now, $\exists$ a proper subgroup M of G such that $M = \left\{\begin{pmatrix}a & b & 0\\ d & e & 0\\ g & h & x\end{pmatrix} : \begin{vmatrix}a & b & 0\\ d & e & 0\\ g & h & x\end{vmatrix} \neq 0\right\}$, where $x \neq 0$ and

$|M| = q^2(q^2-1)(q^2-q)(q-1)$ and as a result, H is a proper subgroup of M. Thus, $H < M < G$ and by Theorem 1.1.11, G is imprimitive since H is not maximal. $\square$

CHAPTER FOUR
SUBORBITAL GRAPHS

4.1 Introduction

This chapter is partitioned into six sections. Section 4.2 to 4.4 contains the graphs, methodology of construction and properties corresponding to suborbital graphs of the action of $GL(2,q)$ on $F_q^2 \setminus \{0\}$ respectively whereas Section 4.5 to 4.7 accounts for the graphs, method of construction and properties corresponding to the action of $GL(3,q)$ on $F_q^3 \setminus \{0\}$ respectively.

4.2 Suborbital Graphs of *GL(2,q)* on $F_q^2 \setminus \{0\}$

Proposition 4.2.1. Suborbital graph of *G=GL(2,2)* acting on $F_2^2 \setminus \{0\}$.

Proof. This action has only one non-trivial suborbit that is Δ_1 as seen in Proposition 3.3.1. By Definition 1.1.2, $|G| = 6$ and the elements are as listed below:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

By Definition 1.1.24, $O_1 = \left\{ g\left(\binom{0}{1}, \binom{1}{0} \right) \mid g \in G \right\}$ where $\binom{1}{0} \in \Delta_1$. This yields:

$$O_1 = \left\{ \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \right.$$
$$\left. \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right\}.$$

Figure 4.1 below is constructed from suborbital O_1.

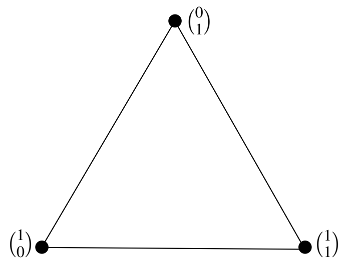

Figure 4.1: Γ_1 corresponding to Δ_1 of *GL(2,2)* on $F_2^2 \setminus \{\mathbf{0}\}$

The graph Γ_1 is connected, complete, undirected, girth is 3, diameter is 1, it is regular with degree 2 and chromatic number is 3. □

Proposition 4.2.2. Suborbital graphs of G=*GL(2,3)* acting on $F_3^2 \setminus \{0\}$.

Proof. By Proposition 3.3.2, this action has two non-trivial suborbits that is Δ_1 and Δ_2. By Definition 1.1.24, $O_1 = \left\{ g \left(\binom{0}{1}, \binom{0}{2} \right) \mid g \in G \right\}$, where $\binom{0}{2} \in \Delta_1$. The elements of O_1 are listed below.

$$O_1 = \left\{ \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \right), \left(\begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right), \left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix} \right), \left(\begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right), \right.$$
$$\left. \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right), \left(\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \right), \left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right), \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right) \right\}$$

The graph corresponding to the above suborbital is constructed below.

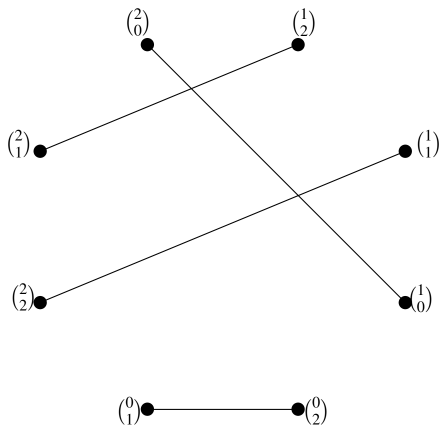

Figure 4.2: Γ_1 corresponding to Δ_1 of *GL(2,3)* on $F_3^2 \setminus \{\mathbf{0}\}$

The graph Γ_1 is undirected, regular with degree 1, chromatic number is 2, it is disconnected, has 4 components, is a forest with trees of length 2 and diameter is ∞. Similarly, the remaining non-trivial suborbital corresponding to Δ_2 is given by, $O_2 = \left\{ g\left(\binom{0}{1}, \binom{1}{0}\right) \mid g \in G \right\}$, where $\binom{1}{0} \in \Delta_2$. The resultant edges from O_2 are listed in Appendix I and Figure 4.3 below displays Γ_2.

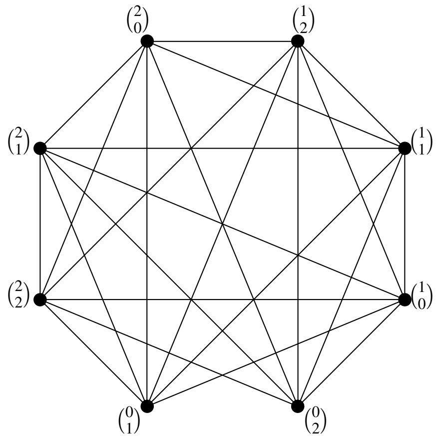

Figure 4.3: Γ_2 corresponding to Δ_2 of *GL(2,3)* on $F_3^2 \setminus \{0\}$

The graph Γ_2 is connected, undirected, regular with degree 6, chromatic number is 4, diameter is 2, has 32 triangles and girth is 3. □

Proposition 4.2.3. Suborbital graphs of *G=GL(2,5)* acting on $F_5^2 \setminus \{0\}$.

Proof. By Proposition 3.3.3, this action has four non-trivial suborbits that is Δ_1, Δ_2, Δ_3 and Δ_4 and by Definition 1.1.24, $O_1 = \left\{ g\left(\binom{0}{1}, \binom{0}{2}\right) \mid g \in G \right\}$, where $\binom{0}{2} \in \Delta_1$. The edges of Γ_1 resulting from O_1 are listed in Appendix II and Figure 4.4 below represents Γ_1.

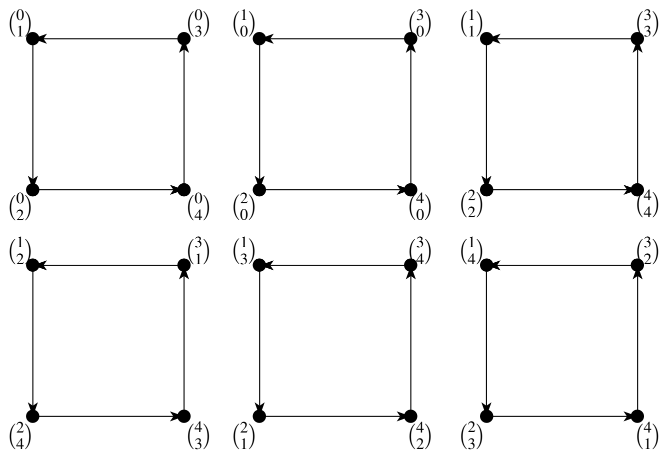

Figure 4.4: Γ_1 corresponding to Δ_1 of *GL(2,5)* on $F_5^2 \setminus \{0\}$

The graph Γ_1 is directed, disconnected, has 6 connected components, diameter is ∞, girth is 4 and the chromatic number is 2.
Next, by definition 1.1.24, $O_2 = \left\{ g\left(\binom{0}{1}, \binom{0}{4}\right) \mid g \in G \right\}$, where $\binom{0}{4} \in \Delta_2$. This yields the edges of Γ_2 below which are listed in Appendix II. It follows that Γ_2 is as in Figure 4.5.

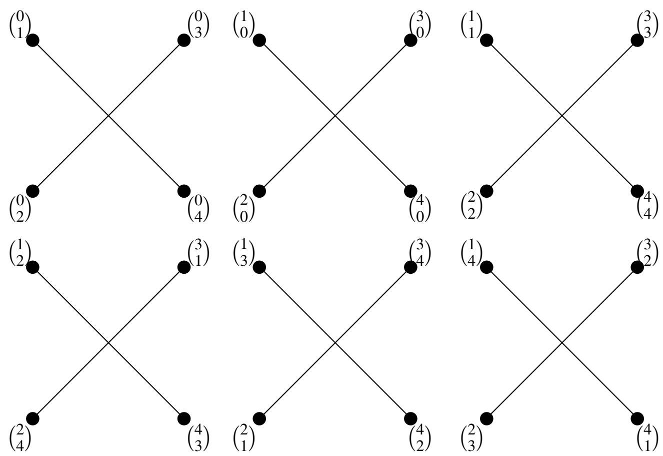

Figure 4.5: Γ_2 corresponding to Δ_2 of *GL(2,5)* on $F_5^2 \setminus \{0\}$

The graph Γ_2 is disconnected, undirected, regular with degree 1, diameter is ∞, is forest with trees of length 2 and the chromatic number is 2.

Also, by Definition 1.1.24, $O_3 = \left\{ g\left(\binom{0}{1}, \binom{0}{3} \right) \mid g \in G \right\}$, where $\binom{0}{3} \in \Delta_3$. The edges of Γ_3 resulting from O_3 are listed in Appendix II and Γ_3 is displayed below in Figure 4.6.

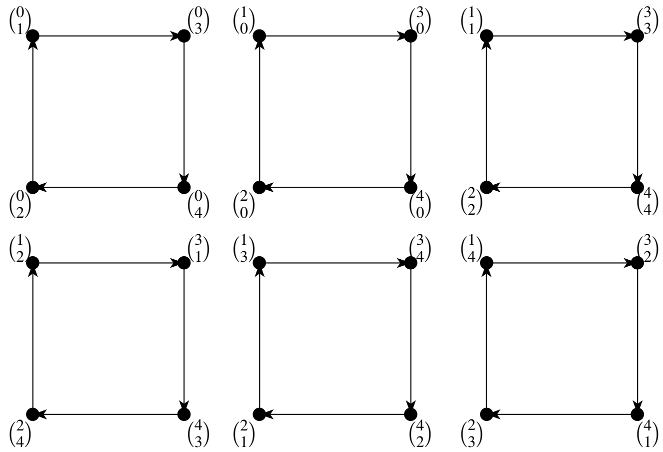

Figure 4.6: Γ_3 corresponding to Δ_3 of *GL(2,5)* on $F_5^2 \setminus \{\mathbf{0}\}$

The graph Γ_3 is directed, disconnected, comprises 6 connected components, diameter is ∞, girth is 4 and the chromatic number is 2.
The suborbital corresponding to Δ_4 is given by $O_4 = \left\{ g\left(\binom{0}{1}, \binom{1}{0} \right) \mid g \in G \right\}$, where $\binom{1}{0} \in \Delta_4$. The resultant edges resulting from O_4 are listed in Appendix II whereas Figure 4.7 below displays Γ_4.

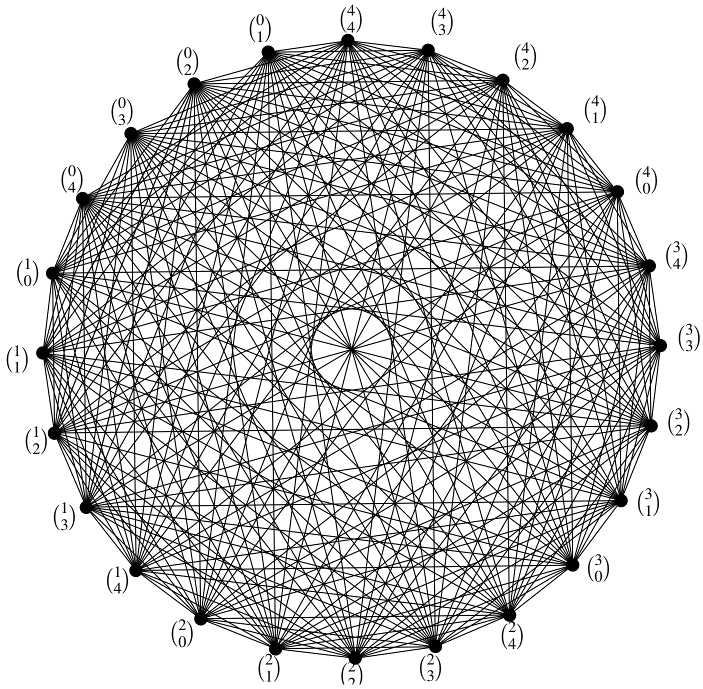

Figure 4.7: Γ_4 corresponding to Δ_4 of *GL(2,5)* on $F_5^2 \setminus \{0\}$

The graph Γ_4 is connected, undirected, regular with degree 20, diameter is 2, chromatic number is 6 and girth is 3. □

4.3 Constructing Suborbital Graphs of *G=GL(2,q)* on $F_q^2 \setminus \{0\}$

Let G act on $Y = F_q^2 \setminus \{0\}$. From Theorem 3.3.4, this action has $q-1$ suborbits of length 1 and one suborbit of length $q^2 - q$.

4.3.1 Suborbital Graphs corresponding to Suborbits of length 1

By Proposition 1.1.26, the components of Γ_i corresponding to non-trivial suborbit $\Delta_i(x)$, where $|\Delta_i| = 1$ are either trees or cycles and by Theorem 1.1.29, these components are isomorphic and therefore after computing the first component, the other components will be deduced from it. Now, let α be a primitive element in F_q. By Theorem 3.3.4, there are $q-1$ suborbits of length 1 i.e. $\Delta_0 = \left\{\binom{0}{1}\right\}$, $\Delta_1 = \left\{\binom{0}{\alpha}\right\}$, $\Delta_2 = \left\{\binom{0}{\alpha^2}\right\}$,..., $\Delta_{q-2} = \left\{\binom{0}{\alpha^{q-2}}\right\}$. Let $O_i = \left\{g\left(\binom{0}{1}, \binom{0}{\alpha^i}\right) | g \in G\right\}$ by Definition 1.1.24, then $\binom{0}{1} \to \binom{0}{\alpha^i}$ is a directed edge of Γ_i. To get another edge involving $\binom{0}{\alpha^i}$, the matrix $g = \begin{pmatrix} \alpha^i & 0 \\ 0 & \alpha^i \end{pmatrix} \in G$ is used since $\begin{pmatrix} \alpha^i & 0 \\ 0 & \alpha^i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^i \end{pmatrix}$. The other edges will be computed as shown in the

computations below. From definition 1.1.24, $O_1 = \left\{g\left(\binom{0}{1}, \binom{0}{\alpha}\right) : g \in G\right\}$. It follows that $\binom{0}{1} \to \binom{0}{\alpha}$ is a directed edge of Γ_1.
Let $g_{11} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, then, $g_{11} \in G$. Thus, $g_{11}\left(\binom{0}{1} \to \binom{0}{\alpha}\right)$ gives the following edge:
$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \to \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix}$.
$= \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \in \Gamma_1$.
To get the other edge, compute $g_{12} = (g_{11})^2$.
Now, $g_{12} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}$. Multiplying, $g_{12}\left(\binom{0}{1} \to \binom{0}{\alpha}\right)$ gives the following edge:
$\begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \to \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^3 \end{pmatrix}$.
$= \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^3 \end{pmatrix} \in \Gamma_1$.
To get another edge, compute $g_{13} = (g_{11})^3 = \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^3 \end{pmatrix}$.
Now, multiplying $g_{13}\left(\binom{0}{1} \to \binom{0}{\alpha}\right)$ gives the following,
$\begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^3 \end{pmatrix} \to \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^3 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix}$.
$= \begin{pmatrix} 0 \\ \alpha^3 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix} \in \Gamma_1$.
Continuing this way, the following edges are obtained:

$$C_1 : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^3 \end{pmatrix} \to \ldots \to \begin{pmatrix} 0 \\ \alpha^{q-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.1)$$

This is a cycle of length equal to the order of α in F_q^*, therefore its of length $q-1$. By Theorem 1.1.29, the other cycles are isomorphic. Now, to get a cycle containing an element $\begin{pmatrix} x \\ y \end{pmatrix}$, the cycle C_1 is multiplied with the matrix $h = \begin{pmatrix} a & x \\ b & y \end{pmatrix} \in G$ since, $\begin{pmatrix} a & x \\ b & y \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Now, hC_1 can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \to \begin{pmatrix} \alpha^2 x \\ \alpha^2 y \end{pmatrix} \to \ldots \to \begin{pmatrix} \alpha^{q-2} x \\ \alpha^{q-2} y \end{pmatrix} \to \begin{pmatrix} \alpha^{q-1} x \\ \alpha^{q-1} y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \alpha\begin{pmatrix} x \\ y \end{pmatrix} \to \alpha^2\begin{pmatrix} x \\ y \end{pmatrix} \to \ldots \to \alpha^{q-2}\begin{pmatrix} x \\ y \end{pmatrix} \to \alpha^{q-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \quad (4.2)$$

Therefore, $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right)$ is an edge of Γ_1 iff $\begin{pmatrix} c \\ d \end{pmatrix} = \alpha\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{aligned} \text{Hence, } O_1 &= \left\{g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha \end{pmatrix}\right) : g \in G\right\}. \\ &= \left\{\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) : \begin{pmatrix} c \\ d \end{pmatrix} = \alpha\begin{pmatrix} a \\ b \end{pmatrix}\right\}. \quad (4.3) \end{aligned}$$

The edges of Γ_2 are in the set $O_2 = \left\{g\left(\binom{0}{1}, \binom{0}{\alpha^2}\right) : g \in G\right\}$. It follows that $\binom{0}{1} \to \binom{0}{\alpha^2}$ is a directed edge of Γ_2.

Let $g_{21} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}$. Then, $g_{21} \in G$. Now, $g_{21}\left(\binom{0}{1} \to \binom{0}{\alpha^2}\right)$ will give the next edge as shown below.

$\begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \to \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix}$.

$= \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix} \in \Gamma_2$.

To get the other edge involving $\binom{0}{\alpha^4}$, $g_{22} = (g_{21})^2 = \begin{pmatrix} \alpha^4 & 0 \\ 0 & \alpha^4 \end{pmatrix}$ is first determined, then proceed to get $g_{22}\left(\binom{0}{1} \to \binom{0}{\alpha^2}\right)$ as follows:

$\begin{pmatrix} \alpha^4 & 0 \\ 0 & \alpha^4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix} \to \begin{pmatrix} \alpha^4 & 0 \\ 0 & \alpha^4 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^6 \end{pmatrix}$.

$= \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^6 \end{pmatrix} \in \Gamma_2$.

Continuing this way, we will obtain the edges:

$$C_2 : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^4 \end{pmatrix} \to \begin{pmatrix} 0 \\ \alpha^6 \end{pmatrix} \to \dots \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.4}$$

This is a cycle of length equal to the order of α^2 in F_q^*.

The other cycles are isomorphic to C_2. The cycle containing vertex $\begin{pmatrix} x \\ y \end{pmatrix}$ is obtained from C_2 by multiplying it with $h = \begin{pmatrix} a & x \\ b & y \end{pmatrix}$ since $\begin{pmatrix} a & x \\ b & y \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and thus the cycle below is obtained:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \alpha^2\begin{pmatrix} x \\ y \end{pmatrix} \to \alpha^4\begin{pmatrix} x \\ y \end{pmatrix} \to \dots \to \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.5}$$

$$\begin{aligned} \therefore O_2 &= \left\{g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha^2 \end{pmatrix}\right) : g \in G\right\}. \\ &= \left\{\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) : \begin{pmatrix} c \\ d \end{pmatrix} = \alpha^2\begin{pmatrix} a \\ b \end{pmatrix}\right\}. \end{aligned} \tag{4.6}$$

Continuing this way, it is observed that $\forall \alpha^i \in F_q^*$ there exists a suborbital graph whose cycles are of length equal to the order of α^i in F_q^*. Therefore,

$$\begin{aligned} O_i &= \left\{g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha^i \end{pmatrix}\right) : g \in G\right\}, \text{ where } \begin{pmatrix} 0 \\ \alpha^i \end{pmatrix} \in \Delta_i \\ &= \left\{\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) : \begin{pmatrix} c \\ d \end{pmatrix} = \alpha^i\begin{pmatrix} a \\ b \end{pmatrix}\right\} \text{for } 0 \le i \le q-2. \end{aligned} \tag{4.7}$$

4.3.2 Suborbital Graphs corresponding to suborbit of length q^2-q

By Theorem 3.3.4, Δ_{q-1} is the only suborbit of length not equal to 1. Now, $O_0 \cup O_1 \cup O_2 \cup O_3 \cup ... \cup O_{q-2} \cup O_{q-1} = F_q^2 \setminus \{0\} \times F_q^2 \setminus \{0\}$ gives all possible edges in an action. The suborbitals: $O_0, O_1, O_2, ..., O_{q-2}$ give all the edges such that the vertices are scalar multiple vectors. This means that all other edges whose vertices are not scalar multiple vectors are in O_{q-1}. It follows that O_{q-1} comprises of edges joining vertices that are not scalar multiple vectors. Suppose $\binom{1}{0} \in F_q^2 \setminus \{0\}$, then $\binom{1}{0} \in \Delta_{q-1}$ by Theorem 3.3.4. Thus,

$$O_{q-1} = \left\{ g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) : g \in G \right\}.$$

$$= \left\{ \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right) : \begin{pmatrix} a \\ b \end{pmatrix} \text{and} \begin{pmatrix} c \\ d \end{pmatrix} \text{are not scalar multiples of one another} \right\}. \quad (4.8)$$

4.4 Properties of *GL(2,q)* Suborbital Graphs

Theorem 4.4.1. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$ and α be a primitive element in F_q. Then, Γ_i corresponding to Δ_i, where $|\Delta_i| = 1$ has girth equal the order of α^i in F_q^*.*

Proof. By Section 4.3.1, the cycles are of length $\left|\alpha^i\right|$ implying that girth is also $\left|\alpha^i\right|$. □

Corollary 4.4.2. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$. Then, $\Gamma_{\frac{q-1}{2}}$ corresponding to $\Delta_{\frac{q-1}{2}}$, where $\left|\Delta_{\frac{q-1}{2}}\right| = 1$ and $2 \nmid q$ is acyclic.*

Proof. Let α be a primitive element in F_q. By Section 4.3.1, each component of $\Gamma_{\frac{q-1}{2}}$ is of length $\left|\alpha^{\frac{q-1}{2}}\right|$. Since $\left(\alpha^{\frac{q-1}{2}}\right)^2 = \alpha^{q-1} = 1$, it follows that $\left|\alpha^{\frac{q-1}{2}}\right| = 2$. It therefore implies that these are components of length 2 and hence acyclic. □

Theorem 4.4.3. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$ and let Γ_i be a suborbital graph corresponding to Δ_i, where $|\Delta_i| = 1$ and α be a primitive element in F_q. Then, the number $\psi(\Gamma_i)$ of components is given by:*

$$\psi(\Gamma_i) = \frac{q^2-1}{|\alpha^i|}. \quad (4.9)$$

Proof. From Section 4.3.1, each component of Γ_i is of length equal to $\left|\alpha^i\right|$. It also follows that each component of Γ_i has $\left|\alpha^i\right|$ vertices and by Theorem 1.1.29, these components are isomophic and since $|Y| = q^2 - 1$ then,

$$\begin{aligned}\psi(\Gamma_i) &= \frac{|Y|}{|\alpha^i|} \\ &= \frac{q^2-1}{|\alpha^i|}.\end{aligned}$$

□

Corollary 4.4.4. *Let $GL(2,q)$ act on $Y = F_q^2 \setminus \{0\}$, then the suborbital graph Γ_i corresponding to Δ_i, where $|\Delta_i| = 1$ is disconnected.*

Proof. By Proposition 1.1.26, suborbital graphs corresponding to non-trivial suborbits of length 1 are either trees or cycles and by Theorem 4.4.3, the number of components $\psi(\Gamma_i) = \dfrac{q^2-1}{|\alpha^i|} > 1$ since $1 \leq \left|\alpha^i\right| \leq q-1$ for $0 \leq i \leq q-2$. Therefore Γ_i is disconnected. □

Theorem 4.4.5. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$ and $\beta \in F_q^*$. Then, the girth of suborbital graph Γ_{q-1} corresponding to suborbit Δ_{q-1} is 3.*

Proof. By Equation 4.8, $\left(\binom{x}{y}, \binom{w}{z}\right)$ is an edge of Γ_{q-1} if $\binom{x}{y}$ and $\binom{w}{z}$ are not scalar multiples of each other. Therefore, the smallest cycle $\binom{0}{\beta} \to \binom{\beta}{0} \to \binom{\beta}{\beta} \to \binom{0}{\beta} \in \Gamma_{q-1}$ is of length 3. □

Theorem 4.4.6. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$. Then Γ_{q-1} corresponding to Δ_{q-1} is connected.*

Proof. Let $\binom{a}{b}, \binom{c}{d}$ be vertices in Γ_{q-1} . If the two vertices are not scalar multiples of each other, then they are adjacent by Equation 4.8. If they are scalar multiples of each other, then there exists another vertex $\binom{u}{v} \in \Gamma_{q-1}$ such that $\binom{u}{v}$ is not a scalar multiple to both $\binom{a}{b}$ and $\binom{c}{d}$. Hence there exists a path from $\binom{a}{b}$ to $\binom{c}{d}$ via $\binom{u}{v}$. Therefore Γ_{q-1} is connected. □

Theorem 4.4.7. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$ and α be a primitive element in F_q. Then, the suborbital graph Γ_i corresponding to a suborbit of length 1 is paired with suborbital graph Γ_j if i + j = q-1.*

Proof. Let O_i and O_j correspond to Δ_i and Δ_j respectively and $|F_q^*| = q-1$. Suppose $i + j = q-1$, then $\alpha^i \cdot \alpha^j = \alpha^{q-1} = 1$. Then, $\alpha^i = \left(\alpha^j\right)^{-1}$ and $\alpha^j = \left(\alpha^i\right)^{-1}$. It follows that $\alpha^i v_1 = v_2$ and $\alpha^j v_2 = v_1$, where v_1, v_2 are vertices. Thus O_i is the reverse of O_j and vice versa. Hence, Γ_i is paired to Γ_j. □

Corollary 4.4.8. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$ and α be a primitive element in F_q. Then, the suborbital graph $\Gamma_{\frac{q-1}{2}}$ corresponding to $\Delta_{\frac{q-1}{2}}$, where $|\Delta_{\frac{q-1}{2}}| = 1$ and $2 \nmid q$ is self-paired.*

Proof. Let $O_{\frac{q-1}{2}}$ correspond to $\Delta_{\frac{q-1}{2}}(x)$, where $2 \nmid q$. Then, from Theorem 4.4.7 above $i = \frac{q-1}{2}$ and $j = \frac{q-1}{2}$. So $i = j$ and hence $\Gamma_{\frac{q-1}{2}}$ is self-paired. □

Corollary 4.4.9. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$. Then, $\Gamma_{\frac{q-1}{2}}$ corresponding to $\Delta_{\frac{q-1}{2}}$, where $2 \nmid q$ is a forest.*

Proof. By Theorem 4.4.2, each component of $\Gamma_{\frac{q-1}{2}}$ is of length equal to 2 and by Corollary 4.4.8, $\Gamma_{\frac{q-1}{2}}$ is self-paired. It follows that these components are trees of length 2. Thus, $\Gamma_{\frac{q-1}{2}}$ is a forest. □

Theorem 4.4.10. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$. Then, the suborbital graph Γ_{q-1} corresponding to Δ_{q-1} is self-paired.*

Proof. By Equation 4.8, $\left(\binom{a}{b}, \binom{c}{d}\right) \in O_{q-1}$ iff $\binom{a}{b}$ and $\binom{c}{d}$ are not scalar multiples of each other. This also implies that $\left(\binom{c}{d}, \binom{a}{b}\right) \in O_{q-1}$. Hence Γ_{q-1} is self-paired. □

Theorem 4.4.11. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$ and Γ_i be the non-trivial suborbital graph corresponding to Δ_i. Then, diameter $diam(\Gamma_i)$ of Γ_i is given by,*

$$diam(\Gamma_i) = \begin{cases} 1 & \text{if } q = 2 \text{ and } |\Delta_i| = q^2 - q \\ 2 & \text{if } q > 2 \text{ and } |\Delta_i| = q^2 - q \\ \infty & \text{if } q > 2 \text{ and } |\Delta_i| = 1 \end{cases}$$

Proof. For the case where $q = 2$ and $|\Delta_i| = q^2 - q$, the action has only 1 non-trivial complete suborbital graph as displayed in Figure 4.1. Thus, $diam(\Gamma_i) = 1$.
For the case where $q > 2$ and $|\Delta_i| = q^2 - q$, if vertices $\binom{x}{y}, \binom{a}{b} \in \Gamma_i$ are not scalar multiples of each other, then the distance between $\binom{x}{y}$ and $\binom{a}{b}$ is 1 since they are adjacent. If vertices $\binom{x}{y}, \binom{a}{b} \in \Gamma_i$ are scalar multiples of each other, then there exists another vertex $\binom{u}{v}$ which is not a scalar multiple to both $\binom{x}{y}$ and $\binom{a}{b}$, implying it is adjacent to both $\binom{x}{y}$ and $\binom{a}{b}$ and as a result, distance between $\binom{x}{y}$ and $\binom{a}{b}$ via $\binom{u}{v}$ is 2. It therefore follows that $diam(\Gamma_i) = 2$.
For the case where $q > 2$ and $|\Delta_i| = 1$, the suborbital graphs are disconnected by Corollary 4.4.4 and hence $diam(\Gamma_i) = \infty$. □

Theorem 4.4.12. *The action of GL(2,q) on $F_q^2 \setminus \{0\}$ for q=2 is primitive.*

Proof. This action only has 1 non-trivial suborbital graph Γ_1 which is complete and hence connected as displayed in Figure 4.1. It follows from Theorem 1.1.27, that the action is primitive. □

Theorem 4.4.13. *The action of G=GL(2,q) on $F_q^2 \setminus \{0\}$ for $q \geq 3$ is imprimitive.*

Proof. Some of the non-trivial suborbital graphs of G are disconnected and hence by Theorem 1.1.27, the action is imprimitive. □

Theorem 4.4.14. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$. Then, the chromatic number $\chi(\Gamma_i)$ of suborbital graph Γ_i corresponding to non-trivial suborbit Δ_i, where $|\Delta_i| = 1$ is either 2 or 3.*

Proof. By Proposition 1.1.26, suborbital graphs corresponding to non-trivial suborbits of length one are either trees or cycles. In case the components are trees, then they are of length two by Corollary 4.4.2. Therefore, $\chi(\Gamma_i) = 2$ since they are bipartite. In the case where components are cycles of even length, then $\chi(\Gamma_i) = 2$ since they are also bipartite. In addition, let C be a cycle of odd length with vertices $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$. The vertices v_1 to v_{n-1} are alternately colored with two colors such that no adjacent vertices possess the same color. Now since v_n is also adjacent to v_1 and both would possess the same color, it therefore leads to v_n being assigned a third color so as no adjacent vertices in C will possess the same color and thus $\chi(\Gamma_i) = 3$. □

Theorem 4.4.15. *Let G=GL(2,q) act on $Y = F_q^2 \setminus \{0\}$. Then, the chromatic number $\chi(\Gamma_{q-1})$ of suborbital graph Γ_{q-1} corresponding to Δ_{q-1} is $q+1$.*

Proof. From Equation 4.8, $\left(\binom{a}{b}, \binom{c}{d}\right) \in O_{q-1}$ if $\binom{a}{b}$ and $\binom{c}{d}$ are not scalar multiples of each other. For any vector $\binom{x}{y} \in Y$, there are $q-1$ scalar multiples of it. So, Y can be partitioned into $\hbar$ subsets each of length $q-1$ and whose elements are scalar multiples of each other. The vectors in each $\hbar$ subset will possess same color since they are not adjacent to each other. Since $|Y| = q^2 - 1$ then,

$$\begin{aligned} \hbar &= \frac{q^2-1}{q-1} \\ &= \frac{(q+1)(q-1)}{q-1} \\ &= q+1 \end{aligned}$$

Therefore, the set Y is partitioned into $q+1$ subsets of length $q-1$ and since the elements of each subset possess the same color, there are $q+1$ different colors implying $\chi(\Gamma_{q-1}) = \hbar = q+1$. □

4.5 Suborbital Graphs of *GL(3,q)* on $F_q^3 \setminus \{0\}$

Proposition 4.5.1. Suborbital graph of *G=GL(3,2)* acting on $F_2^3 \setminus \{0\}$.

Proof. By Proposition 3.6.1, this action has only 1 non-trivial suborbit that is Δ_1. By Definition 1.1.24, $O_1 = \left\{ g\left(\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \right) \mid g \in G \right\}$, where $\begin{pmatrix}1\\0\\0\end{pmatrix} \in \Delta_1$. The edges of Γ_1 resulting from O_1 are listed in Appendix III and Figure 4.8 below displays Γ_1.

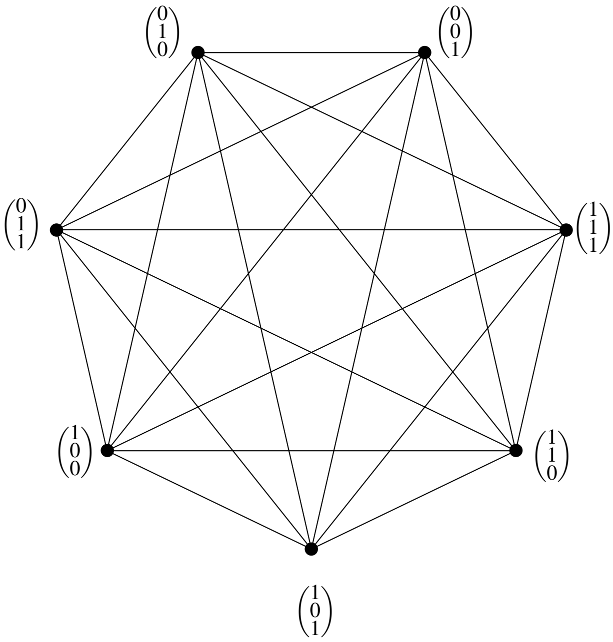

Figure 4.8: Γ_1 corresponding to Δ_1 of *GL(3,2)* on $F_2^3 \setminus \{0\}$

The graph Γ_1 is connected, complete, undirected, regular with degree 6, diameter is 1, girth is 3 and chromatic number is 7. □

Proposition 4.5.2. Suborbital graphs of *G=GL(3,3)* acting on $F_3^3 \setminus \{0\}$.

Proof. By Proposition 3.6.2, this action has 2 non-trivial suborbits that is Δ_1 and Δ_2. By Definition 1.1.24, $O_1 = \left\{ g\left((\begin{smallmatrix}0\\0\\1\end{smallmatrix}), (\begin{smallmatrix}0\\0\\2\end{smallmatrix}) \right) \mid g \in G \right\}$, where $\begin{pmatrix}0\\0\\2\end{pmatrix} \in \Delta_1$. This gives the edges of Γ_1 resulting from O_1 which are outlined in Appendix IV, whereas Γ_1 is displayed in Figure 4.9 below.

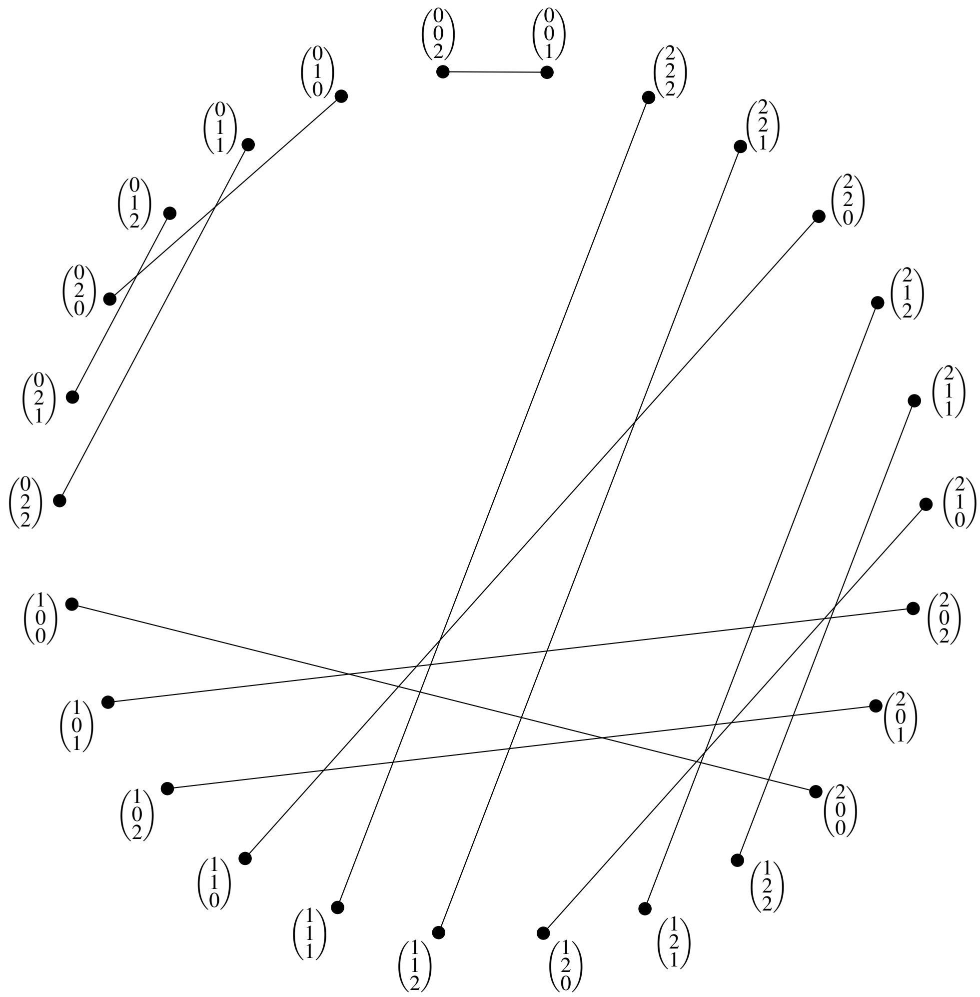

Figure 4.9: Γ_1 corresponding to Δ_1 of *GL(3,3)* on $F_3^3 \setminus \{0\}$

The graph Γ_1 is undirected, disconnected, regular with degree 1, diameter is ∞, is a forest with trees of length 2 and has chromatic number 2.

Similarly, since $\begin{pmatrix}1\\0\\0\end{pmatrix} \in \Delta_2$, by Definition 1.1.24, $O_2 = \left\{ g \left(\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \right) \mid g \in G \right\}$. The edges of Γ_2 resulting from O_2 are listed in Appendix IV, whereas Γ_2 is displayed in Figure 4.10 below.

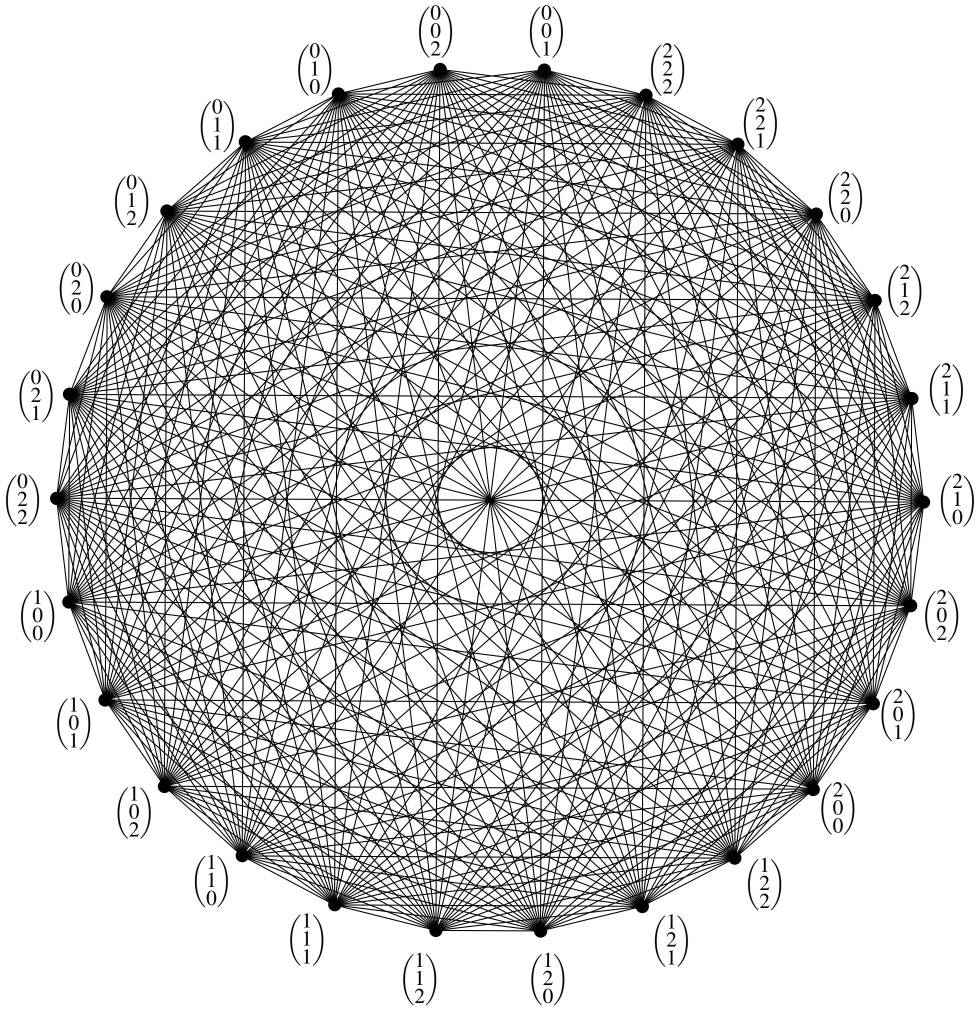

Figure 4.10: Γ_2 corresponding to Δ_2 of *GL(3,3)* on $F_3^3 \setminus \{0\}$

The graph Γ_2 is connected, undirected, regular with degree 24, diameter is 2, girth is 3 and chromatic number is 13. □

4.6 Constructing Suborbital Graphs of *G=GL(3,q)* on $F_q^3 \setminus \{0\}$

4.6.1 Suborbital Graphs corresponding to Suborbits of length 1

Just like in the previous action, G has a total of q suborbits. By Proposition 1.1.26, the components of Γ_i corresponding to Δ_i are either trees or cycles and by Theorem 1.1.29,

these components are isomorphic. Therefore, after computing the first component, the other components will be deduced from it. Let α be a primitive element in F_q, by Theorem 3.6.3, there are $q-1$ suborbits of length 1 that is:
$\Delta_0 = \left\{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)\right\}$, $\Delta_1 = \left\{\left(\begin{smallmatrix}0\\0\\\alpha\end{smallmatrix}\right)\right\}$, $\Delta_2 = \left\{\left(\begin{smallmatrix}0\\0\\\alpha^2\end{smallmatrix}\right)\right\}$, $\ldots$, $\Delta_{q-2} = \left\{\left(\begin{smallmatrix}0\\0\\\alpha^{q-2}\end{smallmatrix}\right)\right\}$. Let $O_i = \left\{g\left(\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\0\\\alpha^i\end{smallmatrix}\right)\right) | g \in G\right\}$. By Definition 1.1.24, $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^i\end{pmatrix}$ is a directed edge of Γ_i. To get another edge involving vertex $\left(\begin{smallmatrix}0\\0\\\alpha^i\end{smallmatrix}\right)$, the matrix $g = \begin{pmatrix}\alpha^i & 0 & 0\\0 & \alpha^i & 0\\0 & 0 & \alpha^i\end{pmatrix} \in G$ is used since $\begin{pmatrix}\alpha^i & 0 & 0\\0 & \alpha^i & 0\\0 & 0 & \alpha^i\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^i\end{pmatrix}$.
The edges of Γ_1 corresponding to Δ_1 are computed using Definition 1.1.24 as follows, $O_1 = \left\{g\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\\alpha\end{pmatrix}\right) : g \in G\right\}$. Then $\exists$ an edge $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha\end{pmatrix} \in \Gamma_1$

Let $g_{11} = \begin{pmatrix}\alpha & 0 & 0\\0 & \alpha & 0\\0 & 0 & \alpha\end{pmatrix} \in G$, it follows that,

$$\begin{pmatrix}\alpha & 0 & 0\\0 & \alpha & 0\\0 & 0 & \alpha\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha\end{pmatrix} \to \begin{pmatrix}\alpha & 0 & 0\\0 & \alpha & 0\\0 & 0 & \alpha\end{pmatrix}\begin{pmatrix}0\\0\\\alpha\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix}$$

Therefore, the edge $\begin{pmatrix}0\\0\\\alpha\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \in \Gamma_1$.

In order to get the next edge in this cycle, $g_{12} = (g_{11})^2$ is computed and multiplied by the edge $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha\end{pmatrix}$ as follows below:

$$\begin{pmatrix}\alpha^2 & 0 & 0\\0 & \alpha^2 & 0\\0 & 0 & \alpha^2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \to \begin{pmatrix}\alpha^2 & 0 & 0\\0 & \alpha^2 & 0\\0 & 0 & \alpha^2\end{pmatrix}\begin{pmatrix}0\\0\\\alpha\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^3\end{pmatrix}$$

Therefore, $\begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^3\end{pmatrix} \in \Gamma_1$.
Similarly, to get the next edge, first, $g_{13} = (g_{11})^3$ is obtained. This is followed by computing $g_{13}\left(\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha\end{pmatrix}\right)$. Since $g_{13} = \begin{pmatrix}\alpha^3 & 0 & 0\\0 & \alpha^3 & 0\\0 & 0 & \alpha^3\end{pmatrix}$, it follows that

$$\begin{pmatrix}\alpha^3 & 0 & 0\\0 & \alpha^3 & 0\\0 & 0 & \alpha^3\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^3\end{pmatrix} \to \begin{pmatrix}\alpha^3 & 0 & 0\\0 & \alpha^3 & 0\\0 & 0 & \alpha^3\end{pmatrix}\begin{pmatrix}0\\0\\\alpha\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^4\end{pmatrix}$$

Hence, $\begin{pmatrix}0\\0\\\alpha^3\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^4\end{pmatrix} \in \Gamma_1$.

Proceeding on this way, the cycle containing these edges will be as follows:

$$C_1 : \begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^3\end{pmatrix} \to \dots \to \begin{pmatrix}0\\0\\\alpha^{q-1}\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad (4.10)$$

This is a cycle whose length is equal to the order of α in F_q^*. Thus C_1 is of length $q-1$. By Theorem 1.1.29, the other cycles are isomorphic to C_1, therefore to get a cycle containing the element $\begin{pmatrix}x\\y\\z\end{pmatrix}$, we multiply C_1 with the matrix $h = \begin{pmatrix}a & b & x\\c & d & y\\e & f & z\end{pmatrix} \in G$ since $\begin{pmatrix}a & b & x\\c & d & y\\e & f & z\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. Thus hC_1 is:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \to \alpha\begin{pmatrix}x\\y\\z\end{pmatrix} \to \alpha^2\begin{pmatrix}x\\y\\z\end{pmatrix} \to \alpha^3\begin{pmatrix}x\\y\\z\end{pmatrix} \to \dots \to \alpha^{q-1}\begin{pmatrix}x\\y\\z\end{pmatrix}. \quad (4.11)$$

Hence, $\left(\begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}\right)$ is an edge iff $\begin{pmatrix}x\\y\\z\end{pmatrix} = \alpha\begin{pmatrix}a\\b\\c\end{pmatrix}$.

$$\begin{aligned}\text{Therefore, } O_1 &= \left\{ g\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\\alpha\end{pmatrix}\right) : g \in G \right\}.\\ &= \left\{ \left(\begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix}x\\y\\z\end{pmatrix} = \alpha\begin{pmatrix}a\\b\\c\end{pmatrix}\right) \right\}. \quad (4.12)\end{aligned}$$

The edges of Γ_2 which corresponds to Δ_2 are computed by applying the Definition 1.1.24, $O_2 = \left\{ g\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix}\right) : g \in G \right\}$. From this, it follows that the edge $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \in \Gamma_2$. Let $g_{21} = \begin{pmatrix}\alpha^2 & 0 & 0\\0 & \alpha^2 & 0\\0 & 0 & \alpha^2\end{pmatrix} \in G$, then,

$$\begin{pmatrix}\alpha^2 & 0 & 0\\0 & \alpha^2 & 0\\0 & 0 & \alpha^2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \to \begin{pmatrix}\alpha^2 & 0 & 0\\0 & \alpha^2 & 0\\0 & 0 & \alpha^2\end{pmatrix}\begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} = \begin{pmatrix}0\\0\\\alpha^4\end{pmatrix}$$

$\therefore \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^4\end{pmatrix} \in \Gamma_2$.

To get the next edge, first, compute $g_{22} = (g_{21})^2 = \begin{pmatrix}\alpha^4 & 0 & 0\\0 & \alpha^4 & 0\\0 & 0 & \alpha^4\end{pmatrix}$, then determine $g_{22}\left(\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\\alpha^2\end{pmatrix}\right)$ as calculated below

$$\begin{pmatrix}\alpha^4 & 0 & 0\\ 0 & \alpha^4 & 0\\ 0 & 0 & \alpha^4\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\\alpha^4\end{pmatrix}\to\begin{pmatrix}\alpha^4 & 0 & 0\\ 0 & \alpha^4 & 0\\ 0 & 0 & \alpha^4\end{pmatrix}\begin{pmatrix}0\\0\\\alpha^2\end{pmatrix}=\begin{pmatrix}0\\0\\\alpha^6\end{pmatrix}$$

$$\therefore \begin{pmatrix}0\\0\\\alpha^4\end{pmatrix}\to\begin{pmatrix}0\\0\\\alpha^6\end{pmatrix}\in\Gamma_2.$$

Continuing this way, the following cycle is obtained:

$$\begin{aligned} C_2 &= \begin{pmatrix}0\\0\\1\end{pmatrix}\to\begin{pmatrix}0\\0\\\alpha^2\end{pmatrix}\to\begin{pmatrix}0\\0\\\alpha^4\end{pmatrix}\to\begin{pmatrix}0\\0\\\alpha^6\end{pmatrix}\to\ldots\to\begin{pmatrix}0\\0\\1\end{pmatrix}. \\ &= \begin{pmatrix}0\\0\\1\end{pmatrix}\to\alpha^2\begin{pmatrix}0\\0\\1\end{pmatrix}\to\alpha^4\begin{pmatrix}0\\0\\1\end{pmatrix}\to\alpha^6\begin{pmatrix}0\\0\\1\end{pmatrix}\to\ldots\to\begin{pmatrix}0\\0\\1\end{pmatrix} \end{aligned} \tag{4.13}$$

The cycle C_2 is of length equal to the order of α^2 in F_q^*. The other cycles of Γ_2 are isomorphic to C_2. Thus, to obtain a cycle containing element $\begin{pmatrix}x\\y\\z\end{pmatrix}$, C_2 is multiplied with the matrix $h=\begin{pmatrix}a & b & x\\ c & d & y\\ e & f & z\end{pmatrix}\in G$. Thus hC_2 is as follows below:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\to\alpha^2\begin{pmatrix}x\\y\\z\end{pmatrix}\to\alpha^4\begin{pmatrix}x\\y\\z\end{pmatrix}\to\ldots\to\begin{pmatrix}x\\y\\z\end{pmatrix}. \tag{4.14}$$

$$\therefore O_2=\left\{\left(\begin{pmatrix}a\\b\\c\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}\right):\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha^2\begin{pmatrix}a\\b\\c\end{pmatrix}\right\}. \tag{4.15}$$

Proceeding on the same way, it is concluded that $\forall\alpha^i\in GF(q)^*$, there exist a suborbital graph whose cycles are of length equal to the order of α^i in F_q^*. Thus,

$$O_i = \left\{g\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\\alpha^i\end{pmatrix}\right):g\in G\right\},\text{ where }\begin{pmatrix}0\\0\\\alpha^i\end{pmatrix}\in\Delta_i \tag{4.16}$$

$$= \left\{\left(\begin{pmatrix}a\\b\\c\end{pmatrix},\begin{pmatrix}x\\y\\z\end{pmatrix}\right):\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha^i\begin{pmatrix}a\\b\\c\end{pmatrix}\right\}\text{for }0\le i\le q-2. \tag{4.17}$$

4.6.2 Suborbital Graph corresponding to suborbit of length q^3-q

By Theorem 3.6.3, Δ_{q-1} is the only suborbit of length not equal to 1. Now, $O_0\cup O_1\cup O_2\cup\ldots\cup O_{q-2}\cup O_{q-1}=F_q^3\setminus\{0\}\times F_q^3\setminus\{0\}$. The suborbitals $O_0,O_1,O_2,\ldots,O_{q-2}$ whose edges are such that vertices are scalar multiple vectors are accounted in Equation 4.16. This implies that O_{q-1} comprises all edges formed by vertices which are not scalar multiples of one another. Thus,

$$O_{q-1} = \left\{ \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) : g \in G \right\}, \text{ where } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \Delta_{q-1}$$

$$= \left\{ \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ are not scalar multiples of each other} \right\} \quad (4.18)$$

4.7 Properties of *GL(3,q)* Suborbital Graphs

Theorem 4.7.1. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$ and α be a primitive element in F_q. Then, Γ_i corresponding to Δ_i, where $|\Delta_i| = 1$ has girth equal the order of α^i in F_q^*.*

Proof. By Section 4.6.1, each component has length $\left|\alpha^i\right|$ implying that girth is also $\left|\alpha^i\right|$. □

Corollary 4.7.2. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$ and α be a primitive element in F_q. Then, $\Gamma_{\frac{q-1}{2}}$ corresponding to $\Delta_{\frac{q-1}{2}}$, where $\left|\Delta_{\frac{q-1}{2}}\right| = 1$ and $2 \nmid q$, is acyclic.*

Proof. By Section 4.6.1, each component of $\Gamma_{\frac{q-1}{2}}$ is of length $\left|\alpha^{\frac{q-1}{2}}\right|$. Since $\left(\alpha^{\frac{q-1}{2}}\right)^2 = \alpha^{q-1} = 1$. It follows that $\left|\alpha^{\frac{q-1}{2}}\right| = 2$. Thus, it implies that these are components of length 2 and hence acyclic. □

Theorem 4.7.3. *Let GL(3,q) act on $Y = F_q^3 \setminus \{0\}$ and let Γ_i correspond to Δ_i, where $|\Delta_i| = 1$ and α be a be primitive element in F_q. Then, the number $\psi(\Gamma_i)$ of components is given by:*

$$\psi(\Gamma_i) = \frac{q^3 - 1}{|\alpha^i|}. \quad (4.19)$$

Proof. From Section 4.6.1, the components of Γ_i are cycles of length equal to $|\alpha^i|$ where $\alpha^i \in GF(q)^*$. This implies that each component of Γ_i has $|\alpha^i|$ vertices and by Theorem 1.1.29, the components are isomorphic. Since $|Y| = q^3 - 1$, then

$$\begin{aligned} \psi(\Gamma_i) &= \frac{|X|}{|\alpha^i|} \\ &= \frac{q^3 - 1}{|\alpha^i|}. \end{aligned}$$

□

Corollary 4.7.4. *Let $GL(3,q)$ act on $Y = F_q^3 \setminus \{0\}$, then the suborbital graph Γ_i corresponding to Δ_i, where $|\Delta_i| = 1$ is disconnected.*

Proof. By Proposition 1.1.26, suborbital graphs corresponding to suborbits of length 1 are either trees or cycles and by Corollary 4.7.3, $\psi(\Gamma_i) = \dfrac{q^3-1}{|\alpha^i|} > 1$ since, $1 \leq \left|\alpha^i\right| \leq q-1$ for $0 \leq i \leq q-2$ and thus Γ_i is disconnected. □

Theorem 4.7.5. *Let GL(3,q) act on $F_q^3 \setminus \{0\}$ and $\beta \in F_q^*$. Then, the girth of Γ_{q-1} is 3.*

Proof. By Equation 4.18, any two vertices $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in Γ_{q-1} are adjacent if they are not scalar multiples of each other. Then $\begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \to \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} \to \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}$ is a cycle of length 3 in Γ_{q-1}. Since a cycle of length 3 is the smallest, then girth is 3. □

Theorem 4.7.6. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$. Then Γ_{q-1} corresponding to Δ_{q-1} is connected.*

Proof. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be any two vertices in Γ_{q-1}. By Equation 4.18, if the two vertices are not scalar multiples of each other then they are adjacent and hence there exists a path between the two. If they are scalar multiples of each other, then there exists another vertex $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ such that $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are not scalar multiples of one another. Thus, $\left(\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)$ is an edge of Γ_{q-1}. Also $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and therefore, $\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) \in \Gamma_{q-1}$. Hence there exists a path from $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ via $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$. Thus, Γ_{q-1} is connected. □

Theorem 4.7.7. *The action of GL(3,q) on $F_q^3 \setminus \{0\}$ for q=2 is primitive.*

Proof. This action has 1 non-trivial suborbital graph which is complete and hence connected as displayed in Figure 4.8. Therefore, by Theorem 1.1.27, the action is primitive. □

Theorem 4.7.8. *The action of G=GL(3,q) on $F_q^3 \setminus \{0\}$ for $q \geq 3$ is imprimitive.*

Proof. Some non-trivial suborbital graphs of G are disconnected. It therefore follows by Theorem 1.1.27, that the action is imprimitive. □

Theorem 4.7.9. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$ and α be a primitive element in F_q, Then Γ_i corresponding to suborbit of length 1 is paired with Γ_j if i + j = q - 1.*

Proof. Let O_i and O_j correspond to Δ_i and Δ_j respectively and $|F_q^*| = q-1$. Suppose $i+j = q-1$, then $\alpha^i \cdot \alpha^j = \alpha^{q-1} = 1$. Then, $\alpha^i = \left(\alpha^j\right)^{-1}$ and $\alpha^j = \left(\alpha^i\right)^{-1}$. It follows that $\alpha^i v_1 = v_2$ and $\alpha^j v_2 = v_1$, where v_1, v_2 are vertices. Thus O_i is the reverse of O_j and hence Γ_i is paired to Γ_j. □

Corollary 4.7.10. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$ and α be a primitive element in F_q. Then, the suborbital graph $\Gamma_{\frac{q-1}{2}}$ corresponding to $\Delta_{\frac{q-1}{2}}$, where $|\Delta_{\frac{q-1}{2}}| = 1$ and $2 \nmid q$ is self-paired.*

Proof. Let $O_{\frac{q-1}{2}}$ correspond to $\Delta_{\frac{q-1}{2}}$, where $2 \nmid q$. Then, from Theorem 4.7.9 above $i = \frac{q-1}{2}$ and $j = \frac{q-1}{2}$. So $i = j$ and hence $\Gamma_{\frac{q-1}{2}}$ is self-paired. □

Theorem 4.7.11. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$. Then, the suborbital graph Γ_{q-1} corresponding to Δ_{q-1} is self-paired.*

Proof. By Equation 4.18, $\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \in O_{q-1}$ iff $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are not scalar multiples of each other. This also implies that $\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \in O_{q-1}$. Hence Γ_{q-1} is self-paired. □

Theorem 4.7.12. *Let G=GL(3,q) act on $F_q^3 \setminus \{0\}$ and Γ_i be the suborbital graph corresponding to Δ_i. Then, the diameter $diam(\Gamma_i)$ of Γ_i is given by,*

$$diam(\Gamma_i) = \begin{cases} 1 & \text{if } q = 2 \text{ and } |\Delta_i| = q^3 - q \\ 2 & \text{if } q > 2 \text{ and } |\Delta_i| = q^3 - q \\ \infty & \text{if } q > 2 \text{ and } |\Delta_i| = 1 \end{cases}$$

Proof. In the case where $q = 2$ and $|\Delta_i| = q^3 - q$, the action has only 1 non-trivial complete suborbital graph as shown in Figure 4.8 and hence $diam(\Gamma_i) = 1$.
Now, where $q > 2$ and $|\Delta_i| = q^3 - q$, vertices $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are adjacent if they are not scalar multiples of each other. This implies that distance is 1. If the two vertices are scalar multiples of each other, then they are not adjacent and hence there exists a vertex $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ which is not a scalar multiple to both $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and as a result it follows that distance from $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ via $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is 2. Thus, $diam(\Gamma_i) = 2$.
For the case where $q > 2$ and $|\Delta_i| = 1$, the suborbital graphs are disconnected by Corollary 4.7.4 and thus $diam(\Gamma_i)$ is ∞. □

Theorem 4.7.13. *Let G=GL(3,q) act on $Y = F_q^3 \setminus \{0\}$. Then, the chromatic number $\chi(\Gamma_i)$ of a suborbital graph Γ_i corresponding to a non-trivial suborbit Δ_i of length 1 is either 2 or 3.*

Proof. This proof is similar to that of Theorem 4.4.14. □

Theorem 4.7.14. *Let GL(3,q) act on $F_q^3 \setminus \{0\}$. Then, the chromatic number $\chi(\Gamma_{q-1})$ of suborbital graph Γ_{q-1} corresponding to Δ_{q-1} is q^2+q+1.*

Proof. By Equation 4.18, $\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \in O_{q-1}$ if $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are not scalar multiples of each other. For any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in Y$, there are $q-1$ scalar multiples of it. Now, the set Y can be partitioned into $\hbar$ subsets, each of length $q-1$ and whose elements are scalar multiples of each other. Now, the vectors in each $\hbar$ subset will possess the same color since they are not adjacent to each other. Since $Y = q^3-1$, then

$$\begin{aligned} \hbar &= \frac{q^3-1}{q-1} \\ &= \frac{(q^2+q+1)(q-1)}{q-1} \\ &= q^2+q+1. \end{aligned}$$

Hence, Y is partitioned into q^2+q+1 subsets of length $q-1$. Since the elements of each subset possess the same color and there are q^2+q+1 subsets, it implies that $\chi(\Gamma_{q-1}) = \hbar = q^2+q+1$. □

CHAPTER FIVE
SUMMARY, CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

5.1 Introduction

This chapter is divided into three sections. Section 5.2 contains the summary of the study. Section 5.3 gives the conclusion of the study whereas the recommendation for further research is given in Section 5.4.

5.2 Summary

The aim of this thesis was to determine transitivity, ranks, subdegrees and suborbital graphs of the action of $GL(2,q)$ and $GL(3,q)$ on non zero vectors over $GF(q)$.
In Chapter 3, it was determined that $GL(2,q)$ and $GL(3,q)$ act transitively on $F_q^2 \setminus \{0\}$ and $F_q^3 \setminus \{0\}$ respectively. Theorem 3.2.5 and Theorem 3.5.5 were derived, and as a result, achieving the first objective. Also, in this chapter, ranks and subdegrees were determined. The rank of $GL(2,q)$ acting on $F_q^2 \setminus \{0\}$ was q and the subdegrees were proved to be $[1]^{[q-1]}$ and q^2-q. Moreover, the rank of $GL(3,q)$ acting on $F_q^3 \setminus \{0\}$ was q and the subdegrees were computed and determined as $[1]^{[q-1]}$ and q^3-q. Derivation of Theorem 3.3.4 and Theorem 3.6.3 marked the achievement of the second objective.
The third objective was attained in Chapter 4, where suborbital graphs of $GL(2,q)$ acting on $F_q^2 \setminus \{0\}$ and those of $GL(3,q)$ acting on $F_q^3 \setminus \{0\}$ were constructed. In both actions, the non-trivial suborbital graphs were found to be complete where $q=2$. The suborbital graph Γ_{q-1} of $GL(2,q)$ acting on $F_q^2 \setminus \{0\}$ were observed to be connected, undirected, self-paired and have diameter as 1 for $q=2$. The suborbital graph Γ_{q-1} resulting from the action of $GL(3,q)$ on $F_q^3 \setminus \{0\}$ were undirected, connected, regular, self-paired, and girth is 3. In both actions, the non-trivial suborbital graphs corresponding to suborbits of length 1 were regular, directed if they formed cycles and undirected if they formed trees. It was shown that both actions are primitive where $q=2$ and imprimitive when $q \geq 3$.

5.3 Conclusion

In Section 3.2 and 3.5, its evident that both $GL(2,q)$ on $F_q^2 \setminus \{0\}$ and $GL(3,q)$ on $F_q^3 \setminus \{0\}$ are transitive. The ranks and subdegrees associated to both actions are calculated in Section 3.3 and 3.6. In addition, primitivity associated to $GL(2,q)$ on

$F_q^2 \setminus \{0\}$ is determined in Section 3.4 whereas primitivity associated to $GL(3,q)$ on $F_q^3 \setminus \{0\}$ is generalized in Section 3.7.
In Chapter 4, construction, methodologies and properties of suborbital graphs associated to $GL(2,q)$ on $F_q^2 \setminus \{0\}$ and $GL(3,q)$ on $F_q^3 \setminus \{0\}$ are discussed in details.

5.4 Recommendations for Further Research

Having achieved the objectives of this research, one may consider determining transitivity, primitivity, ranks, subdegrees and suborbital graphs associated to:

1. The action of $GL(4,q)$ on $F_q^4 \setminus \{0\}$ or probably consider $GL(n,q)$ on $F_q^n \setminus \{0\}$ over F_q.
2. The action of $SL(n,q)$ on $F_q^n \setminus \{0\}$ over F_q.

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APPENDICES

Appendix I: Suborbitals of *GL(2,3)* on $F_3^2 \setminus \{0\}$.

$$
\begin{aligned}
O_2 = \Bigg\{ & \left(\binom{0}{1},\binom{1}{0}\right),\left(\binom{0}{1},\binom{1}{1}\right),\left(\binom{0}{1},\binom{1}{2}\right),\left(\binom{0}{1},\binom{2}{0}\right),\left(\binom{0}{1},\binom{2}{1}\right),\\
& \left(\binom{0}{1},\binom{2}{2}\right),\left(\binom{0}{2},\binom{1}{0}\right),\left(\binom{0}{2},\binom{1}{1}\right),\left(\binom{0}{2},\binom{1}{2}\right),\left(\binom{0}{2},\binom{2}{0}\right),\\
& \left(\binom{0}{2},\binom{2}{1}\right),\left(\binom{0}{2},\binom{2}{2}\right),\left(\binom{1}{0},\binom{0}{1}\right),\left(\binom{1}{0},\binom{0}{2}\right),\left(\binom{1}{0},\binom{1}{1}\right),\\
& \left(\binom{1}{0},\binom{1}{2}\right),\left(\binom{1}{0},\binom{2}{1}\right),\left(\binom{1}{0},\binom{2}{2}\right),\left(\binom{1}{1},\binom{0}{1}\right),\left(\binom{1}{1},\binom{0}{2}\right),\\
& \left(\binom{1}{1},\binom{1}{0}\right),\left(\binom{1}{1},\binom{1}{2}\right),\left(\binom{1}{1},\binom{2}{0}\right),\left(\binom{1}{1},\binom{2}{1}\right),\left(\binom{1}{2},\binom{0}{1}\right),\\
& \left(\binom{1}{2},\binom{0}{2}\right),\left(\binom{1}{2},\binom{1}{0}\right),\left(\binom{1}{2},\binom{1}{1}\right),\left(\binom{1}{2},\binom{2}{0}\right),\left(\binom{1}{2},\binom{2}{2}\right),\\
& \left(\binom{2}{0},\binom{0}{1}\right),\left(\binom{2}{0},\binom{0}{2}\right),\left(\binom{2}{0},\binom{1}{1}\right),\left(\binom{2}{0},\binom{1}{2}\right),\left(\binom{2}{0},\binom{2}{1}\right),\\
& \left(\binom{2}{0},\binom{2}{2}\right),\left(\binom{2}{1},\binom{0}{1}\right),\left(\binom{2}{1},\binom{0}{2}\right),\left(\binom{2}{1},\binom{1}{0}\right),\left(\binom{2}{1},\binom{1}{1}\right),\\
& \left(\binom{2}{1},\binom{2}{0}\right),\left(\binom{2}{1},\binom{2}{2}\right),\left(\binom{2}{2},\binom{0}{1}\right),\left(\binom{2}{2},\binom{0}{2}\right),\left(\binom{2}{2},\binom{1}{0}\right),\\
& \left(\binom{2}{2},\binom{1}{2}\right),\left(\binom{2}{2},\binom{2}{0}\right),\left(\binom{2}{2},\binom{2}{1}\right)\Bigg\}
\end{aligned}
$$

Appendix II: Suborbitals of *GL(2,5)* on $F_5^2 \setminus \{0\}$.

$$
\begin{aligned}
O_1 = \Big\{ & \left(\binom{0}{1},\binom{0}{2}\right),\left(\binom{0}{2},\binom{0}{4}\right),\left(\binom{0}{4},\binom{0}{3}\right),\left(\binom{0}{3},\binom{0}{1}\right),\left(\binom{1}{0},\binom{2}{0}\right), \\
& \left(\binom{2}{0},\binom{4}{0}\right),\left(\binom{4}{0},\binom{3}{0}\right),\left(\binom{3}{0},\binom{1}{0}\right),\left(\binom{1}{1},\binom{2}{2}\right),\left(\binom{2}{2},\binom{4}{4}\right), \\
& \left(\binom{4}{4},\binom{3}{3}\right),\left(\binom{3}{3},\binom{1}{1}\right),\left(\binom{1}{2},\binom{2}{4}\right),\left(\binom{2}{4},\binom{4}{3}\right),\left(\binom{4}{3},\binom{3}{1}\right), \\
& \left(\binom{3}{1},\binom{1}{2}\right),\left(\binom{1}{3},\binom{2}{1}\right),\left(\binom{2}{1},\binom{4}{2}\right),\left(\binom{4}{2},\binom{3}{4}\right),\left(\binom{3}{4},\binom{1}{3}\right), \\
& \left(\binom{1}{4},\binom{2}{3}\right),\left(\binom{2}{3},\binom{4}{1}\right),\left(\binom{4}{1},\binom{3}{2}\right),\left(\binom{3}{2},\binom{1}{4}\right)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
O_2 = \Big\{ & \left(\binom{0}{1},\binom{0}{4}\right),\left(\binom{0}{4},\binom{0}{1}\right),\left(\binom{0}{2},\binom{0}{3}\right),\left(\binom{0}{3},\binom{0}{2}\right),\left(\binom{1}{0},\binom{4}{0}\right), \\
& \left(\binom{4}{0},\binom{1}{0}\right),\left(\binom{2}{0},\binom{3}{0}\right),\left(\binom{3}{0},\binom{2}{0}\right),\left(\binom{1}{1},\binom{4}{4}\right),\left(\binom{4}{4},\binom{1}{1}\right), \\
& \left(\binom{2}{2},\binom{3}{3}\right),\left(\binom{3}{3},\binom{2}{2}\right),\left(\binom{1}{2},\binom{4}{3}\right),\left(\binom{4}{3},\binom{1}{2}\right),\left(\binom{3}{1},\binom{2}{4}\right), \\
& \left(\binom{2}{4},\binom{3}{1}\right),\left(\binom{1}{3},\binom{4}{2}\right),\left(\binom{4}{2},\binom{1}{3}\right),\left(\binom{2}{1},\binom{3}{4}\right),\left(\binom{3}{4},\binom{2}{1}\right), \\
& \left(\binom{1}{4},\binom{4}{1}\right),\left(\binom{4}{1},\binom{1}{4}\right),\left(\binom{3}{2},\binom{2}{3}\right),\left(\binom{2}{3},\binom{3}{2}\right)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
O_3 = \Big\{ & \left(\binom{0}{1},\binom{0}{3}\right),\left(\binom{0}{3},\binom{0}{4}\right),\left(\binom{0}{4},\binom{0}{2}\right),\left(\binom{0}{2},\binom{0}{1}\right),\left(\binom{1}{0},\binom{3}{0}\right), \\
& \left(\binom{3}{0},\binom{4}{0}\right),\left(\binom{4}{0},\binom{2}{0}\right),\left(\binom{2}{0},\binom{1}{0}\right),\left(\binom{1}{1},\binom{3}{3}\right),\left(\binom{3}{3},\binom{4}{4}\right), \\
& \left(\binom{4}{4},\binom{2}{2}\right),\left(\binom{2}{2},\binom{1}{1}\right),\left(\binom{1}{2},\binom{3}{1}\right),\left(\binom{3}{1},\binom{4}{3}\right),\left(\binom{4}{3},\binom{2}{4}\right), \\
& \left(\binom{2}{4},\binom{1}{2}\right),\left(\binom{1}{3},\binom{3}{4}\right),\left(\binom{3}{4},\binom{4}{2}\right),\left(\binom{4}{2},\binom{2}{1}\right),\left(\binom{2}{1},\binom{1}{3}\right), \\
& \left(\binom{1}{4},\binom{3}{2}\right),\left(\binom{3}{2},\binom{4}{1}\right),\left(\binom{4}{1},\binom{2}{3}\right),\left(\binom{2}{3},\binom{1}{4}\right)\Big\}
\end{aligned}
$$

$$
O_4 = \Big\{ \left(\binom{0}{1},\binom{1}{0}\right),\left(\binom{1}{0},\binom{0}{1}\right),\left(\binom{0}{1},\binom{1}{1}\right),\left(\binom{1}{1},\binom{0}{1}\right),\left(\binom{0}{1},\binom{1}{2}\right),
$$

$$\left(\binom{1}{2},\binom{0}{1}\right),\left(\binom{0}{1},\binom{1}{3}\right),\left(\binom{1}{3},\binom{0}{1}\right),\left(\binom{0}{1},\binom{1}{4}\right),\left(\binom{1}{4},\binom{0}{1}\right),$$
$$\left(\binom{0}{1},\binom{2}{0}\right),\left(\binom{2}{0},\binom{0}{1}\right),\left(\binom{0}{1},\binom{2}{1}\right),\left(\binom{2}{1},\binom{0}{1}\right),\left(\binom{0}{1},\binom{2}{2}\right),$$
$$\left(\binom{2}{2},\binom{0}{1}\right),\left(\binom{0}{1},\binom{2}{3}\right),\left(\binom{2}{3},\binom{0}{1}\right),\left(\binom{0}{1},\binom{2}{4}\right),\left(\binom{2}{4},\binom{0}{1}\right),$$
$$\left(\binom{0}{1},\binom{3}{0}\right),\left(\binom{3}{0},\binom{0}{1}\right),\left(\binom{0}{1},\binom{3}{1}\right),\left(\binom{3}{1},\binom{0}{1}\right),\left(\binom{0}{1},\binom{3}{2}\right),$$
$$\left(\binom{3}{2},\binom{0}{1}\right),\left(\binom{0}{1},\binom{3}{3}\right),\left(\binom{3}{3},\binom{0}{1}\right),\left(\binom{0}{1},\binom{3}{4}\right),\left(\binom{3}{4},\binom{0}{1}\right),$$
$$\left(\binom{0}{1},\binom{4}{0}\right),\left(\binom{4}{0},\binom{0}{1}\right),\left(\binom{0}{1},\binom{4}{1}\right),\left(\binom{4}{1},\binom{0}{1}\right),\left(\binom{0}{1},\binom{4}{2}\right),$$
$$\left(\binom{4}{2},\binom{0}{1}\right),\left(\binom{0}{1},\binom{4}{3}\right),\left(\binom{4}{3},\binom{0}{1}\right),\left(\binom{0}{1},\binom{4}{4}\right),\left(\binom{4}{4},\binom{0}{1}\right),$$
$$\left(\binom{0}{2},\binom{1}{0}\right),\left(\binom{1}{0},\binom{0}{2}\right),\left(\binom{0}{2},\binom{1}{1}\right),\left(\binom{1}{1},\binom{0}{2}\right),\left(\binom{0}{2},\binom{1}{2}\right),$$
$$\left(\binom{1}{2},\binom{0}{2}\right),\left(\binom{0}{2},\binom{1}{3}\right),\left(\binom{1}{3},\binom{0}{2}\right),\left(\binom{0}{2},\binom{1}{4}\right),\left(\binom{1}{4},\binom{0}{2}\right),$$
$$\left(\binom{0}{2},\binom{2}{0}\right),\left(\binom{2}{0},\binom{0}{2}\right),\left(\binom{0}{2},\binom{2}{1}\right),\left(\binom{2}{1},\binom{0}{2}\right),\left(\binom{0}{2},\binom{2}{2}\right),$$
$$\left(\binom{2}{2},\binom{0}{2}\right),\left(\binom{0}{2},\binom{2}{3}\right),\left(\binom{2}{3},\binom{0}{2}\right),\left(\binom{0}{2},\binom{2}{4}\right),\left(\binom{2}{4},\binom{0}{2}\right),$$
$$\left(\binom{0}{2},\binom{3}{0}\right),\left(\binom{3}{0},\binom{0}{2}\right),\left(\binom{0}{2},\binom{3}{1}\right),\left(\binom{3}{1},\binom{0}{2}\right),\left(\binom{0}{2},\binom{3}{2}\right),$$
$$\left(\binom{3}{2},\binom{0}{2}\right),\left(\binom{0}{2},\binom{3}{3}\right),\left(\binom{3}{3},\binom{0}{2}\right),\left(\binom{0}{2},\binom{3}{4}\right),\left(\binom{3}{4},\binom{0}{2}\right),$$
$$\left(\binom{0}{2},\binom{4}{0}\right),\left(\binom{4}{0},\binom{0}{2}\right),\left(\binom{0}{2},\binom{4}{1}\right),\left(\binom{4}{1},\binom{0}{2}\right),\left(\binom{0}{2},\binom{4}{2}\right),$$
$$\left(\binom{4}{2},\binom{0}{2}\right),\left(\binom{0}{2},\binom{4}{3}\right),\left(\binom{4}{3},\binom{0}{2}\right),\left(\binom{0}{2},\binom{4}{4}\right),\left(\binom{4}{4},\binom{0}{2}\right),$$
$$\left(\binom{0}{3},\binom{1}{0}\right),\left(\binom{1}{0},\binom{0}{3}\right),\left(\binom{0}{3},\binom{1}{1}\right),\left(\binom{1}{1},\binom{0}{3}\right),\left(\binom{0}{3},\binom{1}{2}\right),$$
$$\left(\binom{1}{2},\binom{0}{3}\right),\left(\binom{0}{3},\binom{1}{3}\right),\left(\binom{1}{3},\binom{0}{3}\right),\left(\binom{0}{3},\binom{1}{4}\right),\left(\binom{1}{4},\binom{0}{3}\right),$$
$$\left(\binom{0}{3},\binom{2}{0}\right),\left(\binom{2}{0},\binom{0}{3}\right),\left(\binom{0}{3},\binom{2}{1}\right),\left(\binom{2}{1},\binom{0}{3}\right),\left(\binom{0}{3},\binom{2}{2}\right),$$
$$\left(\binom{2}{2},\binom{0}{3}\right),\left(\binom{0}{3},\binom{2}{3}\right),\left(\binom{2}{3},\binom{0}{3}\right),\left(\binom{0}{3},\binom{2}{4}\right),\left(\binom{2}{4},\binom{0}{3}\right),$$
$$\left(\binom{0}{3},\binom{3}{0}\right),\left(\binom{3}{0},\binom{0}{3}\right),\left(\binom{0}{3},\binom{3}{1}\right),\left(\binom{3}{1},\binom{0}{3}\right),\left(\binom{0}{3},\binom{3}{2}\right),$$

$$\left(\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right),\left(\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix}\right),\left(\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),\left(\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}\right),\left(\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right),\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}\right),\left(\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right),\left(\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right),\left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right),\left(\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right),\left(\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix}\right),\left(\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),\left(\begin{pmatrix}0\\4\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}\right),\left(\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}0\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}\right),\left(\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right),\left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right),\left(\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right),\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right),\left(\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right),\left(\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix}\right),\left(\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right),\left(\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}\right),\left(\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right),\left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right),\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}\right),\left(\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right),\left(\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right),\left(\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix}\right),\left(\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right),\left(\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}\right),\left(\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right),\left(\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right),\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right),\left(\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right),\left(\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}3\\0\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix}\right),\left(\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}3\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix}\right),\left(\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}\right),\left(\begin{pmatrix}4\\4\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right),\left(\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right),\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}\right),\left(\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right),\left(\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right),\left(\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}4\\0\end{pmatrix},\begin{pmatrix}4\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}4\\1\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right),\left(\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right),\left(\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}4\\3\end{pmatrix}\right),\left(\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix}\right),\left(\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}4\\2\end{pmatrix}\right),\left(\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}\right),\left(\begin{pmatrix}1\\4\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right),$$

$$\left(\binom{2}{3},\binom{1}{1}\right),\left(\binom{1}{1},\binom{3}{2}\right),\left(\binom{3}{2},\binom{1}{1}\right),\left(\binom{1}{1},\binom{4}{1}\right),\left(\binom{4}{1},\binom{1}{1}\right),$$
$$\left(\binom{2}{2},\binom{1}{2}\right),\left(\binom{1}{2},\binom{2}{2}\right),\left(\binom{2}{2},\binom{2}{4}\right),\left(\binom{2}{4},\binom{2}{2}\right),\left(\binom{2}{2},\binom{3}{1}\right),$$
$$\left(\binom{3}{1},\binom{2}{2}\right),\left(\binom{2}{2},\binom{4}{3}\right),\left(\binom{4}{3},\binom{2}{2}\right),\left(\binom{2}{2},\binom{1}{3}\right),\left(\binom{1}{3},\binom{2}{2}\right),$$
$$\left(\binom{2}{2},\binom{2}{1}\right),\left(\binom{2}{1},\binom{2}{2}\right),\left(\binom{2}{2},\binom{3}{4}\right),\left(\binom{3}{4},\binom{2}{2}\right),\left(\binom{2}{2},\binom{4}{2}\right),$$
$$\left(\binom{4}{2},\binom{2}{2}\right),\left(\binom{2}{2},\binom{1}{4}\right),\left(\binom{1}{4},\binom{2}{2}\right),\left(\binom{2}{2},\binom{2}{3}\right),\left(\binom{2}{3},\binom{2}{2}\right),$$
$$\left(\binom{2}{2},\binom{3}{2}\right),\left(\binom{3}{2},\binom{2}{2}\right),\left(\binom{2}{2},\binom{4}{1}\right),\left(\binom{4}{1},\binom{2}{2}\right),\left(\binom{3}{3},\binom{1}{2}\right),$$
$$\left(\binom{1}{2},\binom{3}{3}\right),\left(\binom{3}{3},\binom{2}{4}\right),\left(\binom{2}{4},\binom{3}{3}\right),\left(\binom{3}{3},\binom{3}{1}\right),\left(\binom{3}{1},\binom{3}{3}\right),$$
$$\left(\binom{3}{3},\binom{4}{3}\right),\left(\binom{4}{3},\binom{3}{3}\right),\left(\binom{3}{3},\binom{1}{3}\right),\left(\binom{1}{3},\binom{3}{3}\right),\left(\binom{3}{3},\binom{2}{1}\right),$$
$$\left(\binom{2}{1},\binom{3}{3}\right),\left(\binom{3}{3},\binom{3}{4}\right),\left(\binom{3}{4},\binom{3}{3}\right),\left(\binom{3}{3},\binom{4}{2}\right),\left(\binom{4}{2},\binom{3}{3}\right),$$
$$\left(\binom{3}{3},\binom{1}{4}\right),\left(\binom{1}{4},\binom{3}{3}\right),\left(\binom{3}{3},\binom{2}{3}\right),\left(\binom{2}{3},\binom{3}{3}\right),\left(\binom{3}{3},\binom{3}{2}\right),$$
$$\left(\binom{3}{2},\binom{3}{3}\right),\left(\binom{3}{3},\binom{4}{1}\right),\left(\binom{4}{1},\binom{3}{3}\right),\left(\binom{4}{4},\binom{1}{2}\right),\left(\binom{1}{2},\binom{4}{4}\right),$$
$$\left(\binom{4}{4},\binom{2}{4}\right),\left(\binom{2}{4},\binom{4}{4}\right),\left(\binom{4}{4},\binom{3}{1}\right),\left(\binom{3}{1},\binom{4}{4}\right),\left(\binom{4}{4},\binom{4}{3}\right),$$
$$\left(\binom{4}{3},\binom{4}{4}\right),\left(\binom{4}{4},\binom{1}{3}\right),\left(\binom{1}{3},\binom{4}{4}\right),\left(\binom{4}{4},\binom{2}{1}\right),\left(\binom{2}{1},\binom{4}{4}\right),$$
$$\left(\binom{4}{4},\binom{3}{4}\right),\left(\binom{3}{4},\binom{4}{4}\right),\left(\binom{4}{4},\binom{4}{2}\right),\left(\binom{4}{2},\binom{4}{4}\right),\left(\binom{4}{4},\binom{1}{4}\right),$$
$$\left(\binom{1}{4},\binom{4}{4}\right),\left(\binom{4}{4},\binom{2}{3}\right),\left(\binom{2}{3},\binom{4}{4}\right),\left(\binom{4}{4},\binom{3}{2}\right),\left(\binom{3}{2},\binom{4}{4}\right),$$
$$\left(\binom{4}{4},\binom{4}{1}\right),\left(\binom{4}{1},\binom{4}{4}\right),\left(\binom{1}{2},\binom{1}{3}\right),\left(\binom{1}{3},\binom{1}{2}\right),\left(\binom{1}{2},\binom{2}{1}\right),$$
$$\left(\binom{2}{1},\binom{1}{2}\right),\left(\binom{1}{2},\binom{3}{4}\right),\left(\binom{3}{4},\binom{1}{2}\right),\left(\binom{1}{2},\binom{4}{2}\right),\left(\binom{4}{2},\binom{1}{2}\right),$$
$$\left(\binom{1}{2},\binom{1}{4}\right),\left(\binom{1}{4},\binom{1}{2}\right),\left(\binom{1}{2},\binom{2}{3}\right),\left(\binom{2}{3},\binom{1}{2}\right),\left(\binom{1}{2},\binom{3}{2}\right),$$
$$\left(\binom{3}{2},\binom{1}{2}\right),\left(\binom{1}{2},\binom{4}{1}\right),\left(\binom{4}{1},\binom{1}{2}\right),\left(\binom{2}{4},\binom{1}{3}\right),\left(\binom{1}{3},\binom{2}{4}\right),$$
$$\left(\binom{2}{4},\binom{2}{1}\right),\left(\binom{2}{1},\binom{2}{4}\right),\left(\binom{2}{4},\binom{3}{4}\right),\left(\binom{3}{4},\binom{2}{4}\right),\left(\binom{2}{4},\binom{4}{2}\right),$$

$$\left(\binom{4}{2},\binom{2}{4}\right),\left(\binom{2}{4},\binom{1}{4}\right),\left(\binom{1}{4},\binom{2}{4}\right),\left(\binom{2}{4},\binom{2}{3}\right),\left(\binom{2}{3},\binom{2}{4}\right),$$
$$\left(\binom{2}{4},\binom{3}{2}\right),\left(\binom{3}{2},\binom{2}{4}\right),\left(\binom{2}{4},\binom{4}{1}\right),\left(\binom{4}{1},\binom{2}{4}\right),\left(\binom{3}{1},\binom{1}{3}\right),$$
$$\left(\binom{1}{3},\binom{3}{1}\right),\left(\binom{3}{1},\binom{2}{1}\right),\left(\binom{2}{1},\binom{3}{1}\right),\left(\binom{3}{1},\binom{3}{4}\right),\left(\binom{3}{4},\binom{3}{1}\right),$$
$$\left(\binom{3}{1},\binom{4}{2}\right),\left(\binom{4}{2},\binom{3}{1}\right),\left(\binom{3}{1},\binom{1}{4}\right),\left(\binom{1}{4},\binom{3}{1}\right),\left(\binom{3}{1},\binom{2}{3}\right),$$
$$\left(\binom{2}{3},\binom{3}{1}\right),\left(\binom{3}{1},\binom{3}{2}\right),\left(\binom{3}{2},\binom{3}{1}\right),\left(\binom{3}{1},\binom{4}{1}\right),\left(\binom{4}{1},\binom{3}{1}\right),$$
$$\left(\binom{4}{3},\binom{1}{3}\right),\left(\binom{1}{3},\binom{4}{3}\right),\left(\binom{4}{3},\binom{2}{1}\right),\left(\binom{2}{1},\binom{4}{3}\right),\left(\binom{4}{3},\binom{3}{4}\right),$$
$$\left(\binom{3}{4},\binom{4}{3}\right),\left(\binom{4}{3},\binom{4}{2}\right),\left(\binom{4}{2},\binom{4}{3}\right),\left(\binom{4}{3},\binom{1}{4}\right),\left(\binom{1}{4},\binom{4}{3}\right),$$
$$\left(\binom{4}{3},\binom{2}{3}\right),\left(\binom{2}{3},\binom{4}{3}\right),\left(\binom{4}{3},\binom{3}{2}\right),\left(\binom{3}{2},\binom{4}{3}\right),\left(\binom{4}{3},\binom{4}{1}\right),$$
$$\left(\binom{4}{1},\binom{4}{3}\right),\left(\binom{1}{3},\binom{1}{4}\right),\left(\binom{1}{4},\binom{1}{3}\right),\left(\binom{1}{3},\binom{2}{3}\right),\left(\binom{2}{3},\binom{1}{3}\right),$$
$$\left(\binom{1}{3},\binom{3}{2}\right),\left(\binom{3}{2},\binom{1}{3}\right),\left(\binom{1}{3},\binom{4}{1}\right),\left(\binom{4}{1},\binom{1}{3}\right),\left(\binom{2}{1},\binom{1}{4}\right),$$
$$\left(\binom{1}{4},\binom{2}{1}\right),\left(\binom{2}{1},\binom{2}{3}\right),\left(\binom{2}{3},\binom{2}{1}\right),\left(\binom{2}{1},\binom{3}{2}\right),\left(\binom{3}{2},\binom{2}{1}\right),$$
$$\left(\binom{2}{3},\binom{3}{4}\right),\left(\binom{3}{4},\binom{3}{2}\right),\left(\binom{3}{2},\binom{3}{4}\right),\left(\binom{3}{4},\binom{4}{1}\right),\left(\binom{4}{1},\binom{3}{4}\right),$$
$$\left(\binom{4}{2},\binom{1}{4}\right),\left(\binom{1}{4},\binom{4}{2}\right),\left(\binom{4}{2},\binom{2}{3}\right),\left(\binom{2}{3},\binom{4}{2}\right),\left(\binom{4}{2},\binom{3}{2}\right),$$
$$\left(\binom{3}{2},\binom{4}{2}\right),\left(\binom{4}{2},\binom{4}{1}\right),\left(\binom{4}{1},\binom{4}{2}\right),\left(\binom{2}{3},\binom{1}{2}\right),\left(\binom{1}{2},\binom{3}{2}\right),$$
$$\left(\binom{3}{2},\binom{1}{2}\right),\left(\binom{1}{2},\binom{4}{1}\right),\left(\binom{4}{1},\binom{1}{2}\right),\left(\binom{2}{4},\binom{1}{3}\right),\left(\binom{1}{3},\binom{2}{4}\right)\Big\}$$

Appendix III: Suborbitals of *GL(3,2)* on $F_2^3 \setminus \{0\}$.

$$
\begin{aligned}
O_1 = \Bigg\{ & \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right)\Bigg\}
\end{aligned}
$$

Appendix IV: Suborbitals of *GL(3,3)* on $F_3^3 \setminus \{0\}$.

$$
\begin{aligned}
O_1 = \Bigg\{ & \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
O_2 = \Bigg\{ & \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\\
& \left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),
\end{aligned}
$$

$$\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right)\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right)\left(\begin{pmatrix}2\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right)\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\1\\2\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),$$

$$\left(\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),$$

$$
\begin{aligned}
&\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\\
&\left(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),\left(\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\2\\1\end{pmatrix}\right),\\
&\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right),\\
&\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}\right),\left(\begin{pmatrix}1\\2\\2\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right),\left(\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\2\end{pmatrix}\right)\Bigg\}
\end{aligned}
$$